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Testing the inverse-square law of gravity: Error and design with the upward continuation integral

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It has been reported that the inverse-square law of gravity is violated over a range of a few hundred meters. I present a different method for the analysis of the data from that experiment. In this method, the experimental error can be evaluated analytically and I confirm the previous analysis but show that it is a 2σ effect. The method can also be used to design new experiments that will yield minimum errors for a fixed number of data points.

INTRODUCTION

Eckhardt, Jekeli, Lazarewicz, and Romaides[1] have measured the acceleration of gravity at six locations on the WTVD tower (Raleigh, North Carolina). They report that these measurements do not agree with the Newtonian prediction of gravity that is derived by upwardly continuing (extrapolating) the gravity field measured on the surface of the Earth near the tower. In principle, this is an elegant experiment because Green's theorem guarantees that they can calculate gravity above the Earth if the gravitational potential is known everywhere on the surface of the Earth.

This is probably a more precise test of the inverse square law of gravity than the existing borehole experiments[2-6] because the borehole experiments compare their measurements to a model of the Earth. (Downward continuation of a surface gravity field is very difficult due to the fact that an infinite number of source distributions lead to identical surface potentials. Also, downward continuation necessarily amplifies the noise in the gravity field measured at the surface. Upward continuation dampens the noise.) If a completely accurate model of the Earth was available, then a borehole experiment would be precise. But usually there are insufficient data to build the model. Only shallow mass anomalies out laterally from the hole can be put in the model; mass anomalies deep beneath the hole are simply unknown. All of the published borehole experiments suffer from this problem to one degree or another and the problem is not resolved by performing an experiment in the ocean or in an ice sheet. The deep-seated mass anomalies are still unknown.

UPWARD CONTINUATION

The goal is to measure the acceleration of gravity $g(Z)$ at several points on a tower and to compare these measurements to a Newtonian extrapolation of the surface gravity field $g(r,\phi,Z)$. But $g(Z)$ is not the experimental observable; usually the vertical component of gravity $g_z(Z)$ is measured with a relative instrument such as a LaCoste Romberg gravimeter. And so one measures gravity differences between two points where one of the points must be common to all measurements (e.g., the base of the tower). So in this paper I will derive the upward continuation equations for these differences and the vertical gravity gradient on the tower, $\Gamma(Z)$.

The mathematics of upward continuation is greatly simplified if the gravity data are measured on a spherical Earth with its surface passing through the base of the tower. Since local topography makes this impossible, we assume that a real survey can be extrapolated to this surface without significant error. Then, a straightforward application of Green's theorem,[7-9] in a right-handed spherical coordinate system, yields

$$g_\rho(\rho,0,0) = \frac{a^2(\rho^2-a^2)}{4\pi\rho} \times \int_S \frac{g_\rho(a,\theta,\phi)\, d\Omega}{[\rho^2+a^2-2a\rho\cos(\theta)]^{3/2}} , \qquad (1)$$

where $d\Omega$ is an element of solid angle on the sphere, a is the radius of Earth, ρ is the distance of the point of observation from the center of the sphere, and g_ρ is the vertical component of gravity. In deriving Eq. (1), it is important to recall that the vertical component of gravity in a spherical coordinate system is not harmonic but ρg_ρ is harmonic and will solve Laplace's equation.

If Z is the elevation of the point on the tower above the surface (i.e., $Z\equiv\rho-a$) and r is the distance along the surface of the sphere to a measurement point (i.e., $r\equiv a\theta$) then we can expand Eq. (1) in powers of Z/a, keep leading terms, and find

$$g_\rho(0,0,Z) = \frac{Z(1-Z/a)^2}{2\pi} \times \int_0^\infty r\,dr \times \int_0^{2\pi} d\phi \, \frac{g_\rho(r,\phi,0)}{[\, r^2+Z^2(1-Z/a)\,]^{3/2}} .$$

The perturbations caused by the Z/a terms are small and if $(1-Z/a)$ is set to 1, we have the solution to Laplace's equation above a flat plane. Since the current experiments are not sensitive to these terms, we will drop them for clarity although future experiments may reach sufficient precision for them to be important. We will also drop the subscript ρ and use the notation $g(r,\phi,Z)$ to refer to the vertical component of gravity from now on.

