and

\[ f_1 \equiv f_2 + 1 \equiv f_3 + 1 \equiv f_4 - 1 \mod 3. \]

That is, it is known that these conditions are equivalent to those given by (II)-(iv), (II)-(v), and (II)-(vi) in Table I when \( m_k = 3 \). Thus the restriction on \( j_{\text{max}}(J) \) does not change.

Moreover the set of \( a_1 \) obtained from \( a_{i_1} = (r - 1)a_{i_1} = a_{i_2} = (r - 1)a_{i_4} \) also satisfies (37). Thus the situation as previously described may happen also for \( A(r,2) \). However, we have from (38)

\[ j_1 + 1 \equiv j_2 + 1 \equiv j_3 \equiv j_4 \mod 2, \]

which are equivalent to one of the congruences in (II)-(iv), (II)-(v), or (II)-(vi). Thus no new restriction on \( j_{\text{max}}(J) \) is needed here.

Except for the case of \( a_{i_1} = (r - 1)a_{i_4} = -1 \), we can find several sets of \( a_i \) satisfying (37). However, we cannot find those sets of \( a_i \) in Table I. This fact means that under those conditions \( J \) cannot be divided by an \( A \) that is composed of three or more \( A(r,m_k) \), even if one of them is \( A(r,2) \). Therefore this discussion does not impose any more stringent restriction on \( j_{\text{max}}(J) \).

Case (III)

(III)-(ii): This case has the same condition on \( j_1 \) as that considered by Kondratyev and Trofimov \[1\] for the binary case. It follows from the results obtained there that (13) is a sufficient condition for \( A \neq J \).

Finally we must consider the cases where \( w_s(J) < 4 \). However, the details for these cases are omitted here, because they can be discussed in a similar and even simpler way than that in the case of \( w_s(J) = 4 \). The result obtained is that looser restrictions than (5) and (13) will do.

From all that has been discussed previously and the inequalities

\[ \min_{i_1, i_2} \left( \prod_{k \in i_1} m_k + \prod_{k \in i_2} m_k \right) < \prod_{k \in i_1} m_k - 2 < \prod_{k \in i_1} m_k - 1 \]

we can conclude that the following theorem is valid.

Theorem 2: A radix-\( r \) AN code generated by \( A = \prod_{k \in i} A(r,m_k) \) has distance not less than five under the three conditions stated in Theorem 1.

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References


A Note on the Griesmer Bound

L. D. Baumert and R. J. McEliece

Abstract—Griesmer's lower bound for the word length \( n \) of a linear code of dimension \( k \) and minimum distance \( d \) is shown to be sharp for fixed \( k \), when \( d \) is sufficiently large. For \( k \leq 6 \) and all \( d \) the minimum word length is determined.

I. INTRODUCTION

Denote by \( n(k,d) \) the smallest integer \( n \) such that there exists an \((n,k)\) binary linear code with minimum distance at least \( d \). In 1960 Griesmer \[1\] proved that

\[ n(k,d) \geq \sum_{l=0}^{k-1} \left[ \frac{d}{2^l} \right] \]

and showed that for certain values of \( k \) and \( d \) the inequality (1.1) was in fact an equality. In 1965 Solomon and Stiffler \[2\] simplified Griesmer's proof of (1.1) and at the same time generalized it to linear codes over an arbitrary finite field \( GF(q) \), where it takes the form

\[ n(k,d) \geq \sum_{l=0}^{k-1} \left[ \frac{d}{q^l} \right]. \]

More important, however, Solomon and Stiffler introduced the notion of “puncturing” a \((q^k - 1, k)\) maximal-length shift-register code and showed that for many more values of \( k \) and \( d \) equality holds in (1.2).

In this correspondence we shall use the technique of puncturing to show that for fixed \( k \), when \( d \) is sufficiently large, the Griesmer bound (1.2) is sharp. That is, we will show that for each \( k \) there exists an integer \( D(k) \) such that if \( d \geq D(k) \), then

\[ n(k,d) = \sum_{l=0}^{k-1} \left[ \frac{d}{q^l} \right]. \]

As a matter of fact we will only prove this for \( q = 2 \), the extension to general \( q \) being easy but notationally awkward.

We shall use the notation

\[ g(k,d) = \sum_{l=0}^{k-1} \left[ \frac{d}{2^l} \right] \]

in the rest of the paper.

