Dynamics of Skyrmion Collisions in 3+1 Dimensions

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We calculate classical skyrmion collisions in 3+1 dimensions. Numerical integration of Hamilton’s
equations for the chiral fields is based on a staggered leap-frog method. We study collisions of defensive
hedgehog solitons at various impact parameters for center-of-mass energies of 157, 432, and 885 MeV.
Internal excitations of the skyrmions and meson emission are observed. The time evolution of the pion
field and momentum and baryon densities is shown, as are deflection functions and inelasticities. Some
results for skyrmion-antiskyrmion annihilation are presented.

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Skyrm’s model, which is recognized to have theoretical
connections with QCD, provides a surprisingly suc-
cessful description of baryon properties and the nucleon-
nucleon interaction.\(^1\) However, most calculations in the
Skyrme model have been carried out for either static or
adiabatic configurations with a variational product An-
satz, which has been shown numerically\(^2\) to be unsatis-
factory.

The full dynamics of the model embodies a unified
description encompassing various nonlinear processes
such as meson production, baryon excitation, and baryon
pair production. Numerical evolution of the classical
field equations can be useful in building intuition for the
Skyrme model and can establish contacts with (and
perhaps verify) the conventional static potential ap-
proach that employs adiabaticity assumptions. Also,
since classical solutions are the essential foundation of
any semiclassical approach, they might ultimately allow
direct calculation of the S matrix, circumventing a po-
tential description entirely. In this Letter, we report the
first calculations of skyrmion collisions in 3+1 dimen-
sions. A similar calculation has already been carried out\(^3\) with axial symmetry, effectively in two spatial di-

The Lagrange density of the Skyrme model in Carte-
sian coordinates is

\[
\mathcal{L} = -\frac{1}{2} F_{\mu\nu}^2 (\partial_{\mu} \Phi_{\nu})^2 - \frac{1}{2} e^{-2(\partial_{\mu} \Phi_{\nu})} (\partial_{\mu} \Phi_{\nu})^2 \\
+ \frac{1}{2} e^{-2(\partial_{\mu} \Phi_{\nu})^2} (\partial_{\mu} \Phi_{\nu})^2,
\]

(1)

where the chiral fields \(\Phi_{\mu}\) are defined to form an SU(2)
matrix \(U = \Phi_{\mu} + i \tau_{\nu} \Phi_{\nu}\), with \(U = 1\) at infinity. The equa-
tions of motion can be written as

\[
\begin{align*}
\partial_{\mu} \Phi_{\nu} &= M^{(1)}_{\mu\nu} \Pi_{\nu}, \\
\partial_{\nu} \Pi_{\mu} &= \nabla_{\nu} (M^{(2)}_{\mu\nu} \Phi_{\nu}),
\end{align*}
\]

(2a)

(2b)

where \(\Pi_{\mu}\) are the canonical momenta \(\delta \mathcal{L}/\delta (\partial_{\mu} \Phi_{\nu})\). The sym-
metric 4×4 matrices \(M^{(1)}\) and \(M^{(2)}\) are given by

\[
M^{(1),(2)}_{\mu\nu} = \frac{1}{2} F_{\mu\nu}^2 \delta_{\mu\nu} + e^{-2(\partial_{\mu} \Phi_{\nu})} \delta_{\mu\nu} \\
- e^{-2(\partial_{\mu} \Phi_{\nu})} (\partial_{\mu} \Phi_{\nu}),
\]

(3)

where \(\nu = 1, 2, 3\) \((\nu = 0, 1, 2, 3)\) in Eq. (3) defines \(M^{(1)}\)
\((M^{(2)})\).

We discretize the field equations in space on a uniform
Cartesian lattice using lowest-order centered differen-
tes. The time integration is based on the staggered leap-frog
method, an explicit scheme in which the fields at a time
\(t + \Delta t\) are given in terms of the known field vari-
ables at the previous times \(t\) and \(t - \Delta t\). The time evolu-
tion proceeds through recurring updates of the field vari-
able across the spatial lattice. Since the normalization of
the chiral fields is not preserved by this process, we add a
term of the form \(\lambda \Phi_{\mu} \Phi_{\nu}\) to the Lagrange density.

The symmetries of the chiral fields in the cases we
consider allow us to restrict the computation to one qua-
drant of the total space; boundary conditions at the in-
terface introduce an image skyrmion that we need not
treat explicitly. Our spatial lattice for this quadrant is
\(41 \times 41 \times 21\); with our lattice spacing \(\Delta x = 0.084\) fm, this
corresponds to \(3.36 \times 3.36 \times 1.68\) fm\(^3\). Approximately
1500 time steps are needed to describe one complete
scattering event, with \(\Delta t\) ranging from 0.007 to 0.003
fm/c, depending on the initial velocity. A measure of the
numerical stability of the finite difference equations is
the Courant condition, \(\zeta \equiv \Delta t/\Delta x \ll 1\); we use values of \(\zeta\)
between 0.075 and 0.013. Typical runs of our vectorized
code on the National Magnetic Fusion Energy Computer
Center Cray-2 or on the San Diego Supercomputer
Center Cray XMP/48 take 30 central processing unit
minutes.

In this work we focus on the scattering of defensive
hedgehog skyrmions, although our code evolves an arbi-
trary field configuration. We construct our initial con-
dition by applying Lorentz boosts to well separated static
spherical solutions. The skyrmions translate uniform-
ly before collision and we have no need for an artifi-
cial viscosity term\(^3\) to stabilize the calculation. Total en-
ergy and baryon number are calculated by centered finite-
difference formulas to be within 4% of the expected
values, and each is conserved to within 2% during the
scattering processes.

