Symmetry fractionalization and anomaly detection in three-dimensional topological phases

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In a phase with fractional excitations, topological properties are enriched in the presence of global symmetry. In particular, fractional excitations can transform under symmetry in a fractionalized manner, resulting in different symmetry enriched topological (SET) phases. While a good deal is now understood in 2D regarding what symmetry fractionalization patterns are possible, the situation in 3D is much more open. A new feature in 3D is the existence of loop excitations, so to study 3D SET phases, first we need to understand how to properly describe the fractionalized action of symmetry on loops. Using a dimensional reduction procedure, we show that these loop excitations exist as the boundary between two 2D SET phases, and the symmetry action is characterized by the corresponding difference in SET orders. Moreover, similar to the 2D case, we find that some seemingly possible symmetry fractionalization patterns are actually anomalous and cannot be realized strictly in 3D. We detect such anomalies using the flux fusion method we introduced previously in 2D. To illustrate these ideas, we use the 3D Z_2 gauge theory with Z_2 global symmetry as an example, and enumerate and describe the corresponding SET phases. In particular, we find four nonanomalous SET phases and one anomalous SET phase, which we show can be realized as the surface of a 4D system with symmetry protected topological order.

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I. INTRODUCTION

Fractional excitations in a topological phase are characterized by their fractional statistics when braided around each other. In the presence of a global symmetry, they acquire new topological features. In particular, each fractional excitation can transform under the symmetry in a fractional way. For example, in the $\nu = 1/3$ fractional quantum Hall system with U(1) charge conservation symmetry, a single quasiparticle carries $1/3$ of the electron charge while the underlying electrons always have integer charges [1]. This is the phenomenon of symmetry fractionalization (SF). Systems with the same topological order and the same global symmetry can have different SF patterns, resulting in different symmetry enriched topological (SET) phases [2–11]. An interesting question is to understand what SF patterns are possible and where they can be realized.

Substantial progress has been made in answering this question for 2D topological phases. It was realized that the SF type of a quasiparticle is given by a projective representation of the symmetry [2–4,12,13]. Moreover, the projective representations of different quasiparticles should be consistent with their fusion rules. That is, if two quasiparticles can be fused into a third one, their projective representations should combine into that of the third one, up to some linear representation of the symmetry. Following this rule, the whole set of possible SF patterns of a particular topological order can be exhaustively listed [4].

However, such a counting is overcomplete. It was realized that some of the SF patterns are anomalous, i.e., they cannot be realized in strictly 2D systems [14–26]. Various anomaly detection methods have been proposed to identify such SF patterns. The central idea of most methods is based on introducing symmetry fluxes into the system and trying to gauge the global symmetry [10,22–26]. If the SF pattern can be realized in a strictly 2D model, then the global symmetry can be consistently gauged, resulting in a larger topological theory including the original quasiparticles and the symmetry fluxes and charges. On the other hand, if the SF pattern is anomalous, something goes wrong in the gauging process and the anomaly is revealed. Interestingly, it was found that these anomalous SF patterns can be realized on the surface of 3D systems with nontrivial symmetry protected topological (SPT) order in the bulk. In this case, the anomaly exposed in the gauging process on the surface is canceled by one coming from the bulk, resulting in a consistent theory.

What about 3D topological phases with symmetry? Various topological phases have been found in 3D systems, including gauge theories and their twisted versions [27,28]. What happens when the system also has a global symmetry? This is the question we try to answer in this paper.

In particular, we address the following two parts of the question. (1) How to describe symmetry fractionalization patterns in 3D? (2) How to detect anomalies in the symmetry fractionalization patterns?

New insights are needed to generalize our understanding from 2D to 3D. First, 3D topological phases contain loop like excitations that we refer to as quasistrings. When describing the symmetry action on these excitations, we must take their extended nature into consideration. Secondly, most of the anomaly detection methods proposed depend on the 2D nature of the system and do not generalize in a straight-forward way to 3D. To identify anomalous SF patterns in 3D, a new method is needed.

We address these issues in this paper. In Sec. II, we discuss how to properly describe SF patterns in 3D, in particular the nontrivial symmetry action on loop excitations. Our description is based on dimensional reduction to 2D, in particular, on examining differences in 2D SET order between regions bounded by dimensionally reduced quasistrings. We

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also relate this description to three-loop braiding processes in 3D. Such an understanding enables us to list all possible of SF patterns, although some of them may be anomalous.

In Sec. III, we demonstrate how to use the "flux fusion" method to detect anomalies in 3D SF patterns. We introduced the "flux fusion" idea in Ref. [26] where it was used to identify anomalies in 2D. This method can be straightforwardly generalized to 3D and is used in this paper. We briefly review the basic idea of the method before applying it to 3D. Throughout our discussion, we use the 3D Z_2 gauge theory with unitary Z_2 symmetry as an illustrative example, which we call for simplicity the Z_2Z_2 SET. In particular, we find that there are four nonanomalous SF patterns and one anomalous one in this case. In Sec. IV, we summarize our results and discuss open questions. Three appendices contain a more detailed treatment of the dimensional reduction procedure and description of symmetry fractionalization on quasistrings, accounting for all the fusion and braiding properties characterizing the 2D SET orders.

In previous studies, several classes of 3D SET phases have been analyzed by focusing on the fractional symmetry representations carried by the quasiparticles in the system. For example, Ref. [8] classified 3D Z_2 gauge theories enriched with time reversal symmetry and Ref. [29] classified gapless 3D U(1) spin liquid with time reversal symmetry. In particular, it was pointed out in Ref. [29] that certain types of time reversal symmetric U(1) spin liquids are anomalous.

II. SYMMETRY FRACTIONALIZATION IN 3D

Topological excitations in 3D can be either point like quasiparticles or loop like quasistrings. Symmetry fractionalization on quasiparticles in 3D works in the same way as on quasiparticles in 2D, which we review briefly in Sec. II A. Quasistring excitations exist only in 3D, not in 2D. To understand SF patterns in 3D, the key is to understand how symmetries act on quasistrings. We discuss this in detail in Sec. II B. To illustrate our discussion, we use the Z_2 gauge theory with unitary Z_2 symmetry (the Z_2Z_2 SET) as an example. To distinguish the two Z_2's, we label the gauge group as Z_2^g and the global symmetry group as Z_2^s. The 3D Z_2^g gauge theory has a quasiparticle excitation, which we call the gauge charge e, and a quasistring excitation, which we call the gauge flux loop m. For the Z_2^s global symmetry, we denote the symmetry charge as Q and the symmetry flux loop as Ω. We enumerate all possible ways the unitary Z_2^s symmetry can fractionalize on the e and m excitations.

Braiding processes involving three-loop excitations will play an important role in our discussion, and we use the following notation for such braiding statistics. The statistics angle for a full braid of two loops i, j when they are linked with a base loop k is denoted as $\Phi_{i,j;k}$ (e.g., $\Phi_{m,\Omega;m}$). We can also consider exchange statistics (i.e., a half-braid) of two identical loops i linked with a base loop k; in this case, we denote the statistics angle by $\Phi_{i;k}$. Sometimes we use the label i_k (e.g., Ω_m) to denote i loops linked with a k base loop. We have suppressed these subscripts in $\Phi_{i,j;k}$ (i.e., we are not writing $\Phi_{i_k,j_k;k}$) for simplicity of notation.

