Calculation of the electron magnetic moment in Fried-Yennie-gauge QED

Gregory S. Adkins*

W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125
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The two-loop contribution to the electron magnetic moment is calculated in the Fried-Yennie gauge. This is the first treatment of the magnetic moment beyond one-loop order in a gauge other than the Feynman gauge. The Fried-Yennie gauge is infrared safe, and the calculation is done without introducing an infrared cutoff or photon mass. The Fried-Yennie-gauge result agrees with the Feynman-gauge result, as expected.

I. INTRODUCTION

The electron magnetic moment is one of the oldest and most precise tests of QED. The “anomalous moment” or “(g−2)” correction was first calculated at the one-loop level by Schwinger, with the famous result (in units of the Bohr magneton)

$$\mu^{(1)} = \frac{\alpha}{2\pi}$$  

The two-loop calculation was done by Karplus and Kroll, and was corrected by Sommerfield and Petermann. The two-loop result is

$$\mu^{(2)} = \left[ \frac{\pi^2}{6} \zeta(3) - 3\zeta(2) \ln 2 + \frac{\pi^2}{12} \right] \left( \frac{\alpha}{\pi} \right)^2$$

where $\zeta(n)$ is the Riemann zeta function with

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) = \sum_{m=1}^{\infty} \frac{1}{m^3} = 1.202 056 903$$

Three- and four-loop results are summarized by Kinoshita.

In this report, I describe a calculation of the two-loop magnetic moment in the Fried-Yennie gauge. This gauge is infrared safe: it allows on-shell renormalization without the generation of infrared divergences. All terms in the Fried-Yennie-gauge calculation are infrared finite, and there is no need to introduce an infrared-regulating parameter as was done in the previous evaluations of $\mu^{(2)}$ (Refs. 2–4 and 7–9).

The Fried-Yennie-gauge photon propagator is given by

$$D_{\mu\nu}^{\gamma}(q) = -\frac{1}{q^2} \left( g^{\mu\nu} + \beta g^{\mu} q^\nu q^2 \right)$$

where $\beta=2$. In this work dimensional regularization is used to regulate ultraviolet divergences, and it is convenient to use $\beta=2/(1-2\varepsilon)$, where $n=4-2\varepsilon$ is the number of spacetime dimensions. This choice of $\beta$ results in simple forms for the finite parts of the electron self-energy and vertex functions discussed in Sec. II.

II. CALCULATIONAL TECHNIQUES

A. Extraction of the magnetic moment

The electron magnetic moment can be obtained from the renormalized vertex function $\Gamma_R^{\gamma}(p',p)$ shown in Fig. 1. This function contains a description of the interaction of an electron with an external magnetic field. The general form of the vertex function is

$$\Gamma_R^{\gamma}(p',p) = (\gamma p' - m)M^\lambda + M^\mu (\gamma p - m)$$

$$+ A \gamma^\lambda + B \Sigma^\lambda + C k^\lambda$$

where $M^\lambda, M^\mu, A, B, C$ are functions of $p'$ and $p$, and

$$\Sigma^\lambda = \frac{i e^\lambda k^\lambda}{2m}$$

When the incoming and outgoing electron lines are physical one can write

$$\bar{u}'(p') \Gamma_R^{\gamma}(p',p) u(p) = \bar{u}'(p') [F_1(k^2) \gamma^\lambda + F_2(k^2) \Sigma^\lambda] u(p),$$

since $C$ vanishes by current conservation under these conditions. Here, $F_1$ and $F_2$ are the electric and magnetic form factors, with

$$F_1(0) = 1$$

$$F_2(0) = (\mu - 1) = \frac{\alpha}{2\pi} + O(\alpha^2)$$

The electron magnetic moment is extracted from $\Gamma_R^{\gamma}$ by

![Fig. 1. The vertex function.](image-url)
identifying the term proportional to $\Sigma^\lambda$ when the electron lines are physical in the limit $k^2 \to 0$.

**B. Renormalization**

Although the magnetic moment $\mu$ is contained in the renormalized vertex function $\Gamma^{\lambda}_{R}$, it is convenient to calculate with unrenormalized quantities, so the connection between the two must be considered. The renormalized vertex function is related to the bare vertex function by

$$\Gamma^{\lambda}_{R}(p', p) = Z_1 \Gamma^{\lambda}(p', p) ,$$

where $Z_1 = (1 - B)^{-1}$ is the vertex renormalization constant. Expanding all quantities in powers of the fine-structure constant $\alpha$ one has

$$\Gamma^{\lambda}_{R} = [1 - (B^{(1)} + B^{(2)} + \cdots)]^{-1} \times (\gamma^{\lambda} + \Gamma^{\lambda(1)} + \Gamma^{\lambda(2)} + \cdots) .$$

