STRESS INDUCED ANISOTROPY IN PRESSURIZED THICK WALLED CYLINDERS

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I. Introduction

The most important mechanical features of propellants arise from the presence of a highly packed array of granular particles (filler), and a distribution of adhesive strengths between the rubbery binder and these particles. The first factor leads to dilatation and the formation of voids in any stress field other than pure hydrostatic compression. The second factor virtually guarantees that the pullaway of the binder from the filler is nonuniform, leading in extreme cases to the so-called "zebra-stripe" effect, or localized dewetting. This factor also is associated with stress relaxation due to the slow flow of the binder from regions of high strain concentration into regions of low concentration or into voids.

Finally, because the binder is incompressible, and the filler is for all practical purposes infinitely rigid, most of the macroscopically applied load is concentrated as large strains near the binder-filler interfaces leading to non-linear behavior. At ambient temperature or thereabouts, viscoelasticity as associated with polymer chain uncoiling plays no role in the mechanical behavior of the propellant. Summarizing, the important mechanical features to be expected are

1. Dilatation with void formation when the stress is tensile.
2. Localized dilatation because of nonuniformity of adhesion strengths.
3. Stress relaxation due to binder flow and perhaps due to particle movement at a very slow rate determined by frictional and adhesive effects.
4. Nonlinear stress-strain relations due to high local strains at binder-filler interfaces.

Question: How to deal with such a material, particularly in large motors!

We seek first a description of the material in terms of average macroscopic behavior, and simultaneously to solve problems in which the macroscopic strains do not exceed 20 percent. This rules out the need for nonlinear analysis, for the time being.
Secondly, we choose to work within a relatively short time scale and to use values of mechanical parameters appropriate to this time scale. This rules out the need for viscoelastic analysis.

Thirdly, we assume that the adhesion strengths are uniform. This makes for unrealistic properties in a sense, but it will be shown that conclusions drawn from this analysis can easily be extended to those which would be expected if the adhesion were assumed non-uniform.

Fourthly, mechanical parameters are assumed to be discontinuous on the coordinate planes in principal stress space. Thus, whenever any principal stress changes sign, a modulus changes value.* It will be shown that this leads to a first-order discontinuity in strain or stress, which is not observed in real propellant, because the distribution of adhesion strengths smooths this effect out.

II. The Normal Stress-Strain Equations for Anisotropic Behavior

Consider, for simplicity, the example of a parallelopipedal-shaped body subjected to hydrostatic pressure \( P \), positive longitudinal stress \( \sigma_l \), positive lateral stress \( \sigma_{lat} \), and positive thickness stress \( \sigma_{th} \) with \( P > \sigma_l > \sigma_{lat} > \sigma_{th} \). Assume also initially that:

\[
P (1 - 3 \gamma_c) > \sigma - \gamma_c (\sigma_{lat} + \sigma_{th}) > 0
\]

\[
\sigma_l = \frac{\sigma_{lat} - \sigma}{\sigma_{th} - \sigma_l}
\]

Then the longitudinal strain is compressive and is given by:

\[
\varepsilon = -\frac{P (1 - 3 \gamma_c) - [\sigma - \gamma_c (\sigma_{lat} + \sigma_{th})]}{F_c}
\]

\[
\varepsilon = \frac{[\sigma - \gamma_c (\sigma_{lat} + \sigma_{th})] - \sigma (1 - 3 \gamma_c)}{F_l}
\]

* This condition may be generalized by allowing the discontinuity to occur when the maximum principal stresses exceeds a positive value equal to the adhesion stress between the filler and binder.
The associated strains are given by:

\[
\begin{align*}
\varepsilon & = \left( \begin{array}{ccc}
\frac{1}{E_T} & -\frac{\nu}{E_c} & -\frac{\nu}{E_c} \\
-\frac{\nu}{E_c} & \frac{1}{E_c} & -\frac{\nu}{E_c} \\
-\frac{\nu}{E_c} & -\frac{\nu}{E_c} & \frac{1}{E_c}
\end{array} \right) \begin{pmatrix}
6 - P \\
6_{lat} - P \\
6_{th} - P
\end{pmatrix}
\end{align*}
\]

