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On the Dynamic Response of a Beam to a Randomly Moving Load

The problem considered is that of an infinitely long elastic beam subject to a moving concentrated force whose position is a stochastic function of time, $X(t)$. The expected deflection and expected bending moment are analyzed, with special attention being given to the case of a stationary process $X(t)$ and to the case in which $X(t)$ is a Wiener process.

1 Introduction

THE DYNAMIC behavior of structures under the influence of moving loads is a subject of considerable engineering importance, and much attention has been given to the corresponding mathematical problems. This is especially true for the simpler structures, such as strings and beams, for which analytical complications are at a minimum.

Previously, work[1] on these problems has been largely limited to loads moving with constant velocity. A modest increase in the complexity of the motion of the load leads to a considerable increase in the complexity of the details of the analysis. Our purpose in this paper is to point out that for *highly* complicated motions of the load—those which can be regarded as realizations of random processes—it is possible to obtain information concerning the simplest statistical properties of the transient dynamic response.

More specifically, we shall consider an elastic beam initially at rest and occupying the entire x-axis in its undeflected state. At time $t = 0$, a concentrated transverse force is applied to the beam and, for $t > 0$, the force moves along the beam so that its position at time t is $x = X(t)$. While our interest is in the case of a *random process* $X(t)$, we begin in the next section with an analysis of the resulting deflection when $X(t)$ is a deterministic, but arbitrary, function. We then consider the case of a random process $X(t)$ in Section 3 and, in Section 4, we obtain some general results for processes $X(t)$ which are *stationary*. In Section 5, we examine the response of the beam when $X(t)$ is *normal* as well as stationary. Finally, we treat in Section 6 the case in which $X(t)$ is a Wiener process, corresponding to a moving load whose position at time t is that described by the simplest model of a particle in Brownian motion on the x-axis.

The present analysis, based as it is on the simplest theory of the vibrations of a beam, will presumably reflect the known weaknesses of this theory. Thus, while we might reasonably expect to have confidence in the long-time asymptotic estimates obtained in what follows, the short-time information which comprises part of the results discussed in Section 5 might be altered if the problem were treated on the basis of a more refined theory of beam motion such as that of Timoshenko. An analysis parallel to the present one but using the more accurate theory would be an interesting but involved undertaking.

[1] For relatively recent work, see [1–5],[2] where references to earlier papers will be found.

[2] Numbers in brackets designate References at end of paper.

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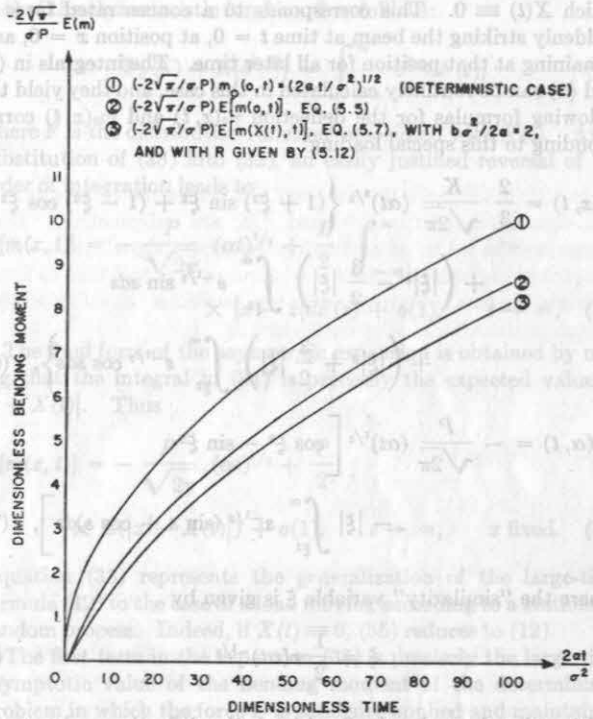

Fig. 1 Comparison of bending moment under the load in the deterministic case with the expected bending moments at the mean position and at the instantaneous position of a load moving according to a stationary normal process