The order-$\alpha^2$ term in $\Gamma^{\lambda}_{R}$ is

$$\Gamma^{\lambda(2)}_{R} = \Gamma^{\lambda(2)} + B^{(1)}\Gamma^{\lambda(1)} + (B^{(2)} + B^{(1)}B^{(1)})\gamma^{\lambda} .$$

Note that there are no infrared divergences induced in $B^{(1)}$ and $C^{(1)}(p)$ as there are in other gauges. The choice $\beta = 2/(1 - 2\varepsilon)$ instead of simply $\beta = 2$ results in a form for $C^{(1)}(p)$ with just a $\gamma^{\mu}$ term and no term proportional to $m$. The one-loop vertex function of Fig. 3(b) is more complicated. It has the form

$$\Gamma_{\mu}^{(1)}(p', p) = -B^{(1)}\gamma^{\mu} + \Gamma_{\mu}^{(1)}(p', p) ,$$

where the subtracted vertex function (in the $\epsilon \to 0$ limit) is

$$\Gamma_{\mu}^{(1)}(p', p) = -\frac{\alpha}{4\pi} \int_0^1 dx \int_0^1 dz \int_0^1 dt \left[ -4x(1-x)S^{\mu} \frac{1}{H^2} + 2R^{\mu} \frac{1}{H} + 2\gamma^{\mu}x^2(H - xm^2) \frac{m^2}{H^2} + 6\gamma^{\mu}x(H - xm^2) \frac{1}{H} \right]$$

with

$$H = xm^2 - xu(1-u)k^2 + (1-x)u(m^2-p'^2) + (1-x)(1-u)(m^2-p^2) ,$$

$$\overline{H} = xm^2 + t(H - xm^2) .$$

**C. Lower-order functions**

The rainbow, side vertex, self-energy, and vacuum-polarization graphs are all reducible, containing lower-order vertex, self-energy, or vacuum-polarization parts. The vacuum-polarization function is gauge independent. It can be found in the standard textbooks and will not be discussed here. The one-loop self-energy function of Fig. 3(a) is given in the dimensionally regularized Fried-Yennie gauge by

$$\Sigma^{(1)}(p) = B^{(1)}(\gamma p - m) + C^{(1)}(p)(\gamma p - m)^2 ,$$

where the one-loop contribution to the renormalization constant is

$$B^{(1)} = -\frac{\alpha}{4\pi} (4\pi)^f \Gamma(e) \left[ \frac{3 - 2\varepsilon}{1 - 2\varepsilon} \right] ,$$

and the finite part is (in the limit $\epsilon \to 0$)

$$C^{(1)}(p) = -\frac{\alpha}{4\pi} (6\gamma p) \int_0^1 dx \int_0^1 dz \frac{x(1-x)}{xm^2(1-x)z(m^2-p^2)} .$$
The gamma matrix factors are
\begin{align}
S^\mu &= q^2 (\gamma p' - m) \gamma^\mu (\gamma p - m) - m q^\mu (\gamma p' - m) (\gamma p - m) \\
&\quad + (\gamma p' - m) \left[ (m \gamma^\mu - 2 (1 - u) p^\mu p q - (1 - u) q^\mu (m^2 - p^2)) + ((m \gamma^\mu - 2 (1 - u) p^\mu p q - u q^\mu (m^2 - p^2)) (\gamma p - m) \\
&\quad + \gamma^\mu (2 p' \gamma p' q - m^2) + (1 - u) (\gamma p' q + 2 m^2 (m^2 - p^2) + u (p \gamma p + 2 m^2 (m^2 - p^2) + 2 (1 - u) m^2 k^2) \\
&\quad + (\gamma p' - m) [(-2 + 3 x - x^2 u) \gamma^\mu + (-2 + 3 x - x^2 u) p^\mu + 2 (1 - u) k^\mu] (\gamma p - m) \\
&\quad + (2 - 3 x - 2 m^2 (1 - u) p^\mu + 2 (1 - u) k^\mu) (\gamma p - m) + \gamma^\mu (2 + 3 x - x^2 u) (1 - u) p^\mu + 2 (1 - u) k^\mu) (\gamma p - m) \\
&\quad + x (1 + x) (1 - u) m \gamma^\mu - x (1 - x) m \gamma^\mu k^\mu, \\
&\quad (18a)
\end{align}
\begin{align}
\bar{R}^\mu &= (3 - 5 x) (\gamma p' - m) \gamma^\mu (\gamma p - m) \\
&\quad + (\gamma p' - m) \left[ 2 - 3 x - 3 x^2 u + 3 x^2 (1 - u) k^\mu \right. \\
&\quad + (1 - u) \left[ 2 - 3 x - 3 x^2 u + 3 x^2 (1 - u) k^\mu \right] (\gamma p - m) \\
&\quad + (2 - 3 x - 3 x^2 u + 3 x^2 (1 - u) k^\mu) (\gamma p - m) + x (1 + x) (1 - u) m \gamma^\mu - x (1 - x) m \gamma^\mu k^\mu, \\
&\quad (18b)
\end{align}

where
\begin{align}
q &= u p' + (1 - u) p, \\
P &= p' + p, \\
k &= p' - p. \\
&\quad (19a)
\end{align}

The one-loop Fried-Yennie-gauge self-energy and vertex functions are similar in complexity to the corresponding Feynman-gauge functions.

