Some Operator and Trace Function Convexity Theorems

Eric A. Carlen\(^1\), Rupert L. Frank\(^2\) and Elliott H. Lieb\(^3\)

1. Department of Mathematics, Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway NJ 08854-8019
2. Department of Mathematics, Caltech, Pasadena, CA 91125
3. Departments of Mathematics and Physics, Jadwin Hall, Princeton University, Washington Road, Princeton, NJ 08544

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Abstract

We consider trace functions \((A, B) \mapsto \text{Tr}[(A^{q/2}B^pA^{q/2})^s]\) where \(A\) and \(B\) are positive \(n \times n\) matrices and ask when these functions are convex or concave. We also consider operator convexity/concavity of \(A^{q/2}B^pA^{q/2}\) and convexity/concavity of the closely related trace functional \(\text{Tr}[A^{q/2}B^pA^{q/2}C^r]\). The concavity questions are completely resolved, thereby settling cases left open by Hiai; the convexity questions are settled in many cases. As a consequence, the Audenaert–Datta Rényi entropy conjectures are proved for some cases.

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1 Introduction

Let \(\mathcal{P}_n\) denote the set of \(n \times n\) positive definite matrices. For \(p, q, s \in \mathbb{R}\), define

\[
\Phi_{p,q,s}(A, B) = \text{Tr}[(A^{q/2}B^pA^{q/2})^s] .
\] (1.1)

We are mainly interested in the convexity or concavity of the map \((A, B) \mapsto \Phi_{p,q,s}(A, B)\), but we are also interested in the operator convexity/concavity of \(A^{q/2}B^pA^{q/2}\). When any of \(p, q\) or \(s\) is zero, the question of convexity is trivial, and we exclude these cases.

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Given any $n \times n$ matrix $K$, and with $p$, $q$, $s$ as above, define

$$\Psi_{K,p,q,s}(A, B) = \text{Tr}\left[(A^{q/2}K^*B^pKA^{q/2})^s\right], \quad (1.2)$$

and note that

$$\Phi_{p,q,s}(A, B) = \Psi_{1,p,q,s}(A, B). \quad (1.3)$$

The main question to be addressed here is this: For which non-zero values of $p$, $q$ and $s$ is $\Psi_{K,p,q,s}(A, B)$ jointly convex or jointly concave on $\mathcal{P}_n \times \mathcal{P}_n$ for all $n$ and all $K$?

We begin with several simple reductions. Since invertible $K$ are dense, it suffices to consider all invertible operators $K$. Then, for $K$ invertible,

$$\Psi_{K,p,q,s}(A, B) = \Psi_{K^{-1},-p,-q,-s}(A, B),$$

and therefore it is no loss of generality to assume that $s > 0$. We always make this assumption in what follows.

Next, the convexity/concavity properties of $\Psi_{K,p,q,s}(A, B)$ are a consequence of those of $\Phi_{p,q,s}(A, B)$, and hence it suffices to study the special case $K = 1$. In fact, more is true as stated in the following Lemma 1.1.

These equivalences may be useful in other contexts. (For $s = 1$ the equivalence of (1) and (4) is in [11] and the equivalence of (1) and (3) is in [4]; the arguments in those papers extend to all $s$, but we repeat them here for completeness.)

1.1 LEMMA (Equivalent formulations). The following statements are equivalent for fixed $p$, $q$, $s$.

1. The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all $K$ and all $n$.
2. The map $(A, B) \mapsto \Psi_{K,p,q,s}(A, B)$ is convex for all unitary $K$ and all $n$.
3. The map $(A, B) \mapsto \Psi_{1,p,q,s}(A, B) = \Phi_{p,q,s}(A, B)$ is convex for all $n$.
4. The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all $K$ and all $n$.
5. The map $A \mapsto \Psi_{K,p,q,s}(A, A)$ is convex for all unitary $K$ and all $n$.

The same is true if convex is replaced by concave in all statements.

Proof. Trivially, (1) implies the other four items.

When $K$ is unitary, $K^*A^qK = (K^*AK)^q$, and hence (3) implies (2) (even for each fixed $n$). By taking $K = 1$, (2) implies (3) (again for each fixed $n$).