It is important to note that Green's theorem applies only to harmonic potentials and the observed gravity field has contributions from nonharmonic, noninverse square potentials (e.g., radial accelerations). So we must subtract an accurate model of the nonharmonic terms from the data, leaving only harmonic terms, before continuing with the upward continuation:

$$g(0,0,Z) - g_{\text{model}}(0,0,Z) = \frac{Z}{2\pi} \int_0^\infty r\,dr \int_0^{2\pi} d\phi \, \frac{g(r,\phi,0) - g_{\text{model}}(r,\phi,0)}{(r^2+Z^2)^{3/2}} . \tag{2}$$

Equation (2) is simplified if we define the gravity anomaly to be $\Delta g \equiv g - g_{\text{model}}$. And if Z_0 is the lowest reliable point of observation on the tower, we can estimate the anomalous gravity gradient $\Delta\Gamma$ between Z and Z_0 by subtracting $\Delta g(0,0,Z_0)$ from both sides of Eq. (2) and rearranging terms:

$$\Delta\Gamma(Z)\,(Z-Z_0) = \frac{1}{2\pi} \int_0^\infty r\,dr \int_0^{2\pi} d\phi \, \Delta g(r,\phi,0) \left[\frac{Z}{(r^2+Z^2)^{3/2}} - \frac{Z_0}{(r^2+Z_0^2)^{3/2}} \right] . \tag{3}$$

To compute this integral, first subdivide the surface into several rings of radii r_j and divide each ring into K_j subsectors of width $\Delta\phi$ (see Fig. 1). Then if each sector is to be approximately "square," the number of sectors per ring is

$$K_j \geq \pi \frac{r_j + r_{j-1}}{r_j - r_{j-1}} , \tag{4}$$

where we choose K_j to be the next largest integer defined by Eq. (4).

The discretized version of Eq. (3) becomes

$$\begin{aligned} \Delta\Gamma(Z)\,(Z-Z_0) \equiv{}& \sum_{j=1}^{N} \frac{\sum_{i=1}^{K_j} \Delta g_{ij}}{K_j} \left[\frac{Z}{(r_{j-1}^2+Z^2)^{1/2}} - \frac{Z}{(r_j^2+Z^2)^{1/2}} + \frac{Z_0}{(r_j^2+Z_0^2)^{1/2}} - \frac{Z_0}{(r_{j-1}^2+Z_0^2)^{1/2}} \right] \\ &+ \frac{1}{2\pi} \int_{r_N}^\infty r\,dr \int_0^{2\pi} d\phi \, \Delta g(r,\phi,0) \left[\frac{Z}{(r^2+Z^2)^{3/2}} - \frac{Z_0}{(r^2+Z_0^2)^{3/2}} \right] . \end{aligned} \tag{5}$$

The discrete sum in Eq. (5) covers the entire range of data, and the integral from r_N to ∞ is the truncation error that is encountered by not being able to measure gravity over the entire surface of the Earth. We will return to an estimate of this error later.