(3)

Only compressive parameters are involved since each principal axis is in compression. Now as \( \theta \) is increased, maintaining \( \varepsilon_{lat} \) and \( \varepsilon_{th} \) fixed, we go through a discontinuity in material properties at \( \varepsilon = P \), and pass over to the state \( \varepsilon > P > \varepsilon_{lat} > \varepsilon_{th} \), which is characterized by:

\[
\begin{align*}
\varepsilon & = \left( \begin{array}{ccc}
\frac{1}{E_T} & -\frac{\nu}{E_i} & -\frac{\nu}{E_i} \\
-\frac{\nu}{E_i} & \frac{1}{E_i} & -\frac{\nu}{E_i} \\
-\frac{\nu}{E_i} & -\frac{\nu}{E_i} & \frac{1}{E_i}
\end{array} \right) \begin{pmatrix}
6 - P \\
6_{lat} - P \\
6_{th} - P
\end{pmatrix}
\end{align*}
\]

(4)

where we have introduced the new mechanical parameters, \( E_T \), the tensile modulus, \( \nu_i \), the interaction Poisson's ratio, and \( E_i \), the interaction modulus. Because the binder is free to pull away from the filler, macroscopically one has \( E_T < E_c \) and \( \nu_i < \nu_c \sim 1/2 \). Moreover, at the point \( \varepsilon = P \), we have:

\[
\begin{align*}
\varepsilon^* & = \frac{\nu}{E_c} \left( 2P - \varepsilon_{lat} - \varepsilon_{th} \right) \\
\varepsilon_{lat}^* & = \frac{\varepsilon_{lat} - \nu_c \varepsilon_{th}}{E_c} - \frac{P}{2\mu_c} \\
\varepsilon_{th}^* & = \frac{\varepsilon_{th} - \nu_c \varepsilon_{lat}}{E_c} - \frac{P}{2\mu_c}
\end{align*}
\]

(5) (6) (7)
Equations (6) and (7) follow from both (3) and (4), whereas (5) generates an inequality since in general \( \nu_i / E_i \neq \nu_c / E_c \). Equation (5) therefore shows that, as the material goes from compression to tension in the longitudinal direction, a sudden increase in strain and a sudden decrease of stiffness of the material will result, providing:

\[
\frac{\nu_i}{E_i} < \frac{\nu_c}{E_c}
\]  
(8)

\[
\nu_i < \nu_c
\]  
(9)

Equation (8) implies that:

\[
E_i \ll E_c
\]  
(9)

\[
\text{since}
\]

\[
\nu_i \ll \nu_c
\]  
(10)

i.e., -- drop in \( E \) is much greater, ratio wise, than the drop in \( \nu \).

A schematic plot of longitudinal strain vs longitudinal stress at fixed lateral and thickness stresses reveals the nature of this discontinuity, under the assumption (8).

\[
\frac{E_i}{\nu_i} < \frac{E_c}{\nu_c}
\]

\[
\frac{E_i}{\nu_i} < \nu_i < \nu_c
\]  
(10)
Figure 1. Discontinuity in Strain Associated with Stress-Induced Anisotropy (P > $\sigma_{\text{lat}}$

$> \sigma_{\text{th}} > 0$, fixed)

The real behavior of the real material is evinced by the curved line which smoothes out the discontinuity because of the distribution of adhesion strengths, which contribute effectively to the hydrostatic pressure $P$. It does not seem physically meaningful to allow for the reversal of the inequality in (8), since this would imply that the material stiffens after pullaway.

Continuing the previous approach, we now increase $\sigma_{\text{lat}}$ maintaining $P$, $\sigma$, $\sigma_{\text{th}}$ fixed until we arrive at the state $\sigma > \sigma_{\text{lat}} > P > \sigma_{\text{th}} > 0$.

In this state, the strains are given by:
where we have now introduced the tensile Poisson's ratio \( \nu_T \).

