A "TRICK" for the Design of FIR Half-Band Filters

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Abstract — Based on a well-known property of FIR half-band filters, this correspondence shows how the design time for equiripple half-band filters can be reduced by a considerable amount. The observation which leads up to this improved procedure also places in evidence new implementation schemes, which simultaneously ensure low passband and stopband sensitivities. Extension of the method to Mth-band filter design is also outlined.

I. INTRODUCTION

Linear-phase FIR half-band filters have found several applications in the past [1], [4]. For instance, in the design of sharp cutoff FIR filters, a multistage design based on half-band filters is very efficient [2]. The efficiency of half-band filters derives from the fact that about 50 percent of the filter coefficients are zero, thus, cutting down the implementation cost. Half-cost filters have also been used in multirate filter bank applications, either directly or indirectly [3], [4].

Let $H(z)$ denote the transfer function of a (linear-phase, FIR) half-band filter of order $N - 1$

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}, \quad h(n) \text{ real. (1)}$$

These filters are restricted to be of Type 1 (i.e., $N - 1$ is even and $h(n) = h(N - 1 - n)$ [12]). The frequency response is thus of the form $H(e^{j\omega}) = e^{-j\omega(N-1)/2}H_0(e^{j\omega})$, where $H_0(e^{j\omega})$ represents the real-valued amplitude response. A typical plot of $H_0(e^{j\omega})$ is shown in Fig. 1, assuming an equiripple type of design. There is a symmetry with respect to the half-band frequency $\pi/2$, i.e., the band edges are related as

$$\omega_p + \omega_s = \pi \quad (2a)$$

and the ripples are related as

$$\delta_1 = \delta_2 = \delta. \quad (2b)$$

In view of this symmetry, the impulse response $h(n)$ satisfies

$$h(n) = \begin{cases} 0, & n = \frac{N - 1}{2} \text{ even and nonzero} \\ 1, & n = \frac{N - 1}{2} \end{cases} \quad (3)$$

The simplest way to design equiripple half-band filters is to invoke the widely used McClellan-Parks algorithm [5] with the specifications satisfying (2a) and (2b). (If equiripple nature is not a requirement, then window designs are the fastest [2].) The resulting filter satisfies (3) with reasonable accuracy. The only disadvantage with this procedure is that those coefficients which are supposed to satisfy (3) are treated as unknowns in the optimization, and, accordingly, the design time is longer than necessary.

In this correspondence, we first indicate a method (the "half-band trick") for considerably reducing the design time by exploiting the partial knowledge (3) about the impulse response coefficients. The technique also leads to a structural interpretation of half-band filters, which enables us to implement these filters in such a way that, if the structure has low passband sensitivity, then it automatically has low stopband sensitivity as well. (This is significant in view of the fact that low passband and low stopband sensitivities are often conflicting requirements [6].) We conclude by indicating how the ideas can be extended for Mth-band filters [7] (which find application in decimation and interpolation filters). Design examples are presented demonstrating the significant features of the results.

II. THE HALF BAND DESIGN TRICK

First notice that, in view of (3), we can always assume $(N - 1)/2$ to be odd. (Indeed, if $(N - 1)/2$ were even, then (3) would imply $h(0) = h(N - 1) = 0$; by redefining $h(1)$ to be $h(0)$, we can cut down the order to $N - 3$.) Given the specifications $\omega_p$, $\omega_s$, and $\delta$, let us first design a one-band prototype linear-phase filter $G(z)$ of order $(N - 1)/2$ with specifications as shown in Fig. 2. $G(z)$ has a zero at $\omega = \pi$, since $(N - 1)/2$ is odd [12]. Its passband extends from 0 to $2\omega_p$ and the transition band is from $2\omega_p$ to $\pi$. If we now define

$$H(z) = G(z^2) + z^{-N/2} \quad (4)$$

then $H(z)$ is a half-band filter, with specifications as in Fig. 1. The conditions (2a), (2b), and (3) are satisfied exactly. The impulse response of $H(z)$ is evidently related to that of $G(z)$ by

$$h(n) = \begin{cases} 0, & n \text{ odd} \\ \frac{1}{2}g\left(\frac{n}{2}\right), & n \text{ even} \\ 1, & n = \frac{N - 1}{2} \end{cases} \quad (5)$$

$G(z)$ can be designed with the help of the McClellan-Parks program. This design time is considerably lower than the time required to design $H(z)$ directly, since the order of $G(z)$ is only $(N - 1)/2$. Moreover, for large $N - 1$, the design-accuracy is better.
III. LOW SENSITIVITY STRUCTURES FOR HALF-BAND FILTERS

It is well known [6], [8] that a digital filter structure having low passband sensitivity does not necessarily have low stopband sensitivity, and vice versa. The coefficient-sensitivity problem in FIR structures has been analyzed in the past [6], [9], [10]. Based on the notion of structural passivity, certain lattice structures are proposed in [10] which can be used to synthesize low-sensitivity structures for any arbitrary FIR transfer function.