Equation (5) can be used to estimate the upwardly continued gravity anywhere. But a high-quality test of the inverse-square law should be done in a geologically flat region where $\Delta\Gamma(Z)$ is constant. This simplification will lead to useful insights and so we integrate both sides of the equation from Z_0 to H, the height of the tower, and normalize by $\int dZ\,(Z-Z_0)$ to yield the average gravity gradient anomaly $\Delta\overline{\Gamma}$:

$$\begin{aligned} \Delta\overline{\Gamma} = \frac{2}{(H-Z_0)^2} \sum_{j=1}^{N} \frac{\sum_{i=1}^{K_j} \Delta g_{ij}}{K_j} \Bigg[& (r_{j-1}^2+H^2)^{1/2} - (r_{j-1}^2+Z_0^2)^{1/2} + (r_j^2+Z_0^2)^{1/2} - (r_j^2+H^2)^{1/2} \\ & + \frac{Z_0(H-Z_0)}{(r_j^2+Z_0^2)^{1/2}} - \frac{Z_0(H-Z_0)}{(r_{j-1}^2+Z_0^2)^{1/2}} \Bigg] . \end{aligned} \tag{6}$$

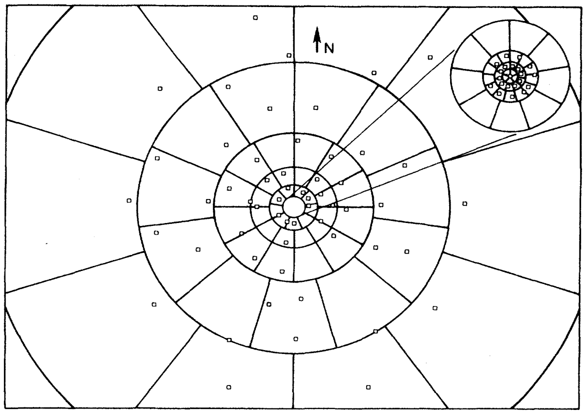

FIG. 1. The optimum survey pattern is compared to the location of the data points selected by Eckhardt, Jekeli, Lazarewicz, and Romaides. Their data are highly optimized but with an obvious lack of data in one ring. The largest ring in the figure has a radius of 7.2 km.

The anomalous gravity gradient is thus a weighted sum of the surface gravity anomalies:

$$\Delta\overline{\Gamma} = \sum_{j=1}^{N}\sum_{i=1}^{K_j} W_{ij}\,\Delta g_{ij} \ .$$

The total sum of weights is zero because the weighting function is negative at small distances and positive at greater distances with the two zones canceling. The data are therefore naturally split into two regions with r_{R0} being the critical radius where the weighting function changes sign. The total weight in either zone is

$$\sum_{j=r_{R0}}^{\infty}\sum_{i=1}^{K_j} W_{ij} = \frac{2}{(H-Z_0)^2} \times \left[(r_{R0}^2+H^2)^{1/2} - (r_{R0}^2+Z_0^2)^{1/2} - \frac{Z_0(H-Z_0)}{(r_{R0}^2+Z_0^2)^{1/2}} \right] \qquad (7)$$

and r_{R0} can easily be found by integrating Eq. (3) over Z and setting the kernel of the integral to zero. More explicitly, r_{R0} is defined to be the solution of the following equation:

$$\frac{1}{(r_{R0}^2+H^2)^{1/2}} - \frac{1}{(r_{R0}^2+Z_0^2)^{1/2}} + \frac{Z_0(H-Z_0)}{(r_{R0}^2+Z_0^2)^{3/2}} \equiv 0 \ . \qquad (8)$$

Equation (8) has an analytic solution but it is not interesting. It is simpler to solve for r_{R0} by iteration.

ERROR ANALYSIS

There are two unavoidable sources of error in the evaluation of Eq. (5). The first is the error in sampling a surface with a finite number of points. The second is the error due to truncating the sum at a finite radius. (There are also errors in the data but they can be assumed to be small.)