2 Deterministic Problem

According to the simplest theory, the deflection $u(x, t)$ of an elastic beam subject to a concentrated force P located at position $x = X(t)$ at time t satisfies the differential equation

$$u_{xxxx} + \frac{1}{\alpha^2} u_{tt} = K\delta[x - X(t)] \tag{1}$$[3]

Here δ is the Dirac delta function, and the constants K and α^2 are given by

$$K = \frac{P}{EI}, \quad \alpha^2 = \frac{EI}{\rho}, \tag{2}$$

E is Young's modulus, I is the moment of inertia of the cross section of the beam, and ρ is the mass per unit length of the beam material. The initial conditions accompanying (1) are

$$u(x, 0) = u_t(x, 0) = 0 \tag{3}$$

We consider the differential equation for $-\infty < x < \infty$ and $t > 0$, subject to the restriction that $u(x, t)$ and its x and t-derivatives tend to zero as $x \to \pm\infty$.

[3] Throughout our discussion, subscripts x and t represent partial derivatives.

For any $X(t)$, a formal solution to the foregoing initial value problem can be constructed with the aid of the Fourier transform. It is given by

$$u(x, t) = \frac{\alpha K}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_0^t d\tau e^{i\lambda[X(\tau)-x]} \frac{\sin \lambda^2\alpha(t-\tau)}{\lambda^2} \quad (4)$$

The bending moment $m(x, t)$ at position x at time t is given by

$$m(x, t) = EIu_{xx}(x, t)$$
$$= -\frac{\alpha P}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_0^t d\tau e^{i\lambda[X(\tau)-x]} \sin \lambda^2\alpha(t-\tau), \quad (5)$$

where we have differentiated (4) twice under the integral signs.[4]

For purposes of subsequent reference and comparison, we record here some results for the special deterministic problem in which $X(t) \equiv 0$. This corresponds to a concentrated force P suddenly striking the beam at time $t = 0$, at position $x = 0$, and remaining at that position for all later time. The integrals in (4) and (5) can be routinely calculated in this case, and they yield the following formulas for the deflection $u_0(x, t)$ and $m_0(x, t)$ corresponding to this special loading:

$$u_0(x, t) = \frac{2}{3} \frac{K}{\sqrt{2\pi}} (\alpha t)^{3/2} \Big\{ (1+\xi^2) \sin \xi^2 + (1-\xi^2) \cos \xi^2 + \left(|\xi|^3 - \frac{3}{2}|\xi| \right) \int_{\xi^2}^{\infty} s^{-1/2} \sin s ds + \left(|\xi|^3 + \frac{3}{2}|\xi| \right) \int_{\xi^2}^{\infty} s^{-1/2} \cos s ds \Big\}, \quad (6)^5$$

$$m_0(\alpha, t) = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} \left[\cos \xi^2 - \sin \xi^2 - |\xi| \int_{\xi^2}^{\infty} s^{-1/2} (\sin s + \cos s) ds \right], \quad (7)^5$$

where the "similarity" variable ξ is given by

$$\xi = \frac{1}{2} x(\alpha t)^{-1/2} \quad (8)$$

Immediately under the fixed load, the deflection and moment predicted by (6) and (7) are

$$u_0(0, t) = \frac{2}{3} \frac{K}{\sqrt{2\pi}} (\alpha t)^{3/2}, \quad (9)$$

and

$$m_0(0, t) = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} \quad (10)$$

It may also be determined that, for any fixed x, the long-time behavior implied by (6) and (7) is

$$u_0(x, t) = \frac{2}{3} \frac{K}{\sqrt{2\pi}} \left\{ (\alpha t)^{3/2} - \frac{7}{8} x^2 (\alpha t)^{1/2} + O(1) \right\}, \quad (11)$$

$$m_0(x, t) = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} + \frac{P}{2}|x| + o(1) \quad (12)$$

as $t \to \infty$.

[4] Under suitable assumptions concerning $X(t)$, it is possible to prove that (4) represents an *actual* solution of the initial value problem, and to justify the operations leading to (5), as well as similar ones occurring later.

[5] The results represented by (6) and (7) may be found in essence in the work of Jones [6], who has also considered the same problem on the basis of Timoshenko's more refined theory of beam motion [7].