Next we show that (2) implies (1), whence (1), (2) and (3) are equivalent. We may suppose, without loss of generality that $K$ is a contraction. Let $K = W|K|$ be its polar decomposition. Then

$$U = \begin{bmatrix} K & W\sqrt{1-|K|^2} \\ -W\sqrt{1-|K|^2} & K \end{bmatrix}$$

is unitary. We consider the case $q < 0$ first. For arbitrary $t > 0$, let

$$A_t = \begin{bmatrix} A & 0 \\ 0 & t1 \end{bmatrix}, \quad B = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}.$$
Then 
\[
\begin{bmatrix}
A^{q/2}K^*B^pKA^{q/2} & 0 \\
0 & 0
\end{bmatrix}
= \lim_{t \to \infty} A_t^{q/2}U^*B^pUA_t^{q/2}.
\]

Thus, recalling that we always assume \( s > 0 \),
\[
\text{Tr}[ (A^{q/2}K^*B^pKA^{q/2})^s ] = \lim_{t \to \infty} \Psi_{U,p,q,s}(A_t,B).
\]

Thus, (2) with \( 2n \) implies (1) with \( n \). The case \( q > 0 \) is treated analogously, letting \( t \to 0 \).

Trivially, (4) implies (5). To show that (5) (with \( 2n \)) implies (3) (with \( n \)), thereby completing the loop, replace \( A \) in (5) by \( \begin{bmatrix} A & 0 \\
0 & B \end{bmatrix} \), and replace \( K \) by the unitary \( \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix} \).

\[\Box\]

2 Known results and our extension of them

Hiai has proved in [8] that if \( p, q \) are both non-zero, and \( s > 0 \), and \( \Phi_{p,q,s} \) is jointly convex in \( A \) and \( B \), then, necessarily, one of the following conditions holds:

(1.) \( 1 \leq p \leq 2 \) and \( -1 \leq q < 0 \) and \( s \geq 1/(p + q) \), or the same with \( p \) and \( q \) interchanged.

(2.) \( -1 \leq p, q < 0 \) and \( s > 0 \).

In the special case \( s = 1 \), condition (1.) was proved to be sufficient in [1] Corollary 6.3, and condition (2.) was proved to be sufficient in [11] Theorem 8; see also [3] for \( s = 1 \) and one of \( p, q \) negative. Hiai [8] has also proved that \( \Phi_{p,q,s} \) is jointly convex in case \( -1 \leq p, q < 0 \) and \( 1/2 \leq s \leq -1/(p + q) \).

Our main focus is on (1.). The joint convexity in this case is known [7] when \( s = 1/(p + q) \), \( p = 1 \) and \( -1 \leq q < 0 \), and of course, with \( p \) and \( q \) interchanged.

Concerning concavity, Hiai has shown [8] that if \( p, q \) are both non-zero, and \( s > 0 \), and \( \Phi_{p,q,s} \) is jointly concave in \( A \) and \( B \), then, necessarily, the following condition holds:

(3.) \( 0 < p, q \leq 1 \) and \( 0 < s \leq 1/(p + q) \).

In the special case \( s = 1 \), this condition was proved to be sufficient in [11] Theorem 1; Hiai [8] showed sufficiency for \( 1/2 \leq s \leq 1/(p + q) \).

Our contribution to the subject is to fill in parts of the table of sufficient/necessary conditions in the following manner. We were motivated in this endeavor by a recent paper of Audenaert and Datta [2], (and Datta’s Warwick lecture on it) and we prove some of their conjectures.

All the results mentioned above refer to trace inequalities. There are some operator convexity/concavity inequalities to be considered as well, and we will present some in the following.

\[1\] After this work was submitted, Hiai posted the preprint arXiv:1507.00853 in which he extended our method to prove joint convexity under condition (2.).
As far as convexity of $\Phi_{p,q,s}$ is concerned we can summarize our results as follows. We are concerned with the region $p \in [1, 2]$, $q \in [-1, 0)$ and $s \geq 1/(p+q)$. (Clearly, $s$ cannot be smaller than $1/(p+q)$ by homogeneity.) We prove joint convexity for $s \geq \min \left\{ \frac{1}{p-1}, \frac{1}{1+q} \right\}$ (Thm. 4.1). Moreover, we prove joint convexity for $p = 1$ and $p = 2$ in the optimal range $s \geq 1/(p+q)$ (Thm. 4.2).

For $p \in (1, 2)$, $q \in [-1, 0)$, the missing regions, where we believe joint convexity also holds, is $1/(p+q) \leq s < 1$ and $1 < s \leq \min \left\{ \frac{1}{p-1}, \frac{1}{1+q} \right\}$. (Ando’s theorem \cite{1} covers the cases $1/(p+q) \leq s = 1$.)

On the other hand, our results completely close the gap between necessary and sufficient conditions for concavity to hold. The trace function $\Phi_{p,q,s}$ is jointly concave if and only if $0 < p, q \leq 1$ and $0 \leq s \leq 1/(p+q)$ (Thm. 4.4). This completes Hiai’s results discussed above.