In order to define what we mean by a Z_2^g gauge theory, we need to specify the braiding properties of e and m. These properties have nothing to do with the Z_2^s symmetry, and persist if the symmetry is broken. We take e to be a boson; there are also Z_2^g gauge theories with fermionic quasiparticles carrying the gauge charge [13,28,30–32], but we do not consider these theories here. The e quasiparticle feels m as a π flux, so that a statistical phase of π is acquired when e winds around a m quasistring. Finally, for three-loop braiding of m quasistrings, $\Phi_{m,m;m} = 0$. Considering exchange statistics of the two m loops linked to a m base loop, there are two consistent possibilities, $\Phi_{m;m} = 0,\pi$. However, these possibilities are not distinct, as they are related by a natural relabeling of excitations. We can shift $\Phi_{m;m} \to \Phi_{m;m} + \pi$ by binding e particles to the two m loops linked to the base loop. Then, because these loops cannot be shrunk to a point, there is not a natural labeling of m versus em, and we are free to relabel $m \leftrightarrow em$.

When we apply the dimensional reduction procedure, we will need to discuss 2D braiding statistics. The statistics angle for a full braid of point excitations i and j in 2D is denoted $\theta_{i,j}$, while we write θ_i for the exchange statistics of two i excitations.

A. Symmetry fractionalization on quasiparticles

The local action of symmetry on quasiparticle excitations only needs to satisfy the group multiplication relation up to a phase factor [2,4,12,13]. (Here we assume that symmetry does not permute the quasiparticles, which is sufficient to discuss the Z_2Z_2 SET.) We write

$$U_a(g_1)U_a(g_2) = e^{i\alpha_a(g_1,g_2)}U_a(g_1g_2), \tag{1}$$

where $U_a(g)$ gives the action of symmetry operation g on a quasiparticle of type a.

The phase angle $\alpha_a(g_1,g_2)$ is not arbitrary. If n copies of a fuse into the vacuum, then $e^{i\alpha_a(g_1,g_2)}$ has to be an nth root of unity. On the other hand, we can redefine $U_a(g) \to \lambda_a(g)U_a(g)$ for $\lambda_a(g)$ a nth root of unity, and any two sets of α related in this way should be considered equivalent. Therefore $U_a(g)$ forms a projective representation of the symmetry group G with coefficients in Z_n and $e^{i\alpha_a(g_1,g_2)}$ specifies an element in $H^2(G,Z_n)$. Moreover, if a and b fuse into c, then $\{U_a(g) \otimes U_b(g), g \in G\}$ should be equivalent to $\{U_c(g), g \in G\}$, in the sense that both representations are characterized by the same element of $H^2(G,Z_n)$.

In the 3D Z_2Z_2 SET, the quasiparticle e can transform in two different ways under the Z_2^s symmetry: applying the nontrivial Z_2^s symmetry operation twice to e may result in a phase factor of $+1$ or -1. Correspondingly, e is said to carry integer or half odd integer Z_2^s charges, which we label as $e0$ and eC, respectively. In both cases, two e particles together always carry integer Z_2^s charge, which must occur because they fuse into the trivial sector.

B. Symmetry fractionalization on quasistrings

How can symmetry fractionalize on quasistring excitations? We try to answer this question in this section. First, we discuss the possibility of quasistring excitations carrying fractional representations, or gapless modes protected by the symmetry. While these are possible nontrivial ways quasistrings can transform under symmetry, a more complete perspective is provided by viewing quasistrings as boundaries between 2D SET phases upon dimensional reduction to 2D. We explain this point in detail and then count all possible symmetry fractionalization patterns in the Z_2Z_2 SET.

1. *Quasistring carrying fractional symmetry representations*

One possible way for quasistrings to transform nontrivially under symmetry is to carry fractional symmetry representations, just like quasiparticles. However, this is not an intrinsic feature of quasistrings.

First, we consider a loop of quasistring that is not linked with any other loops. Such a quasistring can be shrunk down to a point, and the loop becomes a point excitation while any fractional symmetry representation it carries remains unchanged. That is to say, there is some quasiparticle excitation in the theory that carries the same fractional representation. Then, by attaching the antiparticle to the quasistring, we can cancel the fractional symmetry representation carried by the quasistring. Therefore fractional symmetry representation carried by quasistrings can always be removed, and, as we shall do in the following discussion, we are free to focus on quasistrings carrying no fractional symmetry representation without any loss of generality.

In fact, unlike quasiparticles, some quasistring excitations can appear on their own, not in pairs. For example, this is the case for the m flux loop in the Z_2^g gauge theory, although the composite of em has to come in pairs. Quasistring excitations that can exist on their own cannot carry fractional symmetry representations because they can shrink down to nothing. Quasistring excitations that come in pairs can carry fractional representations, but that reduces to the fractional representation carried by quasiparticles once the strings are shrunk down to a point.

Quasistrings that are linked to other loops cannot be shrunk down to a point, so it is not clear, at this point, how to define the fractional symmetry representation they carry.

2. *Quasistring carrying gapless modes protected by symmetry*

Quasistrings are one dimensional excitations. A more intrinsic way for them to transform under symmetry is to carry 1D gapless modes protected by the symmetry. Such 1D gapless modes appear on the edge of 2D symmetry protected topological (SPT) phases. That is to say, we can imagine the quasistring excitation as bounding a 2D surface to which a 2D SPT state is attached. This is natural, because quasistrings in 3D gauge theories can be viewed as edges of strongly fluctuating surfaces with vanishing surface tension. Note that the gapless modes on quasistrings exist on top of the excitation gap of the quasistrings, which scales linearly with the length. Therefore the quasistring is still a gapped excitation with linear energy cost. If a static, infinitely long quasistring is forced into the system, then the spectrum is gapless.

For example, in our Z_2Z_2 SET example, we can have the m loop transform as the boundary of a 2D Z_2^s SPT state. If we create an m loop in the bulk of the system, as shown in Fig. 1(a), naively we would expect it to be gapless unless the Z_2^s symmetry is broken.

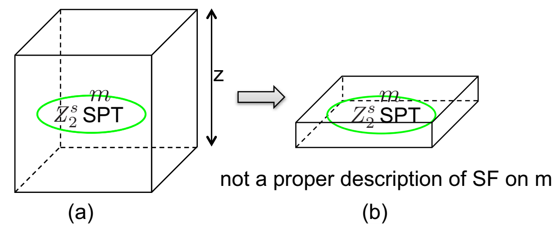

FIG. 1. One possible way for Z_2^s symmetry to transform nontrivially on Z_2^g flux loop m: (a) m loop carries gapless modes protected by Z_2^s symmetry and bounds a 2D Z_2^s SPT state; (b) Compressing the dimension of the system perpendicular to the surface bounded by the m loop reduces the system to 2D. However, depicting the m loop as the boundary of a 2D SPT state is an incomplete and inaccurate description of the symmetry fractionalization pattern on m.

In an attempt to examine such a symmetry action in more detail, it is natural to consider a dimensional reduction procedure where we compress the z dimension of the system while keeping the other two dimensions infinite, as shown in Fig. 1(b). It is tempting to conclude that if an m loop is inserted in the xy plane, then the region inside the string is in the nontrivial 2D Z_2 SPT phase, while the outside is in the trivial phase. However, this is not an accurate description of the dimensionally reduced system, and thus does not provide a description of symmetry fractionalization on quasistrings. In order to have a better understanding, we need to look at the dimensional reduction process in a more careful way. This is similar to the dimensional reduction approach used in Refs. [33–35] to study 3D topological phases but here we add symmetry to the discussion.

3. *Quasistring as boundary between 2D SET phases*

A more careful analysis of the dimensional reduction procedure illustrated in Fig. 1(b) shows that a proper description of symmetry fractionalization on quasistrings is obtained by viewing them as boundaries between 2D SET phases, and not SPT phases. That is, it is important to take into account the nontrivial topological order of the dimensionally reduced system.

The first step is to understand what topological order the dimensionally reduced system has, and for this purpose we can temporarily ignore the Z_2^s symmetry. Suppose that we start from a 3D Z_2^g gauge theory and compress the system down in the z direction, as shown in Fig. 2. We assume periodic boundary conditions in all three directions. The height of the system in the z direction is finite, but larger than any correlation lengths, while the extent in the x and y directions is infinite.