3 Stochastic Problem

We now regard the position $X(t)$ of the load as a sample function of a stochastic process whose statistical properties are known, and we view (4) and (5) as sample functions of the deflection and bending moment processes produced by the randomly moving load. Our concern will be with the calculation of the mean values, or expectations, of $u(x, t)$ and $m(x, t)$, by way of (4) and (5), from the assumed statistical properties of $X(t)$.

Taking the expectation[6] of (4) yields

$$E[u(x, t)] = \frac{\alpha K}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_0^t d\tau \Phi_{X(\tau)}(\lambda) e^{-i\lambda x} \frac{\sin \lambda^2\alpha(t-\tau)}{\lambda^2} \quad (13)$$

where

$$\Phi_{X(t)}(\lambda) = E[\exp i\lambda X(t)] \quad (14)$$

is the characteristic function of the given process $X(t)$. Since any characteristic function $\Phi_{X(t)}(\lambda)$ satisfies the inequality $|\Phi_{X(t)}(\lambda)| \leq 1$ for all real λ, the λ-integral in (13) is absolutely convergent.

The expected value of the bending moment $m(x, t)$ is similarly determined from (5) as

$$E[m(x, t)] = -\frac{\alpha P}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_0^t d\tau \Phi_{X(\tau)}(\lambda) \times e^{-i\lambda x} \sin \lambda^2\alpha(t-\tau) \quad (15)$$

Equations (13) and (14) provide the expected deflection and expected moment, respectively, at a fixed station x. It is clear from these results that such expectations depend on a full knowledge of the distribution of $X(t)$ at *each* instant t, but they do not require for their determination any information concerning the way the values of $X(t)$ are correlated at *different* instants of time.

It is also of interest to examine the expected values of the deflection and moment under the load. For this purpose, we set $x = X(t)$ in (4), (5) prior to taking expectations, obtaining

$$E[u(X(t), t)] = \frac{\alpha K}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_0^t d\tau \Phi_{X(\tau), X(t)}(\lambda, -\lambda) \times \frac{\sin \lambda^2\alpha(t-\tau)}{\lambda^2} \quad (16)$$

$$E[m(X(t), t)] = \frac{-\alpha P}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_0^t d\tau \Phi_{X(\tau), X(t)}(\lambda, -\lambda) \times \sin \lambda^2\alpha(t-\tau), \quad (17)$$

where $\Phi_{X(\tau), X(t)}(\lambda, \mu)$ is the joint characteristic function of the process $X(\cdot)$ at the two instants t and τ. It is defined by

$$\Phi_{X(\tau), X(t)}(\lambda, \mu) = E\{\exp [i\lambda X(\tau) + i\mu X(t)]\} \quad (18)$$

for all real λ and μ.

The variance of $u(x, t)$ is given by

$$\operatorname{var} u(x, t) = E\{[u(x, t)]^2\} - \{E[u(x, t)]\}^2, \quad (19)$$

where the second term on the right would be obtained from (13), but the first would have to be calculated by squaring u in (4) and then taking the expectation. Thus

$$E\{[u(x, t)]^2\} = \left(\frac{\alpha K}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu \int_0^t d\tau \int_0^t d\rho \times e^{-i(\lambda+\mu)x} \Phi_{X(\tau), X(\rho)}(\lambda, \mu) \times \frac{\sin \lambda^2\alpha(t-\tau)}{\lambda^2} \cdot \frac{\sin \mu^2\alpha(t-\rho)}{\mu^2} \quad (20)$$

[6] For the definitions and an elementary discussion of the various probabilistic terms employed here, the reader is referred to the textbook [8] of Parzen.

Although it is possible in principle to obtain the variance from these formulas, the complexity of (20) seems to be prohibitive. Our efforts in what follows will be confined to a study of the first moments (13), (15), (16), and (17) for particular classes of processes X.