### III. RESULTS

In this section, I will briefly discuss the calculation of the various diagrams and give the results.

The crossed diagram \( C \) is completely infrared and ultraviolet safe, so one can set \( \epsilon = 0 \) immediately. There are two photon propagators in \( C \), so there are terms with two, one, and zero factors of the gauge parameter \( \beta \). The terms with two and one factors of \( \beta \) are easy to evaluate. Their contributions to \( \mu_C \) are \((-1/4)(\alpha/\pi)^2\) and \(1/4 (\xi(2) + 1/2)(\alpha/\pi)^2\). The term with zero factors of \( \beta \) is just the Feynman-gauge result for the crossed graph, which is given by Petermann.\(^4\) The total contribution of the crossed graph is
\begin{align}
\mu_C &= \left[ \frac{3}{2} \xi(3) - \frac{5}{2} \xi(2) \ln 2 + \frac{1}{2} \xi(2) + \frac{1}{2} \right] \left( \frac{\alpha}{\pi} \right)^2. \\
&\quad (20)
\end{align}

The evaluation of the rainbow diagram \( R \) involves a slight subtlety. The \( i \sigma^\mu k^\nu \) term in \( \Gamma^{\mu(1)}_i \) contains no factor that is small in the infrared region. (All the rest of the terms in \( \Gamma^{\mu(1)}_i \) that contribute to \( \mu_R \) do contain such factors.) The fact that the spanning photon in the Fried-Yennie gauge becomes crucial in order to avoid an infrared problem, and one must keep \( \beta = 2/(1 - 2 \epsilon) \) for the spanning photon and perform the momentum integration in \( n = 4 - 2 \epsilon \) dimensions. The result for this diagram is
\begin{align}
\mu_R &= -B^{(1)}  \left( \frac{\alpha}{2 \pi} \right) + \left( \frac{3}{16} \right) \left( \frac{\alpha}{\pi} \right)^2. \\
&\quad (21)
\end{align}

The remaining diagrams are straightforward (although evaluation of the side vertex is lengthy), and one finds

\begin{align}
\mu_{SV} &= -2 B^{(1)}  \left( \frac{\alpha}{2 \pi} \right) \\
&\quad + \left( \frac{3}{16} \right) \left( \frac{\alpha}{\pi} \right)^2, \\
&\quad (22)
\end{align}
\begin{align}
\mu_{SE} &= 2 B^{(1)}  \left( \frac{\alpha}{2 \pi} \right) \\
&\quad + \left( \frac{9}{8} \right) \left( \frac{\alpha}{\pi} \right)^2, \\
&\quad (23)
\end{align}
\begin{align}
\mu_{VP} &= -2 B^{(2)} \left( \frac{\alpha}{2 \pi} \right) \\
&\quad + \left( \frac{15}{32} \right) \left( \frac{\alpha}{\pi} \right)^2. \\
&\quad (24)
\end{align}

The final result for the magnetic moment at order \( \alpha^2 \) is
\begin{align}
\mu^{(2)} &= \left[ \frac{3}{2} \xi(3) - \frac{3}{2} \xi(2) \ln 2 + \frac{3}{2} \xi(2) + \frac{1}{16} \right] \left( \frac{\alpha}{\pi} \right)^2 \\
&\quad = (-0.328479) \left( \frac{\alpha}{\pi} \right)^2. \\
&\quad (25)
\end{align}

in agreement with the earlier calculations.

### IV. CONCLUSION

The gauge independence of physical results guarantees that the Fried-Yennie gauge result for \( \mu^{(2)} \) will agree with the known (and well verified\(^5\)) Feynman-gauge result. The value of the present work lies in two areas. First, it is conceptually simpler than the Feynman-gauge approach since no infrared cutoff or photon mass is needed. Second, it provides a check of the Fried-Yennie gauge formalism. This is important since the Fried-Yennie gauge promises to be of great use in bound-state QED, where control of the infrared without the use of a nonzero photon mass is imperative.

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Permanent address: Department of Physics, Franklin and Marshall College, Lancaster, PA 17604.

1 J. Schwinger, Phys. Rev. 73, 416 (1948); 76, 790 (1949).


10 The gamma matrix conventions and natural units \[ t=\epsilon=1, \]
\[ \alpha=e^2/4\pi=(137)^{-1} \]
of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), are used throughout. The symbol \( m \) represents the electron mass \( m = 0.511 \text{ MeV} \).

11 I have included the effect of a mass counterterm along with the self-energy shown in Fig. 3(a) so that no term of order zero in \( (\gamma p - m) \) appears in \( \Sigma^{1/4}(p) \).
FIG. 1. The vertex function.