In this geometry, the system becomes a 2D Z_2^g gauge theory. To see this, note that the gauge charge e quasiparticles in the 3D bulk become quasiparticle excitations in the 2D bulk, which are free to move in the xy plane. The other type of 2D quasiparticles are m quasistrings that wind once across the system in the finite z direction; these excitations are quasiparticles in the dimensionally reduced system, rather than quasistrings, due to their finite extent and, thus, finite energy cost. In the 2D theory, an e excitation going around one of these quasiparticles is equivalent to an e particle going around a m quasistring in the original 3D bulk, which results

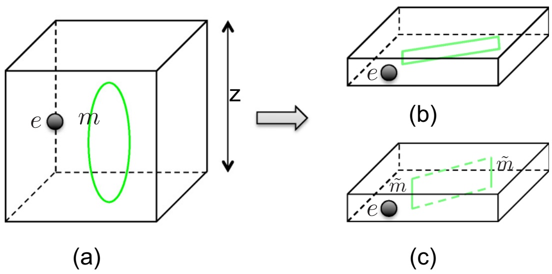

FIG. 2. Dimensional reduction of a 3D Z_2^g gauge theory into a 2D Z_2^g gauge theory: (a) a 3D Z_2^g gauge theory with gauge charge e and gauge flux m. After compressing the system in the z direction, the gauge charge remains a quasiparticle while (b) gauge flux loop with finite extent in z direction becomes nontopological (c) a gauge flux loop which extends across the z direction becomes two quasiparticles, which are the gauge fluxes in 2D.

in a π phase. Therefore the new quasiparticles correspond to the gauge fluxes in the 2D Z_2^g gauge theory and we label them by $\tilde{m}$. The full topological order of the 2D system is that of a Z_2^g gauge theory.

Flux loops of extent less than the system size in the z direction, as shown in Fig. 2(b), become nontopological excitations in 2D. For an e quasiparticle to braid around a segment of such a loop, it has to pass a finite distance from the loop during the braiding process. Such a process can be perturbed by various local perturbations, and there is no well-defined statistical phase. However, starting with such a nontopological flux loop, we can create two $\tilde{m}$ quasiparticles in the 2D theory by stretching the loop in the z direction until its top and bottom segments meet and annihilate, as shown in Fig. 2(c).

We want to note that there is an ambiguity in what 2D Z_2^g gauge theory we can get from this dimensional reduction process. There are two different Z_2^g gauge theories in 2D, one with bosonic gauge flux ($\tilde{m}$) and the other with semionic gauge flux ($\tilde{m}$) [27]. They are called the toric code and the double semion topological order, respectively [30,36]. Exactly which one we obtain depends on the details of the dimensional reduction process. While both are possible, this distinction is not important in our discussion, as we will see below. We note that a more detailed discussion of the dimensional reduction procedure for 3D Z_2^g gauge theory is given in Appendix A.

Now we consider Z_2^s symmetry again, and imagine inserting an m flux loop in the xy plane before compressing the 3D system down to 2D, as shown in Figs. 3(a) and 3(b). Using the same argument as before, we can show that the resulting 2D system is again a 2D Z_2^g gauge theory. Moreover, the dimensional reduction process can be done without breaking the Z_2^s symmetry of the system. Therefore we obtain a 2D system with Z_2^g gauge theory type topological order and Z_2^s symmetry. Then we can ask what type of symmetry enriched topological order does it have. The interesting possibility is that, the inside and the outside of the m loop can have different SET orders. Therefore a proper description of the symmetry fractionalization pattern on m should contain such differences.

First, we show that the two sides of the m loop must have the same type of 2D Z_2^g gauge theory (toric code or double

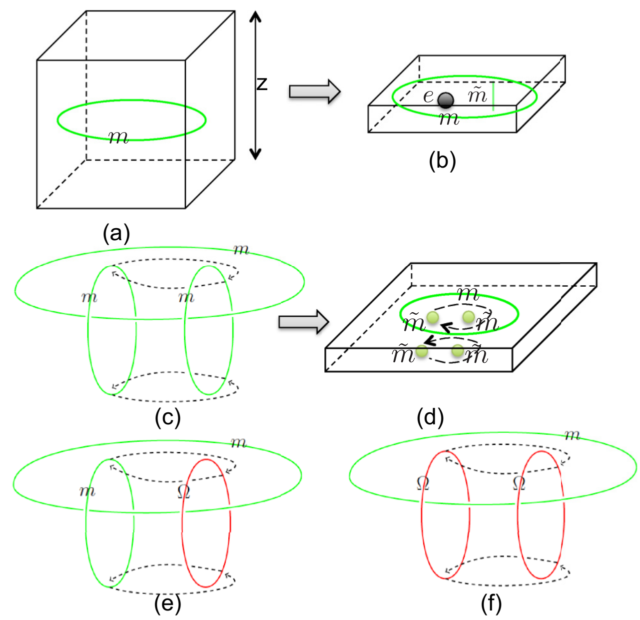

FIG. 3. Dimensional reduction with an m loop in the xy plane: (a) insert an m loop in the xy plane and (b) compress the system down in the z direction. The resulting 2D system has both topological order and symmetry and may carry different SET order inside and outside the m loop. The type of Z_2 gauge theory (toric code or double semion) on the two sides must be the same as (d) the braiding statistics between pairs of $\tilde{m}$ inside and outside of m should cancel which corresponds to (c) the bosonic braiding statistics between two m loops when linked with the m base loop. However, $\tilde{m}$ can carry different symmetry charges which can be detected with the three-loop braiding process shown in (e). Also $\tilde{\Omega}$ may have different topological spin on the two sides of m which can be detected with the three-loop braiding process shown in (f).

semion). That is, $\tilde{m}$ is either both bosonic or both semionic on the two sides. Because two $\tilde{m}$'s fuse to vacuum in the 2D theory, the exchange statistics $\theta_{\tilde{m}}^{[k]}$ must be an integer multiple of $\pi/2$. The superscript $[k]=[I],[m]$ refers to the inside and outside of the base loop, respectively, allowing for the possibility that properties of the 2D theory in these regions may be different. Then, noting that $e\tilde{m}$ is a gauge flux with statistics $\theta_{e\tilde{m}}^{[k]}=\theta_{\tilde{m}}^{[k]}+\pi$, if needed we can redefine $\tilde{m}\to e\tilde{m}$, so that $\theta_{\tilde{m}}^{[I]},\theta_{\tilde{m}}^{[m]}=0,\pi/2$.

In fact, $\theta_{\tilde{m}}^{[I]}=\theta_{\tilde{m}}^{[m]}$. To see this, imagine braiding two pairs of $\tilde{m}$'s, one inside of the m base loop in a clockwise way and one outside of the base loop in a counterclockwise way, as shown in Fig. 3(d). This results in braiding statistics summing to $-\theta_{\tilde{m},\tilde{m}}^{[m]}+\theta_{\tilde{m},\tilde{m}}^{[I]}$. Back to the 3D system before dimensional reduction, such a braiding process corresponds to the braiding of two m loops when they are linked with a m base loop, with braiding statistics $\Phi_{m,m;m}=0$. As shown in Fig. 3(d), the loop on the left expands and passes the right side loop from the outside and then shrinks and goes back to its original position through the inside of the right side loop. Throughout the process, the two m loops are both linked with the m loop. This is the three-loop braiding process described in Refs. [33,34,37]. The $\tilde{m}$ fluxes after dimensional reduction can be thought of as the intersection point of the vertical m

loops with fictitious surfaces spanning the inside and outside of the base loop. Therefore

$$\Phi_{m,m;m} = \theta^{[I]}_{\tilde{m},\tilde{m}} - \theta^{[m]}_{\tilde{m},\tilde{m}} = 0, \tag{2}$$

which, combined with the discussion above, implies $\tilde{m}$ has the same exchange statistics on both sides of the m base loop. Whether $\tilde{m}$ is a boson or a semion depends on the details of the dimensional reduction procedure and is not an intrinsic property of the 3D SET phase.