4 Stationary Processes $X(t)$

In this section, we assume that the random function $X(t)$ is *strictly stationary of first order;* see [8, p. 70]. We also assume that the mean value $E[X(t)]$ of the position of the moving load P is zero. These assumptions describe a situation in which P fluctuates about the origin, and the random mechanism responsible for this fluctuation does not change in time. The probability distribution of $X(t)$ itself is thus independent of t, as is its characteristic function. We indicate this by writing

$$\Phi_{X(t)}(\lambda) = \Phi_X(\lambda) \tag{21}$$

Under these conditions, the τ-integrations involved in (13) and (15) can be carried out explicitly, yielding

$$E[u(x, t)] = \frac{K}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\lambda) e^{-ix\lambda} \frac{1 - \cos \lambda^2 \alpha t}{\lambda^4} d\lambda, \tag{22}$$

$$E[m(x, t)] = -\frac{P}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\lambda) e^{-ix\lambda} \frac{1 - \cos \lambda^2 \alpha t}{\lambda^2} d\lambda \tag{23}$$

It is possible to obtain asymptotic information for large time from (22) and (23) without specializing the characteristic function $\Phi_X(\lambda)$. We illustrate by deriving the asymptotic expansion, as $t \to \infty$, of the expected bending moment (23). For this purpose, we shall assume that, for each t, the random variable $X(t)$ has a finite variance. It follows from this assumption that $\Phi_X(\lambda)$ is twice continuously differentiable[7] for all λ.

A simple transformation allows us to write (23) in the form

$$E[m(x, t)] = \int_0^{\infty} \psi(\lambda, x) \frac{1 - \cos \lambda^2 \alpha t}{\lambda^2} d\lambda, \tag{24}$$

where

$$\psi(\lambda, x) = -\frac{P}{2\pi} [\Phi_X(\lambda) e^{-ix\lambda} + \Phi_X(-\lambda) e^{ix\lambda}] \tag{25}$$

From (24), it follows that

$$E[m(x, t)] = \psi(0, x) \int_0^{\infty} \frac{1 - \cos \lambda^2 \alpha t}{\lambda^2} d\lambda + \int_0^{\infty} \frac{\psi(\lambda, x) - \psi(0, x)}{\lambda^2} d\lambda - \int_0^{\infty} \frac{\psi(\lambda, x) - \psi(0, x)}{\lambda^2} \cos \lambda^2 \alpha t d\lambda \tag{26}$$

Since the characteristic function $\Phi_X(\lambda)$ has the properties

$$\Phi_X(0) = 1, \qquad |\Phi_X(\lambda)| \leq 1, \tag{27}$$

we know from (25) that

$$\psi(0, x) = -\frac{P}{2\pi}, \qquad |\psi(\lambda, x)| \leq \frac{P}{\pi}, \qquad \text{all } \lambda, x \tag{28}$$

We also infer from (25) that

$$\frac{\partial \psi}{\partial \lambda}(0, x) = 0, \qquad \text{all } x, \tag{29}$$

so that, for fixed x,

$$\psi(\lambda, x) = \psi(0, x) + O(\lambda^2) \qquad \text{as } \lambda \to 0 \tag{30}$$

[7] See [9, p. 89].

We are, therefore, assured that the integrals in (26) are finite. The first of them, in fact, can be evaluated explicitly. In the third integral, the change of variables $\lambda = z^{1/2}$, followed by an application of the Riemann-Lebesgue lemma, shows that

$$\int_0^{\infty} \frac{\psi(\lambda, x) - \psi(\lambda, 0)}{\lambda^2} \cos \lambda^2 \alpha t d\lambda = o(1) \qquad \text{as } t \to \infty \tag{31}$$

for fixed x. Consequently, (26), (25), and (31) furnish

$$E[m(x, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} - \frac{P}{2\pi} \int_0^{\infty} \lambda^{-2} \times [\Phi_X(\lambda) e^{-i\lambda x} + \Phi_X(-\lambda) e^{i\lambda x} - 2] d\lambda + o(1) \qquad \text{as } t \to \infty, \quad x \text{ fixed.} \tag{32}$$