The lattice structures in [10] satisfy two crucial properties: first, they provide very low passband sensitivity; second, if the transfer function has linear phase, this linearity is maintained even when the lattice coefficients are quantized. Now assume that we first implement the one-band filter $G(z)$ using such a structure. Then $G(z)$ has low passband sensitivity. When the lattice coefficients are quantized, the magnitude response of the transfer function $G_0(z)$ remains very close to $G(z)$. Since $G_0(z)$ retains linear phase and has odd order $(N - 1)/2$, it continues to have the zero at $\omega = \pi$ despite quantization. Suppose we realize $H(z)$ from this structure for $G(z)$, exactly as suggested by (4) (see Fig. 4). Then the stopband response of $H(z)$ is exactly an image of its passband response, even if the coefficients of the lattice are quantized! Thus, $H_0(z)$ (the response of the quantized lattice) continues to remain a half-band filter and has low passband as well as stopband sensitivities.

Example 2: A half-band filter $H(z)$ with $\omega_p = 0.45\pi$, $\omega_s = 0.55\pi$, and with stopband attenuation of about 86 dB was implemented as in Fig. 4. The required order was $N - 1 = 102$ ($G(z)$ has order 51). The internal details of the lattice structure used to implement $G(z)$ are not relevant here and can be found in [10]. The lattice coefficients were quantized to 5 bits in canonical sign digit code, and the resulting response is as shown in Fig. 5(a). A direct-form implementation of $H(z)$ with coefficients quantized to 5 bits in canonical sign digit code has response as in Fig. 5(b). It is evident that the lattice structure has good passband and stopband sensitivities.

IV. EXTENSION TO THE DESIGN OF $M$th BAND FILTERS

An $M$th band linear-phase (low-pass) FIR filter [12] is a Type 1 filter $H(z)$ with amplitude response $H_0(e^{j\omega})$ as shown in Fig. 6. The impulse response $h(n)$ has one out of $M$ samples equal to zero; the main properties equal to zero; the main properties can be summarized as follows:

$$\omega_p + \omega_s = \frac{2\pi}{M}$$

$$\delta_s \leq \frac{(M - 1)\delta_2}{2}$$

$$h(n) = \left\{ \begin{array}{ll}
0, & n = \frac{N - 1}{2} \text{ nonzero multiple of } M \\
1, & n = \frac{N - 1}{2}
\end{array} \right.$$
A direct design of these filters based on judicious use of [5] is outlined in [7]. It is shown there that such a design requires a good guess of the relative values of $\delta_1$ and $\delta_2$, so that (6c) is satisfied. Since (6c) is only approximately satisfied by the resulting design, its coefficients $h(n)$ are readjusted so that (6c) holds exactly. This results in a loss of equiripple nature. Now, since (6c) prespecifies some of the impulse response coefficients, we can once again, in principle, save design time by eliminating these from the optimization problem. This strategy has the additional advantage that (6c) is then exactly satisfied.

The natural question that arises here is: Is there a logical extension of the half-band trick to the Mth-band case? Basically, we wish to first design a related "prototype" function $G(z)$ of lower order (which does not involve the coefficients in (6c)) and the construct $H(z)$ from $G(z)$. The first issue is how to formulate such an improved procedure, and secondly, how much, if any, improvements of design time is obtainable? We feel that the answers to these are important to know.