The sampling error arises because we will choose to use a single gravity value to estimate the mean value within each sector. Ideally, we would have many measurements per sector and then we could define a parent population of measurements with a mean and standard deviation for each sector. But with only one measurement, the error in the mean is identically the standard deviation of the (unknown) parent population in the sector. Therefore we will have to estimate the characteristics of the parent population from measurements in surrounding sectors. For example, we expect the standard deviation to increase as the size of the sector increases due to changes in the local topography and the underlying geology, so we assume that the variation in values for any sector can be described by a simple function of the size of the sector. (Figure 2 shows the fluctuations in the measurements taken around the WTVD tower in North Carolina. Evidently, the standard deviation is a linear function of the diameter of the sector.) Furthermore, the error in estimating the mean in a sector is independent of the error in estimating the mean in an adjacent sector as long as the measurement points are selected at random within the sectors, or if there are random topographic features in each sector and the points are sampled regularly. Therefore,

$$\sigma_{\Delta\overline{\Gamma}}^2 = \sum_{j=1}^{N}\sum_{i=1}^{K_j} W_{ij}^2 \ \ \sigma_{ij}^2(r_j - r_{j-1}) \ , \qquad (9)$$

where $(r_j - r_{j-1})$ characterizes the size of the sectors.

The truncation error is represented by the integral on the right-hand side of Eq. (5). If we assume that

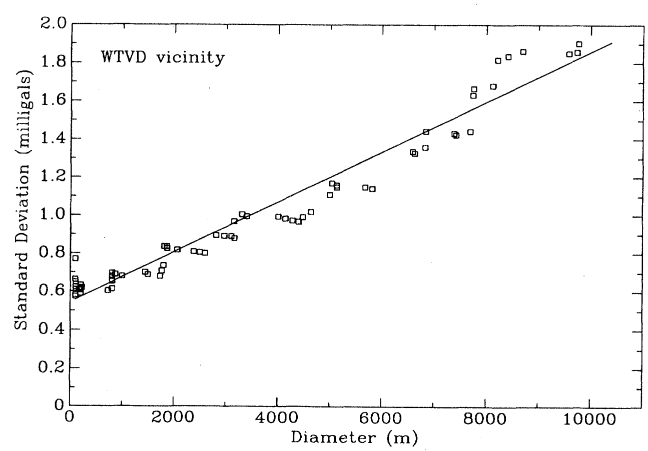

FIG. 2. The mean and standard deviation for all data points falling within a circle of diameter d is easily calculated. The figure suggests that the standard deviation of the free air anomalies surrounding the WTVD tower grows linearly with the size of the circle. The data are plotted as a function of diameter and a straight line fit yields an intercept of 0.54 mGal and a slope of 0.13 mGal/km.

$|\Delta g(r,\phi,0)|$ is bounded in the outer domain, with greatest sensitivity near r_N, then we can evaluate the integral to find

$$\Delta\Gamma_{\text{trunc}}(Z)(Z-Z_0) \le |\Delta g(r,\phi,0)| \times \left[\frac{Z}{(r_N^2+Z^2)^{1/2}} - \frac{Z_0}{(r_N^2+Z_0^2)^{1/2}} \right] \tag{10}$$

as long as $r_N > r_{R0}$. Integrating both sides over Z and taking the limit of large r_N we find that the truncation error in the average gradient anomaly is

$$\Delta\overline{\Gamma}_{\text{trunc}} \lesssim \frac{|\Delta g(r,\phi,0)|}{r_N} . \tag{11}$$

Equations (6), (9), and (11) are sufficient to analyze existing data.

DESIGNING A NEW SURVEY

Deciding where to place gravity measurements in a new survey depends on several factors. One necessary condition is that the total weight per ring, $K_j W_{ij}$, must decrease with increasing ring number so that the sum in Eq. (6) is convergent. Another condition is that the error per ring, $K_j W_{ij}^2 \sigma_{ij}^2$, must decrease with increasing ring number so that Eq. (9) is also convergent. One way to achieve this[10] is to have the sector weights W_{ij} be proportional to $1/\sigma_{ij}^2$. Therefore, we choose

$$|W_{ij}| \equiv \left[\frac{\sum_{i,j} |W_{ij}|}{\sum_{i,j} \frac{1}{\sigma_{ij}^2}} \right] \frac{1}{\sigma_{ij}^2} . \tag{12}$$