Next, we can ask how the Z_2^s symmetry enriches the dimensionally reduced 2D Z_2^g gauge theory. To discuss symmetry enrichment, it is helpful to think about symmetry fluxes. For any unitary, internal symmetry G (Z_2^s in this case), we can minimally couple a system to a static classical G gauge field. A symmetry flux is a defect in this gauge field configuration—a point defect in 2D and a line defect in 3D—with the key property that when an excitation moves around the defect, it is acted on by an element of G associated with the defect. For example, in a system with U(1) charge conservation symmetry, a symmetry flux is a static magnetic flux, and the action of U(1) symmetry on excitations moving around the flux is nothing but the Aharonov-Bohm effect. If we make the gauge field dynamical, the static symmetry fluxes become dynamical gauge flux excitations. In this paper, by "symmetry flux" we refer to either the static version or the dynamical version depending on the context.

An interesting feature of symmetry enrichment is that it can be different on the two sides of the base loop m. In particular, $\tilde{m}$ can carry different Z_2^s symmetry charge in the two regions, and also the Z_2^s symmetry flux $\tilde{\Omega}$ can have different exchange statistics once gauged.

Similar to our discussion above, such differences in the dimensionally reduced SET orders are reflected in nontrivial three-loop braiding processes back in the 3D bulk. The difference in the 2D exchange statistics of $\tilde{\Omega}$ corresponds to the three-loop braiding process shown in Fig. 3(e) with m as the base loop and the two Ω_m loops braiding around each other. Because $Q\tilde{\Omega}$ is another Z_2^s symmetry flux with exchange statistics shifted from that of $\tilde{\Omega}$ by π, $\theta^{[k]}_{\tilde{\Omega}}$ is only well-defined modulo π. There are thus two distinct possibilities: the exchange statistics of $\tilde{\Omega}$ on the two sides of m may be the same or differ by $\pi/2$. We label these two cases by $\Omega_m b$ and $\Omega_m s$, respectively, and the related three-loop braiding statistics in 3D is

$$\Phi_{\Omega;m} = \theta^{[I]}_{\tilde{\Omega}} - \theta^{[m]}_{\tilde{\Omega}} = 0 \quad \text{or } \pi/2. \tag{3}$$

Because loops linked with a base loop m can be braided and exchanged like quasiparticles in 2D, we can also describe these two cases in terms of the topological spin of the loop Ω_m. Using this language, $\Omega_m b$ ($\Omega_m s$) corresponds to topological spin 0 or π ($\pm\pi/2$).

Similarly, the difference in the Z_2^s symmetry charge carried by $\tilde{m}$ in the 2D theory is reflected in the braiding process shown in Fig. 3(f). There, Ω_m and m_m loops are linked to a base loop m, and Ω_m is braided around m_m. This is related to the braiding statistics in the 2D theory by

$$\Phi_{\Omega,m;m} = \theta^{[I]}_{\tilde{m},\tilde{\Omega}} - \theta^{[m]}_{\tilde{m},\tilde{\Omega}}. \tag{4}$$

We note that if $\tilde{m}$ carries integer (half-integer) charge, then the statistics angle $\theta_{\tilde{m}} = 0,\pi$ ($\theta_{\tilde{m}} = \pm\pi/2$). Therefore, if $\tilde{m}$ inside the m loop carries half Z_2^s charge while $\tilde{m}$ outside the m loop carries integer Z_2^s charge (or vice versa), the corresponding three-loop statistics is $\Phi_{\Omega,m;m} = \pm\pi/2$. We label this kind of symmetry fractionalization pattern on the m base loop as $m_m C$. Alternatively, $\tilde{m}$ can carry the same charge (integer or half-integer) on both sides of m, a symmetry fractionalization pattern we label by $m_m 0$. We can also view braiding Ω_m around m_m as a way to detect the Z_2^s symmetry charge of the loop m_m. Therefore we say $m_m 0$ ($m_m C$) corresponds to integer (half-integer) charge of the m_m loop.

A careful analysis of all possible 2D SET phases resulting from the dimensional reduction process is given in Appendix B. Now we can see why describing the quasistring as the boundary between two 2D SPT phases, as we did in Sec. II B 2, is inaccurate. First of all, this is an incomplete description because it does not specify, for example, if $\tilde{m}$ carries different fractional Z_2^s charges on the two sides or not. Moreover, sometimes it is not even a well-defined description, because adding a 2D SPT phase on top of a 2D SET phase (with the same symmetry) may not change the SET order at all. One example of this kind is given in Ref. [7] where the e particle in a 2D Z_2^g gauge theory carries half Z_2^s charge. Adding a Z_2^s SPT to the 2D SET results in the same SET. Therefore if this kind of 2D SET order is realized in the dimensionally reduced 3D system, the difference in 2D SPT order on the two sides of the quasistring cannot be unambiguously defined. In fact, with such symmetry fractionalization patterns, gapless modes carried by the quasistring may not be stable, and may be removed via interaction with the fractional excitations in the SET phase.

From the previous analysis, we arrive at the following observation. A description of symmetry fractionalization on quasistrings can be obtained via dimensional reduction, where quasistrings become boundaries between two 2D SET phases. The difference in the two 2D SET orders describes the symmetry action on the quasistring.

We would like to comment that a similar dimensional reduction procedure can be used to study symmetry fractionalization on quasiparticles, but there are important differences between the quasiparticle case and the quasistring case. Imagine creating a pair of quasiparticles and pulling them apart in the x direction. This can be done either in a 2D SET phase, as shown in Fig. 4(a), or in a 3D SET phase [Fig. 4(b)]. We can compress the other dimensions, i.e., the y dimension in the 2D case and the y,z dimensions in the 3D case, and reduce the system down to 1D. As long as the global symmetry is preserved in the reduction process, we obtain a 1D gapped system with symmetry and we can ask what type of symmetric phase it is in. Fractional quasiparticle and quasistring statistics do not survive dimensional reduction to 1D, but 1D gapped systems with symmetry can have symmetry protected topological (SPT) order. In our dimensionally reduced system, the middle part of the system (between the quasiparticles) can have different SPT order than the outer part, and the quasiparticles become boundary states between these two regions, as shown in Fig. 4(c).

Based on this discussion, the symmetry fractionalization on a quasiparticle in a SET phase can be characterized by

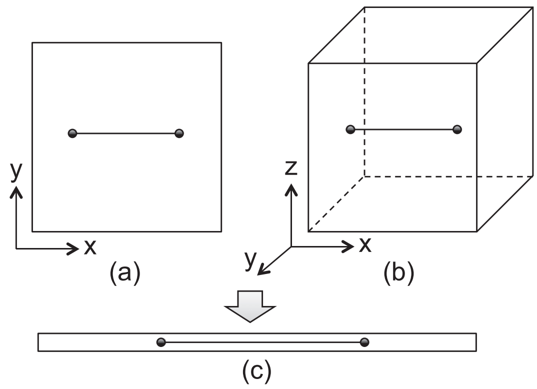

FIG. 4. Dimensional reduction for quasiparticles: create a pair of quasiparticles and pull them apart in the x direction in (a) a 2D system or (b) a 3D system. Compress the system down in y or yz direction and (c) reduce it to a $1D$ chain. The middle part (between the quasiparticles) of the system can have different SPT order than the outer part and the quasiparticle exists as a boundary state between the two.

the projective representation of the symmetry carried by the corresponding boundary state between two 1D SPT orders. However, this characterization is not complete, and some information about symmetry fractionalization is lost upon dimensional reduction to 1D. This raises the possibility that dimensional reduction to 2D may also give an incomplete description of symmetry fractionalization on quasistrings; this point is discussed further in Sec. IV.