Figure 1 shows the time evolution, integrated over the
coordinate perpendicular to the figure, of the baryon
density for two skyrmion-skyrmion (SS) collisions at impact parameter $b = 0.8$ fm. The center-of-mass velocities of the individual skyrmions in these cases are $v = 0.4c$ and $0.75c$ ($E_{c.m.} = 157$ and 885 MeV, respectively). The lower-energy SS collision demonstrates the importance of the three dimensions; the originally spherical solitons scatter in a direction roughly perpendicular to their initial motion. The outgoing skyrmions are distorted by the interaction, which results in a loss of kinetic energy of relative motion. In the higher-energy SS collision, the relativistic contraction of the skyrmions is clearly visible ($\gamma \approx 1.5$) and the inelasticity in the collision has increased (see below). Comparison of these two $b = 0.8$-fm events shows that the scattering angle $\theta$ decreases dramatically with increased bombarding energy, from $\theta \approx 90^\circ$ at $v = 0.4c$ to $\theta \approx 10^\circ$ at $v = 0.75c$; skyrmions become rather transparent at high energies.

Also shown in Fig. 1 is a skyrmion-antiskyrmion (SS) collision at $b = 1.2$ fm and $v = 0.6c$ ($E_{c.m.} = 432$ MeV). We construct the initial antiskyrmion by inverting the hedgehog’s pion field and then rotating it by 180° around the axis parallel to the line of motion. The potential energy of the SS system decreases and the kinetic energy increases in the course of this annihilation until the rapid variation of the chiral fields leads to a failure of the numerical integration. We plan to study this process further as a function of impact parameter and will attempt to extract the SS annihilation cross section.

Figure 2 is a more detailed presentation of a collision at $b = 0.4$ fm and $v = 0.4c$ ($E_{c.m.} = 157$ MeV). This is a nearly central collision, which is expected to probe the “hard core” of the SS interaction; the scattering angle is approximately $125^\circ$. Note that initially not all momentum density vectors of the moving skyrmion point in the direction of motion because of nonvanishing spatial components of the stress tensor for a skyrmion at rest. In the final stages of the event, pion waves are emitted from the skyrmions, as is indicated by the presence of a nonvanishing pion field and momentum density in regions of zero baryon density. A vortex structure can be seen in the momentum density in the third frame of the sequence, and a small rotation of the pion field in each final-state skyrmion is visible, indicating rotational motion as a result of the collision.

A succinct measure of the collision dynamics is the baryon separation coordinate,

$$R(t) = a \int d^3r \left| n \cdot \mathbf{r}_B(r, t) \right.$$  \hspace{1cm} (4)

Here $r$ is measured from the center of mass, $\rho_B$ is the baryon density, and $n$ is the direction in the scattering plane of the principal axis of the baryon inertia tensor.
FIG. 2. The time evolution of the baryon density, $B$, isovector pion field in the collision plane, $\pi$, and momentum density, $P$, for a $SS$ collision at $\nu = 0.4c$ and $b = 0.4$ fm. The density of points shown is one quarter of the density of our lattice. Each arrow represents the value of the pion field or direction of the momentum density at its tail.

FIG. 3. The baryon separation coordinate $R(t)$. We plot trajectories for $b = 0.4$, 0.8, and 1.2 fm at $\nu = 0.4c$ and $\nu = 0.75c$, and for $b = 0.0$ fm at $\nu = 0.4c$. Dots indicate time intervals of 0.70 fm/c. The path of the outgoing skyrmion for $b = 0$ has been displaced slightly. The radius of the circle is calculated by evaluation of Eq. (4) for a spherically symmetric skyrmion with baryon number 1. It shows the minimum value of $R$ expected if the skyrmions interpenetrated without distortion, while the ellipses show the effects of Lorentz contraction.

Having the smaller eigenvalue. In Fig. 3 we give the trajectories of $R(t)$ for various $SS$ collisions. The systematics of the trajectories are generally as expected, with the short-distance repulsion causing a bouncing at small impact parameters that evolves toward a gentle forward deflection in more peripheral collisions. From the behavior of $R(t)$ after the collision, we can extract the final velocity, $v_f$, and deflection angle, $\theta$, of each skyrmion. In Fig. 4, we summarize our results for the deflection function $\theta(b)$ and the inelasticity $v_f(b)/v$. As mentioned earlier, the energy dependence of $\theta(b)$ is pronounced. The inelasticity approaches unity at large impact parameters but is less than 1 at smaller impact parameters (except for the $b = 0$ collision, which appears to be elastic). It assumes its smallest values in collisions with intermediate impact parameters, where rotational modes can be excited in addition to vibrational motion.

In summary, we have presented the first numerical solutions of the (3+1)-dimensional classical field equations describing baryon-baryon collisions in the Skyrme model. We have made a preliminary investigation of the character of the collisions for $E_{\text{c.m.}} \lesssim 885$ MeV and have extracted the inelasticity and scattering angle at various
impact parameters. We have also demonstrated baryon-antibaryon annihilation at nonzero impact parameters. More refined and systematic studies along these lines offer a number of interesting possibilities, including a calculation of the differential $NN$ and $N\bar{N}$ elastic cross sections, together with the $N\bar{N}$ annihilation cross section, an analysis of the time dependence of the interbaryon separation to determine an effective potential, a quantitative comparison of the adiabatic and dynamic field configurations, and a calculation of cross sections for baryon resonance and meson production.

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