II. THE THEOREM OF SOLOMON–STIFFLER

Let \( V_k \) denote a \( k \)-dimensional vector space over \( GF(2) \). Let \( S_1, S_2, \ldots, S_r \) be subspaces of \( V_k \) of dimensions \( k_1, k_2, \ldots, k_r \) such

\[ k = k_1 + k_2 + \cdots + k_r. \]

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\[ 1 \] Actually these bounds were obtained in the form

\[ n(k,d) \geq \sum_{l=0}^{k-1} d_l, \]

where \( d_l = d\) and \( d_l = \lfloor d/l \rfloor q \). It is easy to see, however, that \( d_l = \lfloor d/l \rf \]
that no element (except 0) of \( V_k \) is contained in more than \( h \) of the \( S_i \). Then Solomon and Stiffler showed that there exists an \((n,k)\) binary linear code with minimum distance \( d \), where

\[
n = h(2^k - 1) - \sum_{i=1}^{2^k - 1} (2^h - 1)
\]

\[
d \geq h2^{k-1} - \sum_{i=1}^{2^h - 1} 2^{k-1} = d'.
\]

Furthermore if the \( k_i \) are distinct, \( n = g(k,d) \) and so the code is length optimal; i.e., \( n(k,d) = g(k,d) \). Finally they showed that a sufficient condition for the existence of such subspaces \( S_i \) is that \( \sum k_i \leq kh \).

### III. MAIN RESULT

**Theorem:** For each \( k \) there exists an integer \( D(k) \) such that

\[
n(k,d) = g(k,d), \quad \text{if } d \geq D(k).
\]

**Proof:** We show that \( D(k) = \lfloor (k - 1)/2 \rfloor 2^{k-1} \) will do. Write \( d = d_0 + (h - 1)2^{k-1} \), where \( 1 \leq d_0 \leq 2^{k-1} \). Then if \( d \geq (k - 1)/2 \) it follows that \( h \geq (k - 1)/2 \). Now we write \( 2^{k-1} - d_0 \) in its binary expansion

\[
2^{k-1} - d_0 = \sum_{i=1}^{2^h - 1} 2^{k-1}, \quad 0 < k_1 < k_2 < \cdots < k_t < k.
\]

Then

\[
\sum_{i=1}^{t} k_i \leq 1 + 2 + \cdots + k - 1 = k(k-1)/2 \leq k \cdot h
\]

and so by the results of Solomon-Stiffler quoted in Section II, \( n(k,d) = g(k,d) \).

### IV. NUMERICAL RESULTS

We have been able to calculate the exact values of \( n(k,d) \) for \( k \leq 6 \) and all \( d \). It turns out that the value \( D(k) = \lfloor (k - 1)/2 \rfloor 2^{k-1} \) given in our theorem is extremely conservative; for example, for \( k = 6 \) our theorem only guarantees that if \( d \geq 96 \), \( n(6,d) = g(6,d) \), while \( d \geq 20 \) would do. Much of this disparity arises from our use of the very weak sufficient condition \( \sum k_i \leq kh \) for the existence of subspaces \( S_1, S_2, \ldots \).

Thus consider the example \( k = 6, d = 35 \). Examining the proof in Section III, we write \( 35 = 3 + 1 \cdot 32 = 2^5 + 2^2 + 2^0 \). Thus we need to find subspaces of \( V_6 \) of dimensions \( 5, 4, 3, \) and \( 1 \) that cover each nonzero vector. Thus the Solomon-Stiffler results could not yield a \((37,6)\) code with \( d = 17 \). However, in his original paper (Theorem 5) Griesmer gave a construction that yields such a code.

We conclude the paper with Table I, which shows those values of \( k \) and \( d \) with \( k \leq 6 \) for which \( n(k,d) > g(k,d) \). The column titled “Comments” explains how we calculate \( n(k,d) \).

### References


### A Note on One-Step Majority-Logic Decodable Codes

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**Abstract**—Construction of shortened geometric codes as shown here results in 1-step majority-logic decodable codes. The shortened codes retain the error-correction ability of the parent codes and the decoders for the shortened codes are much simpler than for the parent code. A table of shortened codes is given.

I. SHORTENED FINITE GEOMETRY CODES

A shortened cyclic code retains at least the error-correcting capability of the parent full-length cyclic \((n,k)\) code. In the case

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