A major difference from the quasistring case is that, after the above dimensional reduction for quasiparticles we obtain a 1D bosonic system, which cannot have fractional excitations or long-range entanglement [38,39]. Therefore the dimensionally reduced 1D system can only have SPT order, and the quasiparticle becomes a boundary state between SPT orders. On the other hand, after dimensional reduction for quasistrings, we obtain a long-range entangled 2D system that supports fractional excitations. Therefore the dimensionally reduced 2D system has SET order, and the quasistring exists as a boundary between two 2D SET orders.

C. Z_2^s symmetry fractionalization in 3D Z_2^g gauge theory

Combining the descriptions of symmetry fractionalization on quasiparticles and quasistrings, we can try to list all possible symmetry fractionalization (SF) patterns in 3D SET phases. Naively, one might expect that the total SF pattern can be described by independently choosing the SF pattern for the quasiparticles and the SF pattern for the quasistrings. However, this turns out to be too simplistic. The SF patterns for quasiparticles and quasistrings need to be consistent with each other. Moreover, there may be redundancy in the naive counting, as two seemingly different SF patterns may actually be the same. The general principles are the following.

(1) Consistency. The SF pattern of quasiparticles should be the same both in the 3D bulk and in the dimensionally reduced systems used for the description of the SF pattern on quasistrings.

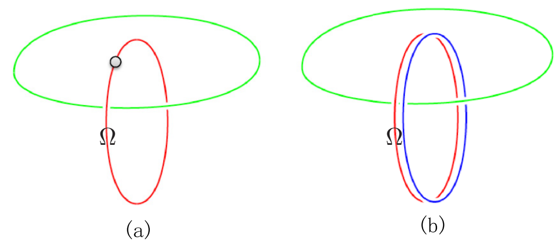

FIG. 5. (a) attaching quasi-particles and (b) attaching quasi-strings to symmetry flux Ω when it is linked with a base loop.

(2) Redundancy. Two SF patterns are the same if they can be related by two kinds of operations: (a) Redefining symmetry fluxes when they are linked with nontrivial base loops by attaching quasiparticles to them, as shown in Fig. 5. Such quasiparticle attachment can be different for different base loops. (b) Redefining symmetry fluxes by attaching quasistrings to them. Such quasistring attachment should be independent of the base loop.

In this section, we explain and apply these principles to describe all SF patterns of the Z_2Z_2 SET example. A more detailed discussion, taking into account the full description of the dimensionally reduced SET orders given in Appendix B, is presented in Appendix C.

As we discussed in Sec. II A, the gauge charge e can carry either integer or half-integer Z_2^s symmetry charge. We label these two cases as $e0$ and eC. On the other hand, the SF pattern on the gauge flux m is described as the difference between two 2D SET phases with Z_2^g topological order and Z_2^s symmetry. Possible types of 2D SET phases with Z_2^g topological order and Z_2^s symmetry have been classified in Ref. [7]. One might want to pick any two possibilities from [7], combine it with either $e0$ or eC and produce a total SF pattern. However, the situation is more complicated.

First, the symmetry charge carried by e should be the same whether in the 3D bulk or in the dimensionally reduced 2D theory. Therefore once we have chosen $e0$ or eC in the 3D bulk, we need to make the same choice in the 2D SET phases used to describe the SF pattern on m. We proceed by discussing the cases of $e0$ and eC in turn.

Consider the $e0$ case, i.e., we suppose that e carries integer Z_2^s charge. Therefore, in the 2D SET phases resulting from dimensional reduction, e carries integer Z_2^s charge both inside and outside of m (Fig. 3). Now, to completely specify the SF pattern on m, we need to consider the difference in the Z_2^s charge carried by $\tilde{m}$ inside and outside of m, and the difference in the topological spin of the Z_2^s flux $\tilde{\Omega}$ inside and outside of m. For the charge difference of $\tilde{m}$, we have the possibilities m_m0 and m_mC, where the charge difference is integer and half-integer, respectively. These cases correspond to statistics angles $\Phi_{\Omega,m;m} = 0,\pi$ (m_m0) and $\Phi_{\Omega,m;m} = \pm\pi/2$ (m_mC) in the three-loop braiding process shown in Fig. 3(c). The difference between 0 and π, and between $+\pi/2$ and $-\pi/2$, corresponds to redefining m_m by binding a symmetry charge Q.

For the difference in topological spin, the possibilities are $\Omega_m b$ and $\Omega_m s$, where the exchange statistics of $\tilde{\Omega}$ modulo π is the same or different (by $\pi/2$) on the two sides of m. These cases correspond to $\Phi_{\Omega,\Omega;m} = 0$ ($\Omega_m b$) and $\Phi_{\Omega,\Omega;m} = \pi$ ($\Omega_m s$).

It seems that there are four possibilities in the $e0$ case, but in fact $e0m_mC\Omega_mb$ is the same as $e0m_mC\Omega_ms$. Starting from one of these cases, if we redefine Ω by attaching a gauge flux m, we change between Ω_mb and Ω_ms. Due to the $\pm\pi/2$ braiding statistics between m_m and Ω_m, such a redefinition changes the braiding statistics between two Ω_m loops, and hence alters the difference in topological spin of $\tilde{\Omega}$ between the two sides of m. This redefinition of Ω does not change the braiding between Ω_m and m_m, so the charge difference m_mC is not affected.

Of course, such a redefinition applies not only to Ω loops linked with a base loop m, but to all Ω loops. We cannot attach quasistrings differently to different Ω loops. This is because any two Ω and $m\Omega$ loops can be distinguished by braiding an e quasiparticle around each loop; the statistical phase acquired differs by π coming from braiding e around m. This holds regardless of whether the Ω and $m\Omega$ loops are linked with any other loops. While it may seem obvious that the redefinition of Ω by attaching quasistrings should be independent of the base loop, we emphasize it here in order to contrast it to the case of attaching quasiparticles discussed below, which can be different for different base loops.

Therefore, in the $e0m_mC$ case, the topological spin difference of $\tilde{\Omega}$ is actually not well defined. This provides a particular example where the difference in Z_2^s SPT order on the two sides of a m base loop is not well defined due to the nontrivial SET order present. On the other hand, this difference is well defined for the $e0m_m0$ case, and is responsible for the distinction between the $e0m_m0\Omega b$ and $e0m_m0\Omega s$ SF patterns. One can check explicitly that redefining Ω by attaching quasiparticles or quasistrings cannot map between $e0m_m0\Omega b$ and $e0m_m0\Omega s$.

Next, we consider the case of eC, where e carries half-integer Z_2^s charge. In this case, e also carries half-integer Z_2^s charge in the dimensionally reduced 2D SET phase, on both sides of the m base loop. To specify the SF pattern on m, we again need to consider the possibilities m_m0 / m_mC and Ω_mb / Ω_ms.

First, we notice that Ω_mb and Ω_ms are always equivalent to each other in this case, because we can redefine Ω_m by attaching an e. This shifts the difference in the topological spin of $\tilde{\Omega}$ by $\pm\pi/2$. It is important to note that this redefinition is done only for those Ω loops linked to an m base loop. In particular, we do not redefine Ω loops that are not linked with any base loop. This is consistent, because Ω loops linked with an m loop cannot be shrunk down to a point, and therefore the quasiparticle type of such loops is not well-defined. One cannot meaningfully compare the quasiparticle type of Ω loops linked with an m base loop with that of the Ω loops linked with other base loops. This point was mentioned in Ref. [33] and discussed in more detail in Ref. [40].