At the discontinuity \( G_{lat} = p \), we have:

\[
\epsilon^k = \frac{G - p}{E_T} + \frac{\nu_T}{F_T} \left( p - G_{th} \right)
\]  

(12)

\[
\epsilon_{\omega} = -\frac{\nu_{i}}{F_i} \left( G - p \right) + \frac{\nu_c}{F_c} \left( p - G_{th} \right) \leq \frac{\nu_T}{F_T} \left( G - p \right) + \frac{\nu_c}{F_c} \left( p - G_{th} \right)
\]  

(13)

\[
\epsilon_f^k = \frac{\nu_{i}}{F_i} \left( G - p \right) - \frac{p - G_{th}}{F_c}
\]  

(14)

Equations (12) and (14) follow both from (4) and (11). Again (13) reveals a discontinuity in strain providing

\[
\frac{\nu_c}{F_c} < \frac{\nu_{i}}{F_i} < \frac{\nu_T}{F_T}
\]  

(15)

where the latter half of (15) implies that:
\[ F_T << E_T \]  \hspace{1cm} (16)

\[ \nu_T < \nu_i \]  \hspace{1cm} (17)

i.e., the drop in \( E \) is much greater ratio-wise than the drop in \( \nu \).

Again the situation corresponding to a reversal of the inequality in (15) is not considered physically meaningful. Figure 1 can now be extended by making a plot of \( \xi \) vs \( \xi_{\text{lat}} \). In this new plot the slope starts out at

\[ \frac{\nu_i}{E_i} \quad \text{and then suddenly increases to} \quad \frac{\nu_T}{F_T} \quad \text{at the discontinuity}. \]

Finally we can then extend the analysis to proceed to the state

\[ 0 > \xi_{\text{lat}} > \xi_{\text{th}} > P > 0 \], resulting from increasing \( \xi_{\text{th}} \). Our needs from this point on will be satisfied by the matrices (3), (4) and (10).

III. Pressurization of an Incompressible Grain in a Thin Elastic Case

We now apply the concepts of section B to a propellant grain (assumed incompressible, but not undilatable) bonded to and pressurized in a thin elastic steel case. Since the signs of the stress field are not known to start, it is first assumed that all normal stresses are negative; then the conditions which must be satisfied in order that the propellant behave isotropic are determined. Taking the inner and outer radii of the grain to be \( a \) and \( b \) and the outer radius of the case to be \( c \), and defining;

\[ C^2 = b^2 (1 + \gamma) \]  \hspace{1cm} (18)
one finds in straightforward fashion:

\[
G_\alpha = \lambda^2 \left( 1 - \alpha_c \right) \quad (19)
\]

\[
G_\beta = \frac{\nu_c}{\nu} \frac{\left( \frac{1-2\nu}{1-\nu} + 1 \right) - \frac{T}{1+\nu}}{1 - \frac{2\nu}{1-\nu}} - \frac{\nu_c}{\nu} \frac{\left( \frac{1-2\nu}{1-\nu} + 1 \right) \frac{c^2}{c^2}}{1+\nu} \quad (20)
\]

\[
G_3 = \frac{\nu_c}{\nu} \frac{\left( \frac{1-2\nu}{1-\nu} + 1 \right) - \frac{T}{1+\nu}}{1 - \frac{2\nu}{1-\nu}} - \frac{\nu_c}{\nu} \frac{\left( \frac{1-2\nu}{1-\nu} + 1 \right) \frac{c^2}{c^2}}{1+\nu} \quad (21)
\]

\[
G_\delta = \frac{\nu_c}{\nu} \frac{\left( \frac{1-2\nu}{1-\nu} + 1 \right) - \frac{T}{1+\nu}}{1 - \frac{2\nu}{1-\nu}} - \frac{\nu_c}{\nu} \frac{\left( \frac{1-2\nu}{1-\nu} + 1 \right) \frac{c^2}{c^2}}{1+\nu} \quad (22)
\]

where the unmarked parameters \( \nu, \gamma \) refer to the case while \( \nu_c \) is the compressive shear modulus of the propellant. Comparison of (20), (21) and (22) indicates that the most stringent restriction on the geometric and mechanical parameters is imposed by the condition that \( G_\beta \) be negative. This guarantees a fortiori that \( G_\delta \) and \( G_\delta \) are negative. Furthermore the inequality \( G_\gamma < 0 \) is tightest at the point \( r = a \); and may be written:

\[
\frac{\nu_c}{\nu} \left( \frac{1-2\nu}{1-\nu} + 1 \right) < \frac{1-2\nu}{1-a} \quad (23)
\]
Assuming that
\[ \nu = .3 \]
\[ \tau = .008 \]
\[ \alpha = .80 \]
\[ \nu = 10^7 \text{ psi} \]
it is found that:

\[ N_0 < 9500 \text{ psi} \quad \text{or} \quad E_x < 38,500 \text{ psi} \]  
(24)