The term in large parentheses is a constant and so we may rearrange Eq. (12), substitute for W_{ij} from Eq. (6), and derive a recursion relation for the various ring radii, r_j. For the special case where $r_0 = 0$ and r_1 is given,

$$\frac{\sigma^2(r_j - r_{j-1})}{K_j} \left[(r_{j-1}^2+H^2)^{1/2} - (r_{j-1}^2+Z_0^2)^{1/2} + (r_j^2+Z_0^2)^{1/2} - (r_j^2+H^2)^{1/2} + \frac{Z_0(H-Z_0)}{(r_j^2+Z_0^2)^{1/2}} - \frac{Z_0(H-Z_0)}{(r_{j-1}^2+Z_0^2)^{1/2}} \right] \equiv \frac{\sigma^2(r_1)}{K_1} \left[(r_1^2+Z_0^2)^{1/2} - (r_1^2+H^2)^{1/2} + \frac{Z_0(H-Z_0)}{(r_1^2+Z_0^2)^{1/2}} \right] . \tag{13}$$

Equation (13) is easily solved with the aid of a computer. [An alternative to Eq. (13) is to define r_{R0} as one of the zone boundaries. This clearly separates the regions of positive and negative weight.] And the total error can be calculated by substituting Eqs. (12) and (7) into Eq. (9) to find

$$\sigma_{\Delta\overline{\Gamma}}^2 = \frac{4}{(H-Z_0)^2} \left[\frac{\sum_{i,j} |W_{ij}|}{\sum_{i,j} \frac{1}{\sigma_{ij}^2}} \right] \left[(r_{R0}^2+H^2)^{1/2} - (r_{R0}^2+Z_0^2)^{1/2} - \frac{Z_0(H-Z_0)}{(r_{R0}^2+Z_0^2)^{1/2}} \right] . \tag{14}$$

These equations yield some surprising results. For example, the surface gravity points should lie inside the sectors shown in Fig. 1. But they do not have to be precisely located since we are only using each point as an estimate of the mean for the sector. (A statistically valid sample of the gravity field can be made by taking the measurements near the center of each sector since it is highly improbable that the underlying geologic structures are correlated to the radial pattern around the tower.) Also, terrain fluctuations determine the required precision of the surface gravity survey. Figure 2 plots the variation in the gravity measurements around the WTVD tower; in this case σ is a linear function of the diameter of the sector. The nonzero offset in Fig. 2 is caused by random terrain fluctuations of $\approx\pm 2$ m. Since this represents "noise" in the data, the elevation of the points only need to be known to some fraction of this noise (e.g., $\frac{1}{2}$ or $\frac{1}{4}$). And since gravity is tightly correlated to elevation, the required precision on the gravity measurements is determined by this elevation noise (± 100 μGal in this case).

CONCLUSIONS

Eckhardt, Jekeli, Lazarewicz, and Romaides have measured the vertical component of gravity at six locations on the WTVD tower in North Carolina.[1,11] The average observed anomalous gravity gradient was 2.1 mGal/km

TABLE I. Data of Eckhardt, Jekeli, Lazarewicz, and Romaides for the WTVD tower are listed beside the results of their upward continuation of the surface gravity field. The observed mean gravity gradient is 2.1 mGal/km. The upward continued mean gradient is 3.1 ±0.1 mGal/km.

Z	$\Delta g(Z)_{\text{observed}}$	$\Delta g(Z)_{\text{upward}}$	Error
93.92	−19.796	−19.612	0.095
192.17	−19.622	−19.321	0.117
283.58	−19.436	−19.024	0.120
379.54	−19.207	−18.709	0.120
473.24	−18.946	−18.406	0.120
562.27	−18.671	−18.124	0.121

TABLE II. Calculated weights for upward continuation of the anomalous gravity gradient around the WTVD tower. The calculated gradient is 3.1 mGal/km. (Values in parentheses are assumed, values in square brackets are unknown.)