Hence, in the case of eC, we only have the distinction between m_m0 and m_mC. One might try to change between these two possibilities by attaching an e to the m_m loop. However, this redefinition causes $\tilde{m}$ to have different exchange statistics on the two sides of the m base loop. We already showed in Sec. II B 3 that $\tilde{m}$ can be taken to have the same statistics everywhere. Making this choice, the charge difference of $\tilde{m}$ across the m base loop is well-defined, and eCm_m0 and eCm_mC describe different SF patterns. In Table I, we list all five possible SF patterns in the 3D Z_2Z_2 SET.

TABLE I. Symmetry fractionalization patterns for 3D Z_2^g gauge theory with Z_2^s symmetry. $e0$ and eC refer to the gauge charge e carrying integer and half-integer symmetry charge, respectively. m_m0 and m_mC refer to the gauge flux loop m carrying integer or half-integer symmetry charge when it is linked with a base loop of m. They correspond to a phase of 0(or π) and $\pm\pi/2$ in the three-loop braiding process shown in Fig. 3(c). Ω_mb and Ω_ms refer to the symmetry flux loop having bosonic or semionic topological spin when linked with a base loop of m. They correspond to a phase of 0 and π in the three-loop braiding process shown in Fig. 1(c). In the last three cases, Ω_m can be either bosonic, fermionic, or semionic depending on the quasiparticle or quasistring attachment.

SF patterns	Z_2^s charge on e	Z_2^s charge on m_m	topo spin of Ω_m
$e0m_m0\Omega_mb$	integer	integer	0 or π
$e0m_m0\Omega_ms$	integer	integer	$\pm\pi/2$
$e0m_mC$	integer	halfinteger	0, π or $\pm\pi/2$
eCm_m0	halfinteger	integer	0, π or $\pm\pi/2$
eCm_mC	halfinteger	halfinteger	0, π or $\pm\pi/2$

III. ANOMALY DETECTION

Can all the SF patterns listed in Table I be realized? In 2D SET phases, we know that some SF patterns may be anomalous, in the sense that they cannot be realized in strictly 2D systems, but can be realized at the boundary of a 3D SPT phase. Is this the case for any of the Z_2Z_2 SET phases we listed in Table I?

To answer this question, we need an anomaly detection method. Several such methods have been proposed, but they mostly work in 2D, and generalization to 3D is not straightforward. In a recent paper, we introduced the flux fusion idea for anomaly detection in some 2D SET phases [26], which other groups have extended and applied to a wider range of 2D SET phases [41–43]. Here, we generalize this idea to 3D and and apply it to the Z_2Z_2 SET example. In Sec. III A, we review the flux fusion idea and use a 2D anomalous SET phase as an example to illustrate how it works. In Sec. III B, we generalize the method to 3D and show that the eCm_mC Z_2Z_2 SET is anomalous. The eCm_mC SET can still be realized, but only as the surface of a 4D SPT phase, which we show using a coupled layer construction in Sec. III D. In Sec. III C, we show that the other four Z_2Z_2 SET phases are nonanomalous, by showing that the Z_2^s symmetry can be consistently gauged, resulting in 3D gauge theories with larger gauge group.

A. The flux fusion idea: recap

The flux fusion method for detecting anomalous SF patterns works as follows [26]. (1) Use the SF pattern to deduce the fusion rules of symmetry fluxes. (2) Consider how symmetry fractionalizes on symmetry fluxes. (3) Determine if *any* SF pattern on symmetry fluxes is consistent with the SF pattern of the fractional excitations and the fusion rules of the symmetry fluxes. If no consistent SF pattern on symmetry fluxes exists, then an anomaly has been detected.

To illustrate how this works in detail, we consider the example of a 2D Z_2 gauge theory with $U(1) \times Z_2^T$ symmetry, where Z_2^T is antiunitary time reversal symmetry. Note that

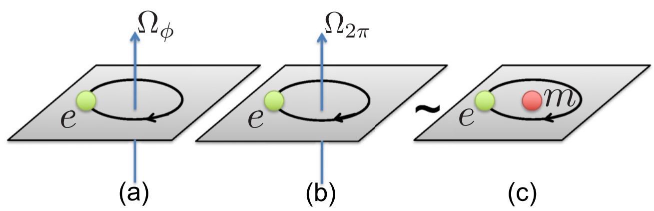

FIG. 6. Relation between flux fusion and fractional excitations in the 2D $eCmT$ theory: (a) braiding an e quasiparticle around a ϕ U(1) flux results in a $\phi/2$ phase and (b) braiding e around a 2π flux results in a phase of π, which is equivalent to (c) braiding e around m.

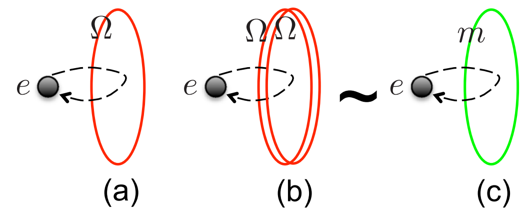

FIG. 7. Relation between flux fusion and fractional excitations in the 3D eCm_mC theory: (a) braiding an e quasiparticle around an Ω Z_2^s flux loop results in a $\pm\pi/2$ phase and (b) braiding e around two Ω flux loops results in a phase of π, which is equivalent to (c) braiding e around an m loop.

the U(1) symmetry commutes with time reversal. Therefore the U(1) charge is reversed under time reversal, and we can think of the U(1) as spin rotation. We consider the SF pattern $eCmT$, which means that the gauge charge e carries half U(1) charge and transforms as $T^2 = 1$ under time reversal, while m carries integer U(1) charge and is a Kramers doublet under time reversal ($T^2 = -1$). We note that m is a quasiparticle here, because this is a 2D example. This SF pattern was argued to be anomalous in Ref. [15]. The same conclusion was obtained via the flux fusion method in Ref. [26]; we follow this discussion here, and more details can be found in Ref. [26].

In the first step, we insert U(1) flux into the system. We consider the configuration shown in Fig. 6(a), where we insert a ϕ flux Ω_ϕ into the system, and bring an e around it. Because e carries half U(1) charge, a statistical phase of $\phi/2$ is accumulated in this braiding process.

Now consider the case of $\phi = 2\pi$. A 2π flux should be equivalent to zero flux; however, bringing an e quasiparticle around it results in a phase of $2\pi/2 = \pi$ instead of 0. Therefore something nontrivial happens with the 2π flux. In fact, a 2π flux in this SET theory is equivalent to an m particle, which explains the π phase when e goes around it. In this way, we complete the first step of the flux fusion method, finding that

$$\Omega_{2\pi} \sim m. \tag{5}$$

Equivalently, we can write it as

$$\Omega_\phi \times \Omega_{2\pi-\phi} \sim m, \tag{6}$$

where $\times$ represents the fusion of symmetry fluxes. This illustrates a general feature of 2D SET phases: the symmetry fluxes Ω_g satisfy a projective fusion rule, with coefficient ω in the set of Abelian anyons of the theory:

$$\Omega_{g_1} \times \Omega_{g_2} = \omega(g_1,g_2)\Omega_{g_1 g_2}. \tag{7}$$

References [10,11,22,44] provide a detailed explanation for this formula and its relation to symmetry fractionalization in 2D. We only want to mention that the projective phase factors $e^{i\alpha_a(g_1,g_2)}$ in Eq. (1) are given by the phase factor resulting from a full braid between a and $\omega(g_1,g_2)$.