As for joint operator convexity, we prove it for $(A, B) \mapsto BA^qB$ if $-1 \leq q < 0$, and show that it does not hold for $(A, B) \mapsto B^{p/2}A^qB^{p/2}$ for any $p < 2$ (Thm. 3.2). (Note that it cannot hold for $p > 2$ since $B \mapsto B^p$ is not operator convex when $p > 2$.)

# 3 Joint operator convexity

We investigate operator convexity and concavity of certain functions on $\mathcal{P}_n \times \mathcal{P}_n$. It is well known \cite{10, 12} that

$$(A, B) \mapsto AB^{-1}A \quad (3.1)$$

is jointly convex. In the scalar case ($n = 1$), $f(a, b) = a^qb^p$ is jointly convex on $(0, \infty) \times (0, \infty)$ if and only if $p \geq 1$, $q \leq 0$ and $p+q \geq 1$, or $q \geq 1$, $p \leq 0$ and $p+q \geq 1$, or $p, q \leq 0$. It is jointly concave if and only if $0 \leq p, q \leq 1$ and $p+q \leq 1$. It is natural to ask for which powers $p$ and $q$

$$(A, B) \mapsto A^{q/2}B^pA^{q/2} \quad (3.2)$$

is jointly operator convex or concave.

This question is closely related to the question: For which values of $p, q, r$ is

$$(A, B, C) \mapsto \text{Tr} A^{q/2}B^pA^{q/2}C^r \quad (3.3)$$

jointly convex or concave in the positive operators $A, B, C$?

**3.1 Lemma.** When the function in (3.3) is convex (or concave) for some choice of $p, q$ and $r$ all non-zero, then the function in (3.2) is operator convex (or concave) for the same $p$ and $q$.

**Proof.** When $r$ is positive, simply take $C$ to be any rank-one projection. When $r$ is negative, let $P$ be any rank-one projection, $t > 0$. Take $C$ to be $P + tP^\perp$, so that $C^r = P + t^rP^\perp$ and let $t$ tend to $\infty$. 

Thus, the operator convexity/concavity of the operator-valued function in (3.2) is a consequence of the seemingly weaker tracial convexity/concavity
of (3.3). In short, (3.3) is stronger than (3.2) for the same values of \( p, q \).
The value of \( r \) is irrelevant as long as it is not zero, and the implication does
not even require convexity/concavity in \( C \), only joint convexity/concavity
in \( A \) and \( B \).

When \( p, r < 0 \), and \(-1 \leq p + r < 0 \), then the map \((A, B, C) \mapsto \text{Tr}AB^pA^rC^q\) is jointly convex for \( B, C \) positive and \( A \) arbitrary. This was
proved in [11, Corollary 2.1]. (This triple convexity theorem is deeper than
the double convexity theorem [11, Theorem 8] referred to in the previous
section because it uses [11, Theorem 2] in an essential way.) By restricting
ourselves to \( A \) positive and taking \( q = 2 \) this function of \( A, B, C \)
reduces to (3.3).

By Lemma 3.1, the function (3.2) is jointly convex when \( q = 2 \) and
\(-1 \leq p < 0 \). Our main result in this section is that there are no other cases
in which this operator-valued function is either convex or concave!

3.2 THEOREM. Let \( p, q \in \mathbb{R} \setminus \{0\} \) and consider the map

\[(A, B) \mapsto A^{q/2}B^pA^{q/2} \quad (3.4)\]

from \( \mathcal{P}_n \times \mathcal{P}_n \) to \( \mathcal{P}_n \) for some fixed \( n \geq 2 \).

(1.) The map (3.4) is jointly operator convex if and only if \( q = 2 \) and
\(-1 \leq p < 0 \).

(2.) The map (3.4) is not jointly operator concave.

3.3 COROLLARY. Let \( p, q \in \mathbb{R} \setminus \{0\} \). The function \((A, B, C) \mapsto \text{Tr}A^{q/2}B^pA^{q/2}C^r\) is never concave, and it is convex if and only if \( q = 2 \),
\( p, r < 0 \) and \(-1 \leq p + r < 0 \).

Proof. By Lemma 3.1 any triple convexity/concavity would imply the corre-
sponding operator convexity/concavity, which is ruled out by the previous
Theorem 3.2 except when \( q = 2 \), \( p, r < 0 \) and \(-1 \leq p + r < 0 \). In this case
convexity is provided by [11, Corollary 2.1].

Our counterexamples to operator convexity and concavity given in The-
orem 3.2 will be based on the following lemma.

3.4 LEMMA. Let \( r \in (-\infty, 0) \cup (0, 1) \), let \( Y \geq 0 \) be rank one and \( n \geq 2 \).
Then the map \( X \mapsto X^rYX^r \) from \( \mathcal{P}_n \) to \( \mathcal{P}_n \) is not operator convex.