To put this in a more appealing form, we utilize the definition (14) of the characteristic function Φ as follows:

$$\Phi_X(\lambda) = E[e^{i\lambda X}] = \int_{-\infty}^{\infty} e^{i\lambda z} dF(z), \tag{33}$$

where F is the distribution function[8] associated with $X(t)$. After substitution of (33) into (32), an easily justified reversal of the order of integration leads to

$$E[m(x, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} + \frac{P}{2} \int_{-\infty}^{\infty} \times |x - z| dF(z) + o(1), \qquad t \to \infty. \tag{34}$$

The final form of the asymptotic expansion is obtained by noting that the integral in (34) is precisely the expected value of $|x - X(t)|$. Thus

$$E[m(x, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} + \frac{P}{2} \times E(|x - X(t)|) + o(1), \qquad t \to \infty, \quad x \text{ fixed.} \tag{35}$$

Equation (35) represents the generalization of the large-time formula (12) to the case of a load moving according to a stationary random process. Indeed, if $X(t) \equiv 0$, (35) reduces to (12).

The first term in the expansion (35) is precisely the large-time asymptotic value of the bending moment of the deterministic problem in which the force P is suddenly applied and maintained at $x = 0$. We conclude that the mean bending moment generated by a moving load whose position is a stationary random function of the time is, in *first* approximation, asymptotically equal for large time to the bending moment produced by the foregoing deterministic loading.[9]

The correction to the first approximation provided by the second term in (35) is independent of time because of the stationarity of $X(t)$. This correction is proportional to the expected distance from the station x under consideration to the randomly moving load, and its contribution to the total mean bending moment is of opposite sign to that of the first term. The magnitude of the second term on the right in (35) never exceeds $\frac{1}{2}|P|(x^2 + \sigma^2)^{1/2}$, where σ is the standard deviation of $X(t)$.

If $X(t)$ is a stationary *normal* process, computation of $E(|x - X(t)|)$ shows that (35) assumes the special form

$$E[m(x, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} + \frac{P}{2} \left\{ x \left(2 \int_{-\infty}^{x/\sigma} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz - 1 \right) + \frac{2\sigma}{\sqrt{2\pi}} e^{-x^2/2\sigma^2} \right\} + o(1), \qquad t \to \infty, \quad x \text{ fixed,} \tag{36}$$

where σ^2 is the variance of $X(t)$.

[8] F is independent of t, since $X(t)$ is stationary.

[9] A corresponding statement may be made for the mean deflection.

Another example of a stationary process is provided by the sine wave with random phase:

$$X(t) = a \sin(\omega t + \delta), \tag{37}$$

where a and ω are given constants, and δ is a random variable with the uniform distribution on $[0, 2\pi]$. When $E(|x - X(t)|)$ is evaluated in this case, (35) becomes

$$E[m(x(, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} + \begin{cases} \frac{P}{\pi}\left[(a^2 - x^2)^{1/2} + |x| \sin^{-1}\frac{|x|}{a}\right], & |x| \le a \\ \frac{P}{2}|x| & |x| > a \end{cases} + o(1) \text{ as } t \to \infty, \quad x \text{ fixed} \tag{38}$$

As a final illustration of the large-time formula (35), we consider a load-motion $X(t)$ which is a discrete random process. Let

$$X(t) = X(0)(-1)^{N(t)}, \tag{39}$$

where $N(t)$ is a Poisson process, and $X(0)$ takes the value a or $-a$ with equal probability; $X(0)$ and $N(t)$ are independent. This process is referred to as a "random telegraph signal" [8]. The load strikes the beam at either $x = a$ or $x = -a$, and then as time progresses, it jumps between these two positions according to the Poisson process $N(t)$. For this example,

$$E(|x - X(t)|) = \frac{1}{2}(|x - a| + |x + a|), \tag{40}$$

so that (35) becomes

$$E[m(x, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} + \frac{P}{4}(|x + a| + |x - a|) + o(1), \quad t \to \infty. \tag{41}$$

In each of the previous examples, the large-time behavior of $E[m(x, t)]$ differs from the deterministic result for small $|x|$ and becomes closer to (12) as $|x|$ increases.