In the second step, we ask about the symmetry fractionalization pattern of the symmetry fluxes. In particular, we ask how time reversal symmetry T fractionalizes on the U(1) fluxes Ω_ϕ. We note that T leaves the flux unchanged, i.e., $T : \Omega_\phi \mapsto \Omega_\phi$, so it is well-defined to talk about the action of T^2 on Ω_ϕ. When $\phi = 0$, naturally time reversal cannot fractionalize and acts in the usual way with $T^2 = 1$. Due to the continuity of ϕ, we would expect this to be true for all ϕ. Therefore we find

$$T^2 = 1 \text{ on all U(1) fluxes } \Omega_\phi. \tag{8}$$

In the third step, we realize that this is not consistent with how time reversal fractionalize on the quasiparticles. In particular, time reversal acts as $T^2 = -1$ on m. However, we found that $\Omega_{2\pi} \sim m$ and $T^2 = 1$ on $\Omega_{2\pi}$. Hence we have a contradiction, and the $eCmT$ SF pattern is anomalous.

B. Identify eCm_mC as anomalous 3D Z_2Z_2 SET

Now we follow the same logic and identify the eCm_mC SF pattern listed in Table I as anomalous. As explained in the previous section, in specifying the m_mC SF pattern, we are focusing on the bosonic m_m loop. When an m loop is linked with a base loop of m, its topological spin can be either bosonic or fermionic. These two types of m_m loops can be mapped to each other by attaching an e charge. m_mC is the SF pattern where the bosonic m_m loop carries half Z_2^s symmetry charge.

First, from the fact that e carries half Z_2^s charge we find that two Z_2^s symmetry fluxes (Ω) fuse into an m loop. To see this, we notice that braiding an e around one Ω loop results in a phase of $\pm\pi/2$ and braiding e around two Ω loops results in a phase of π, as shown in Fig. 7. Therefore the fusion result of two Ω loops is not vacuum. Instead, it is the m loop,

$$\Omega \times \Omega \sim m. \tag{9}$$

This is a generalization of the projective fusion rule for symmetry fluxes in 2D as discussed in the previous section.

Next, we ask how Z_2^s symmetry fractionalizes on the Ω loop. In order to answer this question, we insert an Ω loop in the xy plane and compress the system down in the z direction, as shown in Fig. 8. This is similar to the dimensional reduction procedure illustrated in Fig. 3. The only difference is that we are now inserting an Ω loop instead of an m loop. The Ω loop can be inserted while preserving Z_2^s symmetry, so after dimensional reduction, we get a 2D SET state, with possibly different SET orders inside and outside of the Ω loop. The difference in SET orders describes the SF pattern on the Ω loop.

Compared to the case of dimensional reduction with an m loop, dimensional reduction with an Ω loop gives rise to new possibilities. In particular, we should consider the possibility that the topological orders inside and outside the Ω loop are different. The quasiparticles on the two sides of the Ω base loop are given by e, $\tilde{m}$ and their composites under fusion, as

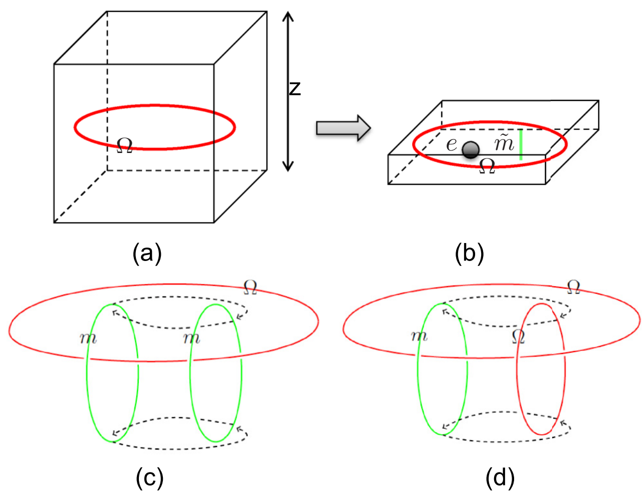

FIG. 8. Dimensional reduction with an Ω loop in the xy plane: (a) insert an Ω loop in the xy plane and (b) compress the system down in the z direction. The $\tilde{m}$ quasiparticle can have different topological spin and different Z_2^s symmetry charge on the inside and outside of Ω corresponding to nontrivial three-loop braiding processes shown in (c) and (d), respectively.

shown in Fig. 8(b). In any region, the topological spin of $\tilde{m}$ can be 0, π or $\pm\pi/2$; this is so because $\tilde{m}^2$ is trivial or a quasiparticle of the 3D theory, and in either case is a boson. The three-loop braiding process shown in Fig. 8(c) measures the difference in topological spin of $\tilde{m}$ between the inside and outside of the Ω base loop, that is,

$$\Phi_{m,m;\Omega} = \theta^{[\Omega]}_{m,m} - \theta^{[I]}_{m,m} = 0,\pi. \qquad (10)$$

We label the two possibilities as $m_\Omega b$ and $m_\Omega s$, respectively.

Moreover, we should also consider the difference in Z_2^s symmetry charge carried by $\tilde{m}$ in the two regions. There are two options: the charges can either be the same (up to an integer charge difference) or differ by a halfinteger charge. The three-loop braiding process shown in Fig. 8(d) results in a phase of 0(or π) and $\pm\pi/2$, respectively, in these two cases,

$$\Phi_{\Omega,m;\Omega} = 0,\pi \quad \text{or} \quad \pm\pi/2. \qquad (11)$$

We label these two possibilities as $m_\Omega 0$ and $m_\Omega C$.

For the third step, we want to see if any of the SF patterns on Ω can be consistent with the SF pattern of m, given the fusion rule $\Omega\times\Omega\sim m$. It turns out that none of the SF patterns are consistent, and this is how we detect an anomaly.

First, we show that $m_\Omega s$ is not possible. Because two Ω loops fuse into an m loop, upon gauging the symmetry we obtain a Z_4 gauge theory in 3D. It is shown in Ref. [33] that $3\times 4\Phi_{\Omega;\Omega} = 0 \bmod 2\pi$, so that $\Phi_{\Omega;\Omega} = 2\pi k/12$ for integer k. Consider two Ω loops, both linked to the same Ω base loop, with the same topological spin (i.e., both characterized by the same value of k). Fusing these two loops together gives a m_Ω loop with exchange statistics $\Phi_{m;\Omega} = 2\pi k/3$. Fusing two such m_Ω loops gives a quasiparticle excitation, with statistics $8\pi k/3$. However, all quasiparticles are bosons, implying that $k = 0 \bmod 3$. Therefore $\Phi_{\Omega;\Omega} = 2\pi k'/4$, for integer k', and the topological spin of m_Ω is always 1. If we take into account of the possibilities of attaching quasiparticles to m_Ω, its topological spin can at most be π. Therefore $m_\Omega s$ is not consistent with $\Omega\times\Omega\sim m$ and only $m_\Omega b$ is possible.

Now we consider the possibilities of $m_\Omega 0$ and $m_\Omega C$. To clarify the meaning of these two possibilities, we need to specify which m_Ω loop we are looking at, because by attaching an e charge we can change the fractional symmetry charge carried by the loop. Here we use the convention that $m_\Omega 0$ and $m_\Omega C$ refers to the fractional symmetry charge carried by the bosonic m_Ω loop. If we attach an e charge to it, the fractional charge carried by the loop changes, and at the same time the topological spin of the loop changes to fermionic, due to the π phase resulting from the braiding of e around m_Ω.

Now we show that neither $m_\Omega 0$ nor $m_\Omega C$ is consistent with the eCm_mC SF pattern on the m loop. We observe that, independent of the Z_2^s charge of m_Ω, m_m always carries integer Z_2^s charge. This follows from the fact that the m base loop can be decomposed into two Ω base loops, and the linearity of three-loop braiding statistics. In particular, we have

$$\Phi_{m,\Omega;m} = \Phi_{m,\Omega;\Omega} + \Phi_{m,\Omega;\Omega} = 2\Phi_{m,\Omega;\Omega}. \qquad (12)$$

Because $\Phi_{m,\Omega;\Omega} = q\pi/2$ for $q = 0,\ldots,3$, this implies $\Phi_{m,\Omega;m} = \pi q$. Moreover, as we have chosen the m loops linked with Ω to be bosonic, their composite m loop linked with $\Omega\times\Omega = m$ is also bosonic. We have thus obtained a contradiction with the m_mC SF pattern, which says that the bosonic m_m loop carries half-integer Z_2^s charge. Therefore none of the possible SF patterns on Ω is consistent with the SF pattern on m and we have found an anomaly.