Proof of Lemma 3.4. First assume that \( r \in (0, 1/2) \). Then for any non-
trivial \( Y \geq 0 \) (not necessarily rank one) the map \( X \mapsto X^rYX^r \) from \( \mathcal{P}_n \) to
\( \mathcal{P}_n \) is not operator convex. This follows simply from the fact that the map
\( x \mapsto x^rY \) from \((0, \infty) \) to \( \mathcal{P}_n \) is not operator convex for \( 0 < r < 1/2 \).
It is, in fact, strictly concave in this region.

Now let \( r \in (-\infty, 0) \). (The proof actually also works for \( r \in (0, 1/2) \),
which is hardly surprising in light of the concavity mentioned above.)
Clearly, we may assume \( n = 2 \). Let \( Y = \langle v|v \rangle \). If the convexity were
true, then for all \( X_1, X_2 \in \mathcal{P}_2 \), with \( X = (X_1 + X_2)/2 \), we would have

\[X^r|v\rangle\langle v|X^r \leq \frac{1}{2}X_1^r|v\rangle\langle v|X_1^r + \frac{1}{2}X_2^r|v\rangle\langle v|X_2^r. \quad (3.5)\]
Without loss of generality, let $|v\rangle = (1, 1)$. If we take $X_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $X_2 = t \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, with $t > 0$, then (3.5) becomes

$$
\begin{bmatrix}
(1 + t)^{2r} & (1 + t)^r (1 + 2t)^r \\
(1 + t)^r (1 + 2t)^r & (1 + 2t)^{2r}
\end{bmatrix}
\leq 2^{2r-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + t^{2r} 2^{2r-1} \begin{bmatrix} 1 & 2r \\ 2r & 2^{2r} \end{bmatrix}.
$$

(3.6)

The vector $|w\rangle = (2r, -1)$ is in the null space of the second matrix on the right in (3.6), and taking the trace of both sides against $|w\rangle\langle w|$ yields

$$
\langle w, \begin{bmatrix}
(1 + t)^{2r} & (1 + t)^r (1 + 2t)^r \\
(1 + t)^r (1 + 2t)^r & (1 + 2t)^{2r}
\end{bmatrix} w\rangle \leq 2^{2r-1} (2^r - 1)^2,
$$

which, in the limit $t \to 0$, becomes $(2^r - 1)^2 \leq 2^{2r-1} (2^r - 1)^2$, so that for $r \neq 0$, we would have $1 \leq 2^{2r-1}$. This is false for all $r < 1/2$, which shows that (3.5) leads to a contradiction for nonzero $r \in (-\infty, 0) \cup (0, 1/2)$.

Our proof for $1/2 \leq r < 1$ is different; this proof actually works in the range $0 < r < 1$. Let $|v\rangle$ be a unit vector in $\mathbb{C}^n$. Then we will show that there is another vector $|w\rangle$ in $\mathbb{C}^n$ such that

$$
X \mapsto |\langle w|X^r|v\rangle|^2
$$

is not convex. Again, we may assume that $n = 2$ and that $|v\rangle = (0, 1)$. Take

$$
X_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.
$$

Let $|w\rangle = (1, -1)$, so that $X_1^r|w\rangle = 0$ and $X_2^r|v\rangle = 0$. Evidently,

$$
\frac{1}{2} |\langle w|X_1^r|v\rangle|^2 + \frac{1}{2} |\langle w|X_2^r|v\rangle|^2 = 0.
$$

However, the eigenvalues of $X = \frac{1}{2} (X_1 + X_2)$ are easily computed to be $\lambda_{\pm} = (3 \pm \sqrt{5})/2$, and then a further simple computation yields

$$
\langle w|X^r|v\rangle = \frac{1}{\sqrt{3}} (\lambda_+^{r-1} - \lambda_-^{r-1}),
$$

and this is strictly negative for all $0 < r < 1$.

\begin{proof}[Proof of Theorem 3.2] As explained above, the convexity assertion in (1.) is a consequence of [11, Corollary 2.1]. Our goal now is to prove that there are no other cases of convexity or concavity.

A number of exponents can be excluded by considering the scalar case. Moreover, since $X \mapsto X^r$ is operator convex on $\mathcal{P}_n$ if and only if $r \in [-1, 0] \cup [1, 2]$, and is operator concave on $\mathcal{P}_n$ if and only if $r \in [0, 1]$, the only cases in which convexity cannot be immediately ruled out are $p \in [1, 2]$, $q \in [-1, 0]$ and $p + q \geq 1$ (or the same with $p$ and $q$ interchanged). Likewise, the only

\end{proof}
cases of in which concavity cannot be immediately ruled out are \( p, q \in [0, 1] \), \( p + q \leq 1 \).