5 Stationary Normal Processes $X(t)$

To examine the case of stationary processes $X(t)$ in more detail, we now assume that $X(t)$ is a stationary *normal* (or Gaussian) process with zero mean; see [8]. The characteristic function is then given by

$$\Phi_{X(t)}(\lambda) = \Phi_X(\lambda) = \exp(-{}^1/_2\sigma^2\lambda^2), \tag{42}$$

where σ^2 is the (constant) variance of $X(t)$. The joint characteristic function defined by (18) is of the special form

$$\Phi_{X(\tau),X(t)}(\lambda, \mu) = \exp\left\{-\frac{\sigma^2}{2}(\lambda^2 + \mu^2) + \lambda\mu R(|t - \tau|)\right\}, \tag{43}$$

where R is the associated covariance kernel:

$$R(|t - \tau|) = E[X(t), X(\tau)] \tag{44}$$

We note in passing that $R(0) = \sigma^2$, and $|R(t)| \le \sigma^2$.

When the characteristic function (42) is substituted into the general formulas (22) and (23), the resulting integrals can be put in various forms, but they are not expressible in closed form for general x. At $x = 0$ (the mean position of the moving load), however, they can be explicitly evaluated. Carrying out such an evaluation at $x = 0$ for (22) and (23) leads to

$$E[u(0, t)] = \frac{K\sigma^2}{8\sqrt{\pi}}\left\{t_0[(1 + t_0^2)^{1/2} - 1]^{1/2} - \frac{2}{3}[(1 + t_0^2)^{1/2} + 1]^{3/2} + \frac{4}{3}\sqrt{2}\right\}, \tag{45}$$

$$E[m(0, t)] = -\frac{P\sigma}{2\sqrt{\pi}}\{[(1 + t_0^2)^{1/2} + 1]^{1/2} - \sqrt{2}\}, \tag{46}$$

respectively, where we have introduced a dimensionless time

$$t_0 = \frac{2\alpha t}{\sigma^2} \tag{47}$$

The results represented by (45) and (46) provide the exact values of the mean deflection and mean bending moment at the *mean* position of the moving load. We may compute the mean deflection and mean bending moment at the *instantaneous* position of the moving load by specializing (16) and (17) to the present case with the aid of (43). For the bending moment, for example, (17) and (43) furnish

$$E[m(X(t), t)] = -\frac{\alpha P}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_0^t d\tau \times \exp\left[-\lambda^2\sigma^2\left(1 - \frac{R(|t - \tau|)}{\sigma^2}\right)\right] \sin\lambda^2\alpha(t - \tau)$$

After a reversal of the order of integration and an elementary change of variables, the resulting Fourier transform can be evaluated; see [10]. We find

$$E[m(X(t), t)] = -\frac{\alpha}{\sigma}\frac{P}{2\sqrt{2\pi}} \times \int_0^t \left[\frac{\{[1 - r(\tau)]^2 + \alpha^2\tau^2/\sigma^4\}^{1/2} - 1 + r(\tau)}{[1 - r(\tau)]^2 + \alpha^2\tau^2/\sigma^4}\right]^{1/2} d\tau, \tag{48}$$

where

$$r(\tau) = R(\tau)/\sigma^2, \quad \tau \ge 0 \tag{49}$$

The expected bending moment under the load thus depends on the covariance kernel of the process $X(t)$.

The expectations (45), (46), and (48) may be compared with the results given in (9) and (10) for the deterministic problem of the load P suddenly applied and maintained at $x = 0$. In the limit as $\sigma \to 0$, (45) reduces to (9), while (46) and (48) reduce to (10), as is to be expected. As $t \to \infty$, the expectations (45) and (46) are asymptotically equal to the corresponding results for the deterministic problem, according to the general discussion of the proceeding section. A direct analysis of (48) also shows that

$$E[m(X(t), t)] = -\frac{P}{\sqrt{2\pi}}(\alpha t)^{1/2} + O(1), \quad \text{as } t \to \infty \tag{50}$$