Referring to Figure II-3 of GALCIT 101-4, it is seen that this inequality is observed providing \( T > -40^\circ \text{F} \). Below this temperature, the tangential stress becomes tensile. In fact, as the temperature is continually decreased, one gradually arrives at the point where the axial stress also becomes tensile. The radial stress, on the other hand, always remains compressive.

Now in order to solve for the stress-displacement field below \(-40^\circ \text{F}\), it is necessary to combine solutions obtained from matrices of the types (3) and (4) and also to solve for the position of the boundary \( r^* \) between the two regions of propellant — the one between \( b \) and \( r^* \) where all stresses are compressive, and the other between \( r \) and \( a \) where only the tangential stress is tensile.

In order to solve for the stress-displacement field below the temperature at which two of the stresses (axial and tangential) become tensile, it is necessary to allow for the possibility of three regions of behavior and this to solve for two floating boundaries. The algebra in both of these problems is quite complex. We shall thus consider a simplified problem in which the case is assumed rigid, but in which both pressurization and thermal stresses are present. It is readily shown that in the absence of pressure stresses, thermal stresses do produce only tensile stresses. Thus in the mixed loading condition, the thermal stresses cause a floating boundary to develop at a transition temperature higher than the one calculated from equation (24).
IV. Thermal and Pressurization Stresses in a Grain Bonded to a Rigid Case

Proceeding as before, one obtains, assuming isotropy;

\[
G_0 = \frac{[F\sigma - P(1-\alpha_t)] + [F\sigma + P(1-\alpha_t)] (\alpha_t) \frac{A^2}{4}}{2-3\nu-\alpha_t} \tag{25}
\]

\[
G_3 = \frac{F\sigma (\alpha_t) - 3\nu P(\alpha_t)}{2-3\nu-\alpha_t} \tag{26}
\]

\[
G_\alpha = \frac{[F\sigma - P(1-\alpha_t)] - [F\sigma + P(1\nu)] (1-\alpha_t) \frac{A^2}{4}}{2-3\nu-\alpha_t} \tag{27}
\]

\[
\delta = \Delta \alpha \left(-\Delta T\right) \tag{28}
\]

In order that all these stresses be positive, a situation which can arise only at the lowest temperatures, it is necessary that the numerator of \(G_\alpha\) be positive. This is the most stringent restriction and leads to:

\[
\left[E_T \delta - P(1-\alpha_t)\right] > \left[E_T \delta + P(1\nu)\right] (1-\alpha_t) \frac{A^2}{4} \tag{29}
\]

Obviously this inequality can not be observed for all \(\nu\), since, at \(\nu = a\), we have:

\[
-(1-a_t) > 1-3\nu \tag{30}
\]

, which is impossible.
In order to guarantee that the remaining two stresses be positive, we have for both $\sigma_r$ and $\sigma_\theta$:

$$E_T \sigma > 3 \nu_T' \rho \left( \frac{1 - \nu_T' \rho}{2 - \nu_T' \rho} \right)$$  \hspace{1cm} (31)

Assuming $P = 900 \text{ psi}$

$$\nu_T = .80$$

$$\delta = \Delta \Delta (-\Delta_T^r) = 10^{-2}$$

$$\nu_T' = 1/4$$

it follows that

$$E > 7500 \text{ psi}$$  \hspace{1cm} (32)

which means that the temperature be lower than $-100^\circ \text{F}$.