For part (1.), it remains for us to show that (3.4) is not jointly operator convex in the following three cases,

(a) \( p \in [-1, 0) \), \( q \in [1, 2) \) and \( p + q \geq 1 \).

(b) \( p \in [1, 2) \), \( q \in [-1, 0) \) and \( p + q \geq 1 \).

(c) \( p \in (-1, 0) \) and \( p + q \geq -1 \).

Let us prove failure of convexity in case (a). Let \(|v|\) be any unit vector in \( \mathbb{C}^n \). Let \( P \) be the orthogonal projection onto the span of \( v \), and let \( P^\perp \) denote the complementary projection. Fix \( t > 0 \), and define \( B_t = P + t P^\perp \).

Then \( B_t^p = P + t^p P^\perp \). If convexity would hold, then for any \(|w|\) the map \( A \mapsto \langle w | A^{q/2} B_t^p A^{q/2} | w \rangle \) would be convex. Since \( \lim_{t \to \infty} B_t^p = \frac{|v| \langle v, |v| \rangle}{|v|^2} \), and since limits of convex functions are convex, it would follow that \( A \mapsto \langle w | A^{q/2} | w \rangle \) would be convex on \( \mathcal{P}_n \) for any \(|w|\). This contradicts Lemma 3.4 with \( r = q/2 \in [1/2, 1) \). The proof for (c) is almost exactly the same, except one uses Lemma 3.5 with \( r = q/2 < 0 \).

The proof in case (b) is similar. Again, we let \(|v|\) be a unit vector in \( \mathbb{C}^n \) and set \( B = |v\rangle \langle v| \). Then \( B^p = |v\rangle \langle v| \) and, if convexity would hold, then for any \(|w|\) the map \( A \mapsto \langle v | A^{q/2} | w \rangle \) would be convex on \( \mathcal{P}_n \). This contradicts Lemma 3.4 with \( r = q/2 \in [-1/2, 0) \).

Finally, we prove (2.), the failure of concavity. According to the discussion above, it remains for us to show that (3.4) is not jointly operator concave for \( p, q \in (0, 1] \) and \( p + q \leq 1 \). Suppose \((A, B) \mapsto A^{q/2} B^p A^{q/2}\) were concave for some \( p, q \) in this range. Then for all non-negative \( A \) and \( B \) we would have

\[
\frac{1}{2} A^{q/2} B^p A^{q/2} + \frac{1}{2} B^{q/2} A^p B^{q/2} \leq \left( \frac{A + B}{2} \right)^{q/2} \left( \frac{B + A}{2} \right)^p \left( \frac{A + B}{2} \right)^{q/2} = 2^{-p-q}(A + B)^{p+q}.
\]

Suppose that \( A \) has a non-trivial null space (here we use the assumption \( n \geq 2 \)), and \(|v|\) is a unit vector with \( A|v\rangle = 0 \). By Jensen’s inequality, since \( p + q \leq 1 \),

\[
\langle v | (A + B)^{p+q} | v \rangle \leq \langle v | (A + B) | v \rangle^{p+q} = \langle v | B | v \rangle^{p+q}.
\]

Thus we would have

\[
\langle v | B^{q/2} A^p B^{q/2} | v \rangle \leq 2^{1-p-q} \langle v | B | v \rangle^{p+q}.
\]

The left side is homogeneous of degree \( q \) in \( B \), while the right side is homogeneous of degree \( p + q \), and hence the inequality cannot be generally valid. (The positivity of the powers is essential here; the argument of course cannot be adapted to yield a counterexample to the convexity proved in the first part of the theorem.)

\( \square \)

3.5 Remark. There is another way to prove the convexity in (3.4) for \( q = 2 \) and \(-1 \leq p < 0 \). For \( p = -1 \) one can use the Schwarz type inequality in 12.
In this section we prove, among other things, two cases of a conjecture of Audenaert and Datta [2]. Much of our analysis is based on the formulas with $C_{\Phi}^{4}$.

Convexity of $p,q,s$\[B\]

We have by (4.1),

Now define $D$.

Proof. First, we prove convexity if $-1 < p < 0$ one can use the integral representation $B^p = C_p \int_0^\infty (B+t)^{-1}t^p dt$ with $C_p > 0$ to reduce matters to the case $p = -1$. Indeed, one can replace $B^p$ by any Herglotz function $\int_{t \geq 0} (B+t)^{-1}d\mu(t)$ with $\mu > 0$.