For small times, we find from (45) and (46) that

$$E[u(0, t)] = \frac{K}{8\sqrt{2\pi}}\frac{\alpha^4 t^4}{\sigma^6} + O(t^6), \quad t \to 0, \tag{51}$$

$$E[m(0, t)] = -\frac{P}{2\sqrt{2\pi}}\frac{\alpha^2 t^2}{\sigma^3} + O(t^4), \quad t \to 0, \tag{52}$$

in contrast to the deterministic t-dependence of (9) and (10). If the covariance kernel R is continuously differentiable, an analysis of the integral (48) shows that

$$E[m(X(t), t)] = -\frac{CP}{\sqrt{2\pi}}[(\alpha t)^{1/2} + O(t^{3/2})], \quad t \to 0, \tag{53}$$

where the constant C depends on the derivative $R'(0)$ of the covariance kernel according to the formula

$$C = \left\{ \frac{\dfrac{R'(0)}{\alpha} + \left[1 + \left(\dfrac{R'(0)}{\alpha} \right)^2 \right]^{1/2}}{1 + \left(\dfrac{R'(0)}{\alpha} \right)^2} \right\}^{1/2} \tag{54}$$

Except for the multiplicative constant C, the small-time behavior (53) is the same as that exhibited by the deterministic bending moment $m_0(0, t)$ in (10). Since $R'(0)$ is negative, C decreases monotonically from unity to zero as $|R'(0)/\alpha|$ increases from zero to infinity. The effect of randomness on the bending moment under the moving load is therefore to diminish its expected value, at least for small times, in comparison with the deterministic result (10).

Fig. 1 provides a comparison of the bending moment under the load in the deterministic problem of the suddenly applied fixed force, equation (10), with the expected bending moment at $x = 0$, equation (46), and at $x = X(t)$, equation (48), for the case of the randomly moving load. The scalings on the axes in Fig. 1 are chosen to capitalize on the fact that $-2\sqrt{\pi}\, m_0(0, t)/\sigma P$ and $-2\sqrt{\pi}\, E[m(0, t)]/\sigma P$ are functions of $2\alpha t/\sigma^2$ only, as (40) and (46) show. In the underlying calculations based on (48), an exponential covariance kernel

$$R(t) = \sigma^2 e^{-bt} \tag{55}$$

has been used.[10]

6 Case of a Wiener Process $X(t)$

An example in which the motion $X(t)$ is a nonstationary random process is obtained by considering the case in which the load moves as if it were a particle in one-dimensional Brownian motion. The simplest model of this situation is that in which $X(t)$ is a Wiener process [8]. The mean position of the moving load is $x = 0$, but the root-mean-square distance from $x = 0$ to the load increases like $\sqrt{t}$. The appropriate characteristic function is

$$\Phi_{X(t)}(\lambda) = \exp\left(-\frac{1}{2}\gamma^2\lambda^2 t \right), \tag{56}$$

corresponding to a normal process with mean zero and variance $\gamma^2 t$. The joint characteristic function required in the general formulas (16) and (17) for $E[u(X(t), t)]$, $E[m(X(t,) t)]$ is given by

$$\Phi_{X(t),X(\tau)}(\lambda, \mu) = \exp\left\{ -\frac{\gamma^2}{2} [\lambda^2 t + \mu^2\tau + 2\lambda\mu \min(t, \tau)] \right\} \tag{57}$$

From (15) and (42), we find that

$$E[m(x, t)] = -\frac{\alpha P}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{-i\lambda x} \int_0^t \times d\tau e^{-1/2\gamma^2\lambda^2\tau} \sin \lambda^2\alpha(t - \tau) \tag{58}$$

The τ-integration in (58) can be carried out immediately; after several subsequent transformations, the result can be put in the form

$$E[m(x, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} \left\{ \frac{1 + \beta^2}{1 + \beta^4} \cos \xi^2 - \frac{1 - \beta^2}{1 + \beta^4} \sin \xi^2 - |\xi| \int_{\xi^2}^{\infty} s^{-1/2} \left(\frac{1 + \beta^2}{1 + \beta^4} \sin s + \frac{1 - \beta^2}{1 + \beta^4} \cos s \right) ds - \frac{\sqrt{2}}{1 + \beta^4} |\xi| \int_{|\xi|/\beta}^{\infty} s^2 e^{-s^2} ds \right\}, \tag{59}$$

where

$$\beta = \gamma(2\alpha)^{-1/2} \tag{60}$$

[10] The author is indebted to J. C. McWilliams for the numerical computations leading to Fig. 1.

and, as in (8),

$$\xi = \frac{1}{2} x(\alpha t)^{-1/2} \tag{61}$$

The expected bending moment thus possesses the same similarity structure as does the actual bending moment in the deterministic problem described in Section 2.