C. Other Z_2Z_2 SETs are nonanomalous

While the eCm_mC Z_2Z_2 SET phase is anomalous, the other four listed in Table I are not. In fact, the Z_2^s symmetry in these SET phases can be consistently gauged, resulting in a larger topological theory in 3D. We discuss each of these four cases in the following.

First, the $e0m_m0\Omega_m b$ case is the simplest, with only trivial Z_2^s symmetry action on the fractional excitations (e or m). Therefore the Z_2^g gauge theory and Z_2^s symmetry sectors are independent of each other. After gauging, we obtain a $Z_2\times Z_2$ gauge theory, which is just composed of two copies of the usual Z_2 gauge theory.

Next, we consider the eCm_m0 case. In this case, e transforms nontrivially under the Z_2^s symmetry, but m is trivial. Using the argument presented in Sec. III B, we find that two Z_2^s symmetry fluxes Ω fuse into one m loop. Therefore, if the Z_2^s symmetry is gauged, we obtain a Z_4 gauge theory with Ω being the fundamental flux loop. e becomes the fundamental gauge charge. The Z_4 gauge theory has only trivial loop braiding processes.

Now let us consider the case of $e0m_m0\Omega_m s$. As e carries integer charge under Z_2^s, two Ω loops should fuse into vacuum instead of the m loop, therefore, we would expect to get some $Z_2\times Z_2$ gauge theory. However, this is a twisted $Z_2\times Z_2$ gauge theory, due to the nontrivial three-loop braiding process between two Ω loops with base loop m. This braiding process results in a phase of π; that is,

$$\Phi_{\Omega,\Omega;m} = \pi, \qquad (13)$$

TABLE II. Nonanomalous Z_2Z_2 SETs and their corresponding gauge theory when the Z_2^s symmetry is gauged. Nontrivial three-loop braiding statistics in the gauge theory is also listed. GT stands for gauge theory.

Z_2Z_2 SET	Gauging result	Nontrivial three-loop braiding
$e0m_m0\Omega_m b$	untwisted $Z_2 \times Z_2$ GT	none
$e0m_m0\Omega_m s$	twisted $Z_2 \times Z_2$ GT	$\Phi_{\Omega,\Omega;m} = \pi$
$e0m_mC$	twisted $Z_2 \times Z_2$ GT	$\Phi_{m,\Omega;m} = \pm\pi/2$
eCm_m0	Z_4 gauge theory	none

which is equivalent to saying that the topological spin of Ω when linked with m is $\pm\pi/2$. The other three-loop braiding processes with base loop m are trivial. Given this information, the general constraints on three-loop braiding obtained in Ref. [33] imply that braiding of two Ω loops linked with an Ω base loop is trivial, and that

$$\Phi_{\Omega,m;\Omega} = \pm\pi/2. \tag{14}$$

The same three-loop braiding statistics were shown in Refs. [5,33] to be realized in a model for a bosonic SPT phase with $Z_2 \times Z_2$ global symmetry, upon gauging the full symmetry. This implies that we can realize the $e0m_m0\Omega_m s$ SET phase by starting with such a bosonic SPT phase, and gauging one of the Z_2 factors, leaving the remaining factor as the Z_2^s global symmetry. This provides a route to realize this SET phase via parton constructions, by putting bosonic partons into an appropriate SPT phase, and may be helpful in identifying physically reasonable realizing this nontrivial SET phase.

Finally, we consider $e0m_mC$. Similar to the previous case, we expect to get a $Z_2 \times Z_2$ gauge theory, but also with some twisting. In particular, due to the fractional Z_2^s charge carried by m_m when linked with base loop m, there is nontrivial braiding between m and Ω when linked with m:

$$\Phi_{m,\Omega;m} = \pm\pi/2. \tag{15}$$

In addition, we argued in Sec. II C that we can take Ω_m to be bosonic when linked with m, so that

$$\Phi_{\Omega,\Omega;m} = 0. \tag{16}$$

This information, combined with the general constraints of Ref. [33], implies

$$\Phi_{m,m;\Omega} = \pi, \tag{17}$$

and that braiding of two Ω loops linked with a Ω base loop is trivial. Again, the same three-loop braiding was obtained in Refs. [5,33] by gauging a SPT phase with $Z_2 \times Z_2$ symmetry. In Table II, we list all the nonanomalous Z_2Z_2 SET phases, and the corresponding gauge theory when the Z_2^s symmetry is gauged.[1]

[1]Note that there are four $Z_2 \times Z_2$ gauge theories in 3D (including twisted and untwisted), while in Table II, only three of them are listed. In fact, the fourth one can be obtained from the one in the third row by redefining Ω as $m\Omega$.

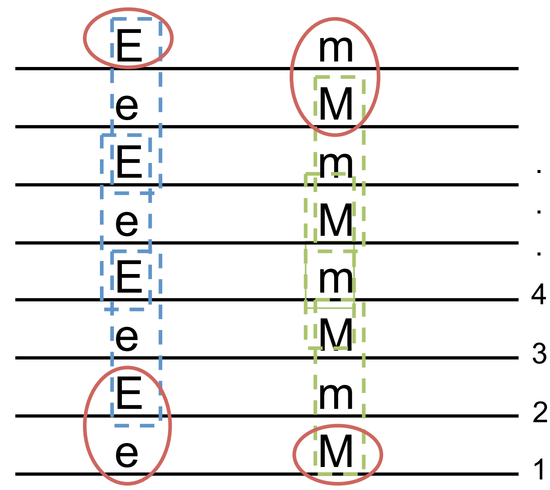

FIG. 9. Coupled layer construction of a 4D system realizing the eCm_mC anomalous SET on its surface. Each layer is a 3D Z_2^g gauge theory. e and E are Z_2^g gauge charges without and with fractional Z_2^s charge, respectively. m and M are Z_2^g gauge fluxes without and with nontrivial Z_2^s symmetry fractionalization. Fractional excitations in the dotted boxes are condensed and those in red circles remain after the condensation as surface anomalous Z_2Z_2 SET.

D. Anomalous Z_2Z_2 SET as surface of 4D Z_2 SPT

The eCm_mC Z_2Z_2 SET is anomalous and cannot be realized in a strictly 3D system. However, it can be realized on the surface of a 4D system. This is similar to anomalous SETs studied in 2D, which can be realized as the surface of a 3D system. In this section, we present a "coupled layer" construction of such a 4D system realizing the eCm_mC SF pattern on its surface, following a similar construction introduced to realized 2D anomalous SETs [15]. Here each "layer" used in the construction is actually a 3D system.

As shown in Fig. 9, each layer is a 3D Z_2^g gauge theory with Z_2^s symmetry. e and E denote Z_2^g gauge charges and m and M denote Z_2^g gauge fluxes. e transforms trivially under Z_2^s while E carries fractional charge of Z_2^s. Also m transforms trivially under Z_2^s while M transforms in the same way as the flux loop in eCm_mC. That is, the Z_2Z_2 SET in the odd-numbered layers (counting from the bottom) has the $e0m_mC$ SF pattern and those in the even-numbered layers have the $eCm_m0\Omega_m b$ SF pattern. Therefore each layer is made up of nonanomalous Z_2Z_2 SETs which can be realized in strictly three dimensions.

Now we condense the composite objects in the dashed boxes as shown in Fig. 9. Note that the composite inside each box is either a bosonic quasiparticle or a quasistring with trivial exchange statistics, whether linked with a base loop or not. Therefore they can