4 Convexity of $\Phi_{p,q,s}(A, B)$

In this section we prove, among other things, two cases of a conjecture of Audenaert and Datta [2]. Much of our analysis is based on the formulas

\[\operatorname{Tr}[X^s] = s \sup_{Z \geq 0} \left\{ \operatorname{Tr}[XZ^{1-1/s}] + \left( \frac{1}{s} - 1 \right) \operatorname{Tr}[Z] \right\} \quad \text{if } s > 1 \quad (4.1)\]

and

\[\operatorname{Tr}[X^s] = s \inf_{Z \geq 0} \left\{ \operatorname{Tr}[XZ^{1-1/s}] + \left( \frac{1}{s} - 1 \right) \operatorname{Tr}[Z] \right\} \quad \text{if } 0 < s < 1; \quad (4.2)\]

see [4], Lemma 2.2]. These formulas have already played an important role in our previous works [4] and [7].

4.1 THEOREM. When $p \in [1, 2]$, $q \in [-1, 0)$, $\Phi_{p,q,s}(A, B)$ is jointly convex for all

\[s \geq \min \left\{ \frac{1}{p-1}, \frac{1}{1+q} \right\}.\]

Here we set $\frac{1}{p-1} = +\infty$ for $p = 1$ and $\frac{1}{1+q} = +\infty$ for $q = -1$. Thus, the theorem implies that, in particular, for $p = 1$, $\Phi_{1,q,s}(A, B)$ is jointly convex in the optimal range $q \in [-1, 0)$ and $s \geq \frac{1}{1+q}$. An optimal result for $p = 2$ will be proved in Theorem 4.2 As discussed in Section 2, for $p \in (1, 2)$, $q \in [-1, 0)$, the region where convexity is not settled is $1/(p+q) \leq s < 1$ and $1 < s < \min \left\{ \frac{1}{p-1}, \frac{1}{1+q} \right\}$.

Proof. First, we prove convexity if $s \geq 1/(1+q)$. Since this implies $s > 1$, we have by (4.1),

\[\Phi_{p,q,s}(A, B) = s \sup_{Z \geq 0} \left\{ \operatorname{Tr}[A^{q/2}B^pA^{q/2}Z^{1-1/s}] + \left( \frac{1}{s} - 1 \right) \operatorname{Tr}[Z] \right\}.
\]

Now define $D^2 = A^{q/2}Z^{(s-1)/s}A^{q/2}$ and note that $Z = (A^{-q/2}D^2A^{-q/2})^{s/(s-1)}$ to write

\[\Phi_{p,q,s}(A, B) = s \sup_{D \geq 0} \left\{ \operatorname{Tr}[DB^pD] + \left( \frac{1}{s} - 1 \right) \operatorname{Tr}[(DA^{-q}D)^{s/(s-1)}] \right\}. \quad (4.3)\]

For $1 \leq p \leq 2$, the map $B \mapsto B^p$ is operator convex and therefore $B \mapsto \operatorname{Tr}[DB^pD]$ is convex. Moreover, by Hiai’s extension of Epstein’s Theorem [8], Thm. 4.1] the map $A \mapsto \operatorname{Tr}[(DA^{-q}D)^{s/(s-1)}]$ is concave as long as $s/(s-1) \leq -1/q$, which is the same as $s \geq 1/(1+q)$. Thus, (4.3) represents $\Phi_{p,q,s}(A, B)$ as a supremum of jointly convex functions and so $\Phi_{p,q,s}(A, B)$ is jointly convex for $s \geq 1/(1+q)$. This proves the first part of the theorem.
We now prove convexity if \( s \geq 1/(p-1) \). Let us first consider the case \( p = 2 \) and \( s = 1 \), where \( \Phi_{2,q,1}(A, B) = \text{Tr}[A^{q/2}B^2A^{q/2}] = \text{Tr}[BA^qB] \). For \(-1 \leq q < 0\), the map \((A, B) \mapsto BA^qB\) is operator convex by Theorem 3.2 and therefore \((A, B) \mapsto \text{Tr}[BA^qB]\) is convex, as claimed. We now assume that \( s > 1 \) (and still \( s \geq 1/(p-1) \)). Then by (4.1), making use of \( \text{Tr}[(A^{q/2}B^pA^{q/2})^s] = \text{Tr}[(B^{1/p}A^qB^{1/p})^s] \),

\[
\Phi_{p,q,s}(A, B) = s \sup_{Z \geq 0} \left\{ \text{Tr}[B^{p/2}A^qB^{p/2}Z^{1-1/s}] + \left( \frac{1}{s} - 1 \right) \text{Tr}[Z] \right\} .
\]

Note that

\[
\text{Tr}[B^{p/2}A^qB^{p/2}Z^{1-1/s}] = \text{Tr}[BA^qB(B^{p/2-1}Z^{1-1/s}B^{p/2-1})].
\]

Define \( D^2 = B^{p/2-1}Z^{(s-1)/s}B^{p/2-1} \), so that \( Z = (B^{1-p/2}D^2B^{1-p/2})^s/(s-1) \).