Although the calculation is tedious, it is possible to obtain the expected deflection from (59) by using the relationship

$$u(x, t) = \int_x^{\infty} \left\{ \int_z^{\infty} u_{xx}(z', t) dz' \right\} dz$$

which leads to

$$E[u(x, t)] = \frac{1}{\rho\alpha^2} \int_x^{\infty} \left\{ \int_z^{\infty} E[m(z', t)] dz' \right\} dz \tag{62}$$

Equations (62) and (59) provide

$$\begin{aligned} E[u(x, t)] = \frac{2}{3} \frac{K}{\sqrt{2\pi}} \frac{(\alpha t)^{3/2}}{1 + \beta^4} \Big\{ & [(1 + \beta^2) + (1 - \beta^2)\xi^2] \sin \xi \\ & + [(1 - \beta^2) - (1 + \beta^2)\xi^2] \cos \xi^2 \\ & + \left[(1 + \beta^2)|\xi|^3 - \frac{3}{2}(1 - \beta^2)|\xi| \right] \int_{\xi^2}^{\infty} s^{-1/2} \sin s ds \\ & + \left[(1 - \beta^2)|\xi|^3 + \frac{3}{2}(1 + \beta^2)|\xi| \right] \int_{\xi^2}^{\infty} s^{-1/2} \cos s ds \\ & - \frac{3}{\sqrt{2}} \beta^3 e^{-\xi^2/\beta^2} - 3\sqrt{2} \left(\beta^2|\xi| + \frac{2}{3}|\xi|^3 \right) \int_{|\xi|/\beta}^{\infty} e^{-s^2} ds \\ & + 3\sqrt{2}\, \beta^2 \int_{|\xi|/\beta}^{\infty} \left(s + \frac{2}{3} s^3 \right) e^{-s^2} ds \Big\} \end{aligned} \tag{63}$$

When $\beta \to 0$, (59) and (63) reduce, respectively, to (7) and (6), as is to be expected.

At the mean position $x = 0$ of the moving load, (59) and (63), respectively, reduce to

$$E[m(0, t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} \left(\frac{1 - \sqrt{2}\,\beta + \beta^2}{1 + \beta^4} \right), \tag{64}$$

$$E[u(0, t)] = \frac{2}{3} \frac{K}{\sqrt{2\pi}} (\alpha t)^{3/2} \left(\frac{1 - \beta^2 + \sqrt{2}\,\beta^3}{1 + \beta^4} \right) \tag{65}$$

The functions of β appearing in (64) and (65) represent the modifications of the deterministic results (10) and (9), respectively, arising from the Brownian motion of the load. These functions decrease from unity to zero as β increases from zero to infinity. Thus the random motion of the load decreases the deflection and bending moment, at the mean position of the load, from their values in the deterministic case.

The expected values of the deflection and bending moment under the moving load can be obtained by using the joint characteristic function (57) in the general expressions (16) and (17). After evaluating the Fourier integrals involved, we find that for all $t \geq 0$

$$E[u(X(t), t)] = \frac{2}{3} \frac{K}{\sqrt{2\pi}} (\alpha t)^{3/2} [(1 + \beta^4)^{1/2} - \beta^2]^{1/2}, \tag{66}$$

while

$$E[m(X(t), t)] = -\frac{P}{\sqrt{2\pi}} (\alpha t)^{1/2} \left[\frac{(1 + \beta^2)^{1/2} - \beta^2}{1 + \beta^4} \right]^{1/2} \tag{67}$$

The remarks in the preceding paragraph concerning the effect of β on the values of $E[u]$ and $E[m]$ also apply to (66) and (67).

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