No.	Radius (m)	K	% weight	Sum %|Wt|	Ring average weight	Δg	Sum weight Δg
0	0	0	0.0	0.0	0.0	0.0	0.0
1	36	4	−7.03	7.03	-2.364×10^{-4}	−19.70	4.659
2	73	11	−14.56	21.59	-4.898×10^{-4}	−19.93	14.42
3	129	13	−17.05	38.64	-5.738×10^{-4}	−19.92	25.85
4	267	9	−11.36	50.00	-3.821×10^{-4}	[−20.000]	33.49
5	578	9	10.50	60.50	3.532×10^{-4}	−20.13	26.38
6	1 024	12	13.18	73.68	4.435×10^{-4}	−19.77	17.61
7	1 890	12	11.07	84.75	3.725×10^{-4}	−19.06	10.51
8	3 680	11	7.24	91.99	2.436×10^{-4}	−18.00	6.12
9	7 234	10	3.91	95.90	1.316×10^{-4}	−17.66	3.80
10	13 945	10	1.97	97.87	6.626×10^{-5}	(−9.15)	3.20
11	25 917	11	0.98	98.85	3.309×10^{-5}	−5.55	3.01
12	45 530	12	0.49	99.35	1.662×10^{-5}	0.38	3.02
13	75 522	13	0.26	99.61	8.721×10^{-5}	8.89	3.10
14	120 195	15	0.15	99.75	4.921×10^{-6}	8.83	3.14
15	182 441	16	0.08	99.84	2.839×10^{-6}	2.09	3.14

and their upward continuation of the surface field yielded 3.1 ± 0.1 mGal/km. (See Table I.) They attribute the difference to a non-Newtonian interaction.

I have attempted to duplicate this calculation with the techniques described above. Figure 1 shows the predicted survey pattern within 2.5 km of the WTVD tower. The locations of the measured gravity points[11] are shown and obviously they were well chosen with the exception of some missing points in one of the rings. Using the methods presented in this paper I find an anomalous gravity gradient of 3.1 mGal/km (see Table II), in good agreement with the previous calculation.

Figure 2 shows the variation in the free air anomalies[7] around the tower. The figure was generated by calculating the mean and standard deviation for all measurements within a circle of diameter d. The standard deviations were then plotted as a function of increasing diameter.

The sampling errors can be estimated by assuming that *any* sector on the ground is subject to the same gravity fluctuations as a sector of similar size centered on the tower. This is a local model which is valid because the integral is weighted so heavily near the tower (e.g., over 90% of the weight [Eq. (6)] comes within 4 km) and in addition the errors propagate as the square of the weight per sector.

Using this model and assuming that all rings were filled with data yields a sampling error of 0.23 mGal/km. But the ring with missing data represents approximately 11% of the integral weight and so may cause an additional systematic shift of the gradient by ±0.1 mGal/km. And, there is a shortage of data between 5 and 10 km (see Table I) which could cause a systematic shift as large as ±0.3 mGal/km. The truncation error also contributes but is less than 0.3 mGal/km if the free air anomalies beyond 200 km do not exceed 50 mGal. Adding all these terms in quadrature yields an error of 0.5 mGal/km; small enough to justify the claim of Eckhardt, Jekeli, Lazarewicz, and Romaides that the difference between the observed anomalous gravity gradient and the upward continued gradient is evidence for a breakdown of Newtonian gravity at the 2σ level.

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[10]One might choose to give equal weight to each sector, or to each ring, or to have either proportional to $1/\sigma$, but these schemes cause the ring-averaged terms in Eq. (9) to diverge. It is easy to show empirically that selecting weights $\propto 1/\sigma^2$ minimizes the random errors, but the author does not know of a general proof of this assertion.

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