Then

\[
\Phi_{p,q,s}(A, B) = s \sup_{D \geq 0} \left\{ \text{Tr}[DBA^qBD] + \left( \frac{1}{s} - 1 \right) \text{Tr}[(B^{1-p/2}D^2B^{1-p/2})^s/(s-1)] \right\} 
= s \sup_{D \geq 0} \left\{ \text{Tr}[DBA^qBD] + \left( \frac{1}{s} - 1 \right) \text{Tr}[(DB^{2-p}D)^s/(s-1)] \right\} .
\]

(4.4)

Since \(-1 \leq q < 0\), \((A, B) \mapsto BA^qB\) is operator convex by Theorem 3.2 so \((A, B) \mapsto \text{Tr}[DBA^qBD]\) is convex. By Hiai’s extension of Epstein’s Theorem (1c) Thm. 4.1], \( B \mapsto \text{Tr}[(DB^{2-p}D)^s/(s-1)]\) is concave as long as \( s/(s-1) \leq 1/(2-p) \), which is the same as \( s \geq 1/(p-1) \). Thus, (4.4) represents \( \Phi_{p,q,s}(A, B) \) as a supremum of jointly convex functions and so \( \Phi_{p,q,s}(A, B) \) is jointly convex for \( s \geq 1/(p-1) \). This completes the proof. 

4.2 THEOREM. When \( p = 2 \), \( \Phi_{p,q,s}(A, B) \) is jointly convex for all \(-1 \leq q < 0 \) and \( s \geq 1/(2 + q) \).

This result yields the optimal range of convexity for \( p = 2 \). It had been conjectured in [2] for \( s = 1/(2 + q) \).

Proof. The convexity for \( s \geq 1 \) follows from Theorem 4.1 and therefore we may assume that \( 1/(p+q) \leq s < 1 \). Then, making use of \( \text{Tr}[(A^{q/2}B^2A^{q/2})^s] = \text{Tr}[(BA^qB)^s] \),

\[
\Phi_{2,q,s}(A, B) = s \inf_{Z \geq 0} \left\{ \text{Tr}[BA^qBZ^{1-1/s}] + \left( \frac{1}{s} - 1 \right) \text{Tr}[Z] \right\} .
\]

(4.5)

The important distinction between this formula and formulas (4.3) and (4.4) is the infimum in place of the supremum. Joint convexity in \( A, B \) no longer suffices. Instead we need joint convexity in \( A, B, Z \), with which we can apply [4] Lemma 2.3.

Note that \( 1 - 1/s \leq 0 \). By [11] Corollary 2.1], \((A, B, Z) \mapsto \text{Tr}[BA^qBZ^{1-1/s}]\) is jointly convex as long as \( q + 1 - 1/s \geq -1 \), which means \( s \geq 1/(2 + q) \). For such \( s \), the argument of the infimum in (4.5) is jointly convex in \( A, B \) and \( Z \). By [4] Lemma 2.3], the infimum itself is jointly convex in \( A \) and \( B \). This proves the assertion for \( 1/(2 + q) \leq s \leq 1 \).
4.3 Remark. In the previous proof for the range \( s \geq 1 \) we referred to Theorem 4.1 which, in turn, was based on Hiai’s extension of Epstein’s theorem. For the case relevant for Theorem 4.2 however, there is a more direct proof. Indeed, let \( A_j, B_j \in \mathcal{P}_n \), \( j = 1, 2 \), and \( \lambda \in (0, 1) \) and set \( A = \lambda A_1 + (1 - \lambda)A_2 \) and \( B = \lambda B_1 + (1 - \lambda)B_2 \). Then by Theorem 4.2 for \(-1 \leq q < 0\),
\[
BA^q B \leq \lambda B_1 A^q_1 B_1 + (1 - \lambda)B_2 A^q_2 B_2 .
\]
For all \( s \geq 0 \), \( X \mapsto \text{Tr}[X^s] \) is monotone on \( \mathcal{P}_n \). Hence, even for all \( s \geq 0 \),
\[
\text{Tr}[(BA^q B)^s] \leq \text{Tr}[(\lambda B_1 A^q_1 B_1 + (1 - \lambda)B_2 A^q_2 B_2)^s] .
\]
Finally, for \( s \geq 1 \), \( X \mapsto \text{Tr}[X^s] \) is convex on \( \mathcal{P}_n \). Therefore,
\[
\text{Tr}[(\lambda B_1 A^q_1 B_1 + (1 - \lambda)B_2 A^q_2 B_2)^s] \leq \lambda^s \text{Tr}[(B_1 A^q_1 B_1)^s] + (1 - \lambda)^s \text{Tr}[(B_1 A^q_1 B_1)^s] .
\]
This proves the convexity for \( s \geq 1 \) and \(-1 \leq q < 0\).