Let us first define a function $V(z)$, related to the $M$th-band filter $H(z)$ of order $N - 1$, as follows:

$$V(z) = H(z) - \frac{1}{M} z^{-((N-1)/2)}.$$  

(7)

Basically, the impulse response $v(n)$ is same as $h(n)$, with the middle term $h((N - 1)/2)$ replaced by zero. Thus

$$v(n) = 0, \quad n - \frac{N - 1}{2} \text{ multiple of } M.$$  

(8)

Let us represent $V(z)$ in terms of its "polyphase components" [2] as follows:

$$V(z) = \sum_{k=0}^{M-1} z^{-k} F_k(z^{M})$$  

(9)

so that $v(n)$ is related to the coefficients $f_k(n)$ of $F_k(z)$ by

$$f_k(n) = v(k + nM), \quad 0 \leq k \leq M - 1.$$  

(10)

Because of the constraint (6c), we conclude that $f_{n_0}(z) = 0$, where $n_0 = N - 1/2 \mod M$. We now define the prototype $G(z)$ to be the function, whose impulse response $g(n)$ is constructed from $v(n)$ by eliminating the zero-valued samples in (8). Thus

$$G(z) = \sum_{k=0}^{M-2} z^{-k} F_k(z^{M-1}) + \sum_{k=n_0}^{M-1} z^{-k} F_{k+1}(z^{M-1}).$$  

(11)

The relation (10) implies

$$F_k(z) = \frac{z^{k}}{M} \sum_{l=0}^{M-1} (z^{1/M} W^{-l})^k V(z^{1/M} W^{-l})$$  

(12)

and therefore

$$G(z) = \frac{1}{M} \left[ \sum_{k=0}^{n_0-1} z^{-k} \sum_{l=0}^{M-1} (z^{(M-1)/M} W^{-l})^k V(z^{(M-1)/M} W^{-l}) \right. \left. + \sum_{k=n_0}^{M-1} z^{-k} \sum_{l=0}^{M-1} (z^{(M-1)/M} W^{-l})^k V(z^{(M-1)/M} W^{-l}) \right]$$  

(13)

where $W = e^{j2\pi/ M}$. The design procedure is now the following: Given the specifications $\omega_p$ and $\omega_s$ for $H(z)$, find the specifications of $G(z)$ by using the relation (13). $G(z)$ has a distorted low-pass specification, with cutoff frequencies

$$\omega'_p = \omega_p \left( \frac{M}{M-1} \right), \quad \omega'_s = \omega_s \left( \frac{M}{M-1} \right).$$  

(14)
The attenuation requirement in a given band (for \(G(z)\)) is not a constant any more because of the complicated relationship featured by (13). However, because of the extreme flexibility of the software in [5], \(G(z)\) satisfying these specifications can indeed be designed. The impulse response \(k(n)\) of the \(M\)th-band filter can then be computed from \(g(n)\) as

\[
h(n) = \begin{cases} 
0, & \text{if } n = 0 \\
\frac{1}{M}, & \text{if } n = n_0 + M \\
g\left(n - \frac{n - n_0 + M}{M}\right), & \text{otherwise.}
\end{cases}
\]

**Example 3:** Consider a 5th-band FIR filter of order \(N - 1 = 56\) with transition band \(\Delta f = 0.05\). There are two stopbands (and a “don’t care” band) in the region \(0 \leq \omega \leq \pi\). We use a relative weighting function 1.5.5 for the designs. Fig. 7 shows the relevant responses for the filter designed using the above method and also by method described in [7].

Table II compares the design time required by the new and conventional methods for \(M\)th-band filters with various specifications. Since \(G(z)\) has fewer coefficients than \(H(z)\), its design is usually faster. However, the complexity involved in mapping the specifications from \(H(z)\) to \(G(z)\) sometimes reduces the effectiveness of this approach, as seen from the table. Our conclusion is that, for \(M = 3\), a marginal improvement is obtained by using the above method, whereas for large \(M\), the improvement is negligible (and often negative).

**V. CONCLUDING REMARKS**

In this paper, we described an improved method to design equiripple linear-phase FIR half-band and \(M\)th-band digital filters. The new design method is considerably faster than the conventional approach for half-band filters. For third-band filters, there is hardly any difference in the design time, as compared to conventional methods. It, however, is slower than the conventional approach for \(M > 3\) due to additional computations needed in the new method. Since half-band and \(M\)th-band FIR filters are widely used in signal decimation, interpolation, filtering, and also in the minimization of intersymbol interference [11], the improved designs presented above are expected to be of wide interest.

**ACKNOWLEDGMENT**

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\[
n = \frac{N - 1}{2} = \text{nonzero multiple of } M, \\
n = \frac{N - 1}{2}, \text{ otherwise.}
\]

**REFERENCES**


**TABLE II**

**Comparison of the New and Conventional Methods in \(M\)th Band Filter Designs**

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<th>(N - 1)</th>
<th>(M)</th>
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<th>(\epsilon_1)</th>
<th>(\epsilon_2)</th>
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