Assuming now that the radial stress is compressive in some region between $\alpha < \alpha < \alpha^-$, solution with the use of matrix (10) leads to

$$\sigma_r = \frac{C}{A^{1+K}} + \frac{D}{A^{1-K}} + \frac{I}{1-K^2}$$  \hspace{1cm} (33)

$$\frac{d}{A} = \frac{C}{A^{1+K}} \left[ \frac{-\nu_k}{E_i} - \frac{k \nu_i}{E_i} + \nu_i \frac{F_T}{E_i} \right] + \nu_i \frac{F_T}{E_i} < \frac{1 + \nu_i \frac{F_T}{E_i}}{1 - \nu_i^2}$$  \hspace{1cm} (34)

$$K = \left( \frac{\nu_i \frac{F_T}{E_i}}{1 - \nu_i^2} + \frac{1}{E_i} \right) \frac{F_T}{E_i} \frac{\nu_i}{E_i} < 1$$  \hspace{1cm} (35)

$$I = \frac{E_T \sigma}{1 - \nu_i^2} \left( 1 + \nu_i \frac{F_T}{E_i} \right)$$  \hspace{1cm} (36)
In the region between \( A < r < b \), where all stresses or tensile, we have

\[
G_A = A - \frac{B}{A^2}
\]  

Equations (33) - (38) involve five unknowns: \( A, B, C, D, f_r \). By setting the radial stresses in (33) and (37) each equal to zero at \( r = r_k \), and by matching the displacements in (34) and (38) at \( r = r_k \), and by setting \( \psi = 0 \) in (38) at \( r = b \), and \( G_T = -p \) in (33) at \( r = a \), it is possible to solve for the unknown constants using a trial-and-error numerical technique.

In order to proceed more rapidly, we assumed the position of the floating boundary and calculated the pressure needed to place it there. The following geometrical and mechanical and thermal parameters were assumed,

\[
\begin{align*}
E_c &= 12000 \text{ psi} \\
E_i &= 6000 \text{ psi} \\
E_T &= 3000 \text{ psi} \\
\nu_c &= \frac{1}{2} \\
\nu_i &= \frac{1}{3} \\
\nu_T &= \frac{1}{4} \\
\Delta r &= 10^{-4} \\
-\Delta T &= 100^\circ F \\
\delta &= 10^{-2} \\
b &= 80 \text{ in} \\
a &= 35.8 \text{ in} \\
\rho &= 58 \text{ in}
\end{align*}
\]

For these assumed values, the operating pressure turns out to be 300 psi.

At the boundary \( r_k^p = 58 \text{ in. which is about halfway between "a" and "b"} \) there is a discontinuity in the tangential and axial stress components as well as in the radial strain. The importance of stress-induced anisotropy, or the marked deviation from the results which would be obtained in the inisotropic situation, is shown by the magnitude of this discontinuity in the tangential stress, which is evaluated to be:

60 psi on the case - side of the boundary
and 45 psi on the port - side of the boundary.
The analytic expression for the discontinuity is readily determined to be:

\[ \Delta g_y = \frac{F_T}{1 - \frac{I}{\sqrt{1 + \frac{1}{b^2}}} \left( 1 - \frac{1}{b^2} \right)} \]

(39)

\[ \Delta g_z = \frac{2 F_T \frac{S}{1 - \frac{I}{\sqrt{1 + \frac{1}{b^2}}} \left( 1 - \frac{1}{b^2} \right)}}{1 - \frac{I}{\sqrt{1 + \frac{1}{b^2}}} \left( 1 - \frac{1}{b^2} \right)} - \frac{S E_T \left( 1 - \frac{1}{b^2} \right)}{1 - \frac{I}{\sqrt{1 + \frac{1}{b^2}}} \left( 1 - \frac{1}{b^2} \right)} \]

(40)

It is observed that the magnitude of these discontinuities is directly proportional to \( F_T S \), and thus become larger with decreasing temperature, while the boundary moves in the direction of increasing radius.

In carrying out this analysis, it was necessary to assign values to \( \nu_i \) and \( E_i \), as well as \( \nu_T \) and \( E_T \). Up to the present, most propellant data has been analyzed in terms of two parameters only. Current efforts among experimentalists at a number of solid rocket plants are being devoted to measuring those so-called bilinear properties of real propellants.