The next result concerns the concavity of \( \Phi_{p,q,s}(A, B) \).

4.4 Theorem. The trace function \( \Phi_{p,q,s}(A, B) \) is jointly concave if and only if \( 0 \leq p, q \leq 1 \) and \( 0 \leq s \leq 1/(p + q) \).

Proof. The necessity of the condition is proved in [8 Prop. 5.1] and the sufficiency for \( 1/2 \leq s \leq 1/(p + q) \) is proved in [8 Thm. 2.1]. Our task is to prove sufficiency in the case \( 0 < s < 1/2 \). We write, using (4.2),
\[
\Phi_{p,q,s}(A, B) = s \inf_{X > 0} \text{Tr} \left\{ A^{q/2} B^p A^{q/2} X^{1-1/s} + \left( \frac{1}{s} - 1 \right) X \right\}
\]
\[
= s \inf_{Y > 0} \text{Tr} \left\{ B^p Y + \left( \frac{1}{s} - 1 \right) \left( A^{q/2} Y^{-1} A^{q/2} \right)^{s/(1-s)} \right\}
\]
\[
= s \inf_{Y > 0} \text{Tr} \left\{ B^p Y + \left( \frac{1}{s} - 1 \right) \left( Y^{-1/2} A^q Y^{-1/2} \right)^{s/(1-s)} \right\} .
\]
Since \( 0 \leq p \leq 1 \), \( B \mapsto B^p \) is operator concave and so \( B \mapsto \text{Tr} B^p Y \) is concave. By the extension of Epstein’s Theorem proved in [8 Theorem 4.1], \( A \mapsto \text{Tr} A^{-1/2} Y^{-1/2} A^{-1/2} Y^{-1/2} \) is concave if \( s/(1-s) \leq 1/q \). This condition is satisfied since \( s \leq 1/2 \leq 1/(1 + q) \). We conclude that \( \Phi_{p,q,s}(A, B) \) as an infimum of concave functions is concave. 

We conclude with a corollary of Theorem 4.2. For \( \rho, \sigma \in \mathcal{P}_n \) and \( \alpha, z > 0 \), we introduce the so-called \( \alpha \)-\( z \)-relative Rényi entropies
\[
D_{\alpha,z}(\rho || \sigma) = \frac{1}{\alpha - 1} \ln \frac{\text{Tr} \left( \sigma^{(1-\alpha)/(2z)} \rho^{\alpha/z} \sigma^{(1-\alpha)/(2z)} \right)^z}{\text{Tr} \rho} .
\]
(For \( \alpha = 1 \), a limit has to be taken.) These functionals appeared in [9 Sec. 3.3] and were further studied in [2], where the question was raised whether the \( \alpha \)-\( z \)-relative Rényi entropies are monotone under completely positive, trace preserving maps. Currently this is known for \( 0 < \alpha \leq 1 \) and \( z \geq \max\{\alpha, 1 - \alpha \} \), and for \( 1 \leq \alpha \leq 2 \) and \( z = 1 \), and for \( 1 \leq \alpha < \infty \) and \( z = \alpha \). See [2] for these cases. In this paper Audenaert and Datta conjecture that monotonicity holds for \( 1 \leq \alpha \leq 2 \) and \( \alpha/2 \leq z < \alpha \), and for \( 2 \leq \alpha < \infty \) and \( \alpha - 1 \leq z < \alpha \). Our contribution here is to prove their conjecture for \( 1 < \alpha = 2z \leq 2 \).
4.5 COROLLARY. Let $\alpha = 2z \in (1, 2]$ and let $\rho, \sigma \in \mathcal{P}_n$. Then for any completely positive, trace preserving map $\mathcal{E}$ on $\mathcal{P}_n$,

$$D_{\alpha,\alpha/2}(\rho||\sigma) \geq D_{\alpha,\alpha/2}(\mathcal{E}(\rho)||\mathcal{E}(\sigma)).$$

Proof. By a classical argument due to Lindblad and Uhlmann, see, e.g., [5, 7], the monotonicity follows once it is shown that

$$(\rho, \sigma) \mapsto \text{Tr} \left( \sigma^{(1-\alpha)/\alpha} \rho^{2} \sigma^{(1-\alpha)/\alpha} \right)^{\alpha/2} = \Phi_{2,2(1-\alpha)/\alpha,\alpha/2}(\sigma, \rho)$$

is jointly convex. For $\alpha \in (1, 2]$ this convexity follows from Theorem 4.2.

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References


