Error correction for a proposed quantum annealing architecture

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Abstract

Recently, Lechner, Hauke and Zoller [1] have proposed a quantum annealing architecture, in which a classical spin glass with all-to-all connectivity is simulated by a spin glass with geometrically local interactions. We interpret this architecture as a classical error-correcting code, which is highly robust against weakly correlated bit-flip noise.

Quantum annealing [2] is a method for solving combinatorial optimization problems by using quantum adiabatic evolution to find the ground state of a classical spin glass. Hoping to extend the reach of quantum annealing in practical devices, Lechner et al. [1] have proposed a scheme, using only geometrically local interactions, for simulating a classical spin system with all-to-all pairwise connectivity. Their scheme may be viewed as a classical low-density parity-check code (LDPC code) [3]; here we point out that the error-correcting power of this LDPC code makes the scheme highly robust against weakly correlated bit-flip noise.

Lechner et al. propose representing \( N \) logical bits \( \vec{b} = \{b_i, i = 1, 2, \ldots, N\} \) using \( K = \binom{N}{2} \) physical bits \( \vec{g} = \{g_{ij}, 1 \leq i < j \leq N\} \), where \( g_{ij} \) encodes \( b_i \oplus b_j \) and \( \oplus \) denotes addition modulo 2. The \( K \) physical variables obey \( K - N + 1 \) independent linear constraints. Hence only \( N - 1 \) physical variables are logically independent; we may, for example, choose the independent variables to be \( \{g_{12}, g_{23}, g_{34}, \ldots, g_{N-1,N}\} \). The linear constants may be chosen to be weight-3 parity checks. If weight-4 constraints are also allowed then the parity checks can be chosen to be geometrically local in a two-dimensional array. Higher-dimensional versions of the scheme may also be constructed [1]; we will discuss only the two-dimensional coding scheme here, but the same ideas also apply in higher dimensions.

While \( g_{ij} \) denotes the value of \( b_i \oplus b_j \) in the ideal ground state of the classical spin glass, we use \( g'_{ij} \) to denote the (possibly noisy) readout of the corresponding physical variable after a run of the quantum annealing algorithm. If the readout is not too noisy, we can exploit the redundancy of the LDPC code to recover the ideal value of \( \{b_i \oplus b_j\} \) from the noisy readout \( \vec{g}' \) with high success probability. Given an error model, we can determine the conditional probability \( p(\vec{g}' | \vec{b}) \) of observing \( \vec{g}' \) given \( \vec{b} \). Assuming that each \( \vec{b} \) has the same \textit{a priori} probability, we decode \( \vec{g}' \) by finding the most likely \( \vec{b} \):

\[
\vec{b}_{\text{decoded}} = \text{MLE}(\vec{g}') = \text{ArgMax}_{\vec{b}} p(\vec{g}' | \vec{b}),
\]  

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where MLE means “maximum likelihood estimate.” In fact, we can only recover the ideal \( \vec{b} \) up to an overall global flip since one bit of information is already lost during encoding.

We adopt the simplifying assumption of independent and identically distributed (i.i.d.) noise: \( g'_{ij} \) is flipped from its ideal value \( g_{ij} \) with probability \( \varepsilon \leq 1/2 \), and agrees with its ideal value with probability \( 1 - \varepsilon \). Though we do not necessarily expect this simple noise model to faithfully describe the errors arising from imperfect quantum annealing, our assumption follows the presentation of [1]. This model might be appropriate if, for example, the noise is dominated by measurement errors in the readout of the final state. It also allows us to estimate \( p(g'_{ij} | \vec{b}) \), either analytically or numerically. Exact MLE decoding is possible in principle, but has a very high computational cost. We will settle instead for decoding methods which are feasible though not optimal.

There is a very simple error correction procedure for which we can easily estimate the probability of a decoding error. For the purpose of decoding (say) \( g_{12} \equiv b_1 \oplus b_2 \), we make use of the following \( N - 2 \) weight-3 parity checks:

\[
0 = (12) \oplus (23) \oplus (13) = (12) \oplus (24) \oplus (14) = \cdots = (12) \oplus (2N) \oplus (1N),
\]

where we’ve used \((ij)\) as a shorthand for \( g_{ij} \). These checks provide us with \( N - 2 \) independent ways to recover the logical value of \( b_1 \oplus b_2 \), namely

\[
b_1 \oplus b_2 = (13) \oplus (23) = (14) \oplus (24) = \cdots = (1N) \oplus (2N).
\]

(Of course, \( g'_{12} \) itself provides another independent way to recover \( b_1 \oplus b_2 \), but to keep our analysis simple we will not make use of \( g'_{12} \) here.) Since \( g'_{ij} \neq g_{ij} \) with probability \( \varepsilon \), each \( g'_{1j} \oplus g'_{2j} \neq g_{ij} \) with probability

\[
\varepsilon^* := 2\varepsilon(1 - \varepsilon) \leq 1/2.
\]

Therefore, \( g_{12} \) is protected by a length-\((N - 2)\) classical repetition code, in which the \( N - 2 \) bits flip independently with probability \( \varepsilon^* \). This repetition code can be decoded through simple majority voting. The probability of a decoding error for \( g_{12} \) can be estimated from the Chernoff bound:

\[
p_{\text{fail}} \leq \exp \left( -2(N - 2) \left( \frac{1}{2} - \varepsilon^* \right)^2 \right).
\]

This is not the tightest possible Chernoff bound, and using additional information such as the observed value of \( g'_{12} \) will only improve the success probability. However, eq.(5) already illustrates our main point: the probability of a decoding error for any \( b_i \oplus b_j \) decays exponentially with \( N \).

A simple union bound constrains the probability with which any of the \( N - 1 \) bits are decoded incorrectly:

\[
p_{\text{total fail}} \leq (N - 1) \exp \left( -2(N - 2) \left( \frac{1}{2} - \varepsilon^* \right)^2 \right).
\]

Including \( g'_{12} \) in the decoding algorithm surely improves the accuracy of our estimate of \( b_1 \oplus b_2 \), and including higher-weight parity checks such as \( 0 = (12) \oplus (23) \oplus (34) \oplus (14) \) can yield further improvements. Following a pragmatic approach to using such information, we have implemented
belief propagation (BP) [4], a fairly standard decoding heuristic for LDPC codes. BP efficiently approximates MLE decoding when the constraint graph is a tree, and sometimes works well in cases where the graph contains closed loops.

In BP, a marginal distribution is assigned to each variable, and updated during each iteration based on the values of neighboring variables. A consistent neighborhood reduces the entropy of the marginal whereas an inconsistent neighborhood may increase the entropy or even change a variable’s most likely value. In our implementation, the initial configuration is the $\tilde{g}$ observed in a given run of the experiment, where each $g_{ij}'$ is assumed to be correct with probability $1 - \varepsilon$; two bits of $\tilde{g}$ are considered to be neighbors if they share an index. A single iteration of belief propagation includes the majority vote on variable pairs $g_{ik}' \oplus g_{jk}'$ which we have already discussed, as well as the value $g_{ij}'$ itself, to locally estimate the likelihood of each value for $g_{ij}$. To decode, beliefs are updated repeatedly until they converge to stable values. The probability of a decoding error is plotted in Fig. 1 as a function of $\varepsilon$ and the number $N$ of encoded spins. We include the Matlab code implementing the BP decoder and producing the benchmark figure as part of the arXiv source.

![Logical error scaling](image)

Figure 1: Performance of iterative BP decoding algorithm. The probability of a decoding error is plotted as a function of the number $N$ of encoded spins, for various values of the physical error probability $\varepsilon$. Each data point was obtained by averaging over 5000 noise realizations, and for each realization the BP algorithm was iterated five times, incorporating information about loops up to length $33 = 2^5 + 1$. The decoding performance is significantly better than for a single BP iteration, where only loops of size $\leq 3$ are considered. The logical error probability starts at $p_{\text{fail}}^{\text{total}} = \varepsilon$ for $N = 2$ and rises with $N$ until the onset of exponential decay, which begins for a smaller value of $N$ than indicated in eq.(6).

We conclude that the architecture proposed in [1], and the decoding method proposed here, provide good protection against i.i.d. noise in the readout of the physical spins, assuming an error probability $\varepsilon$ for each physical spin which is independent of the total number $N$ of encoded spins.
But is this characterization of the noise appropriate in this physical setting? Though quantum error-correcting codes might be invoked to improve accuracy\[5,6\], no truly scalable scheme for quantum annealing has been proposed\[7\]. How well the Lechner et al. architecture performs under realistic laboratory conditions is a question best addressed by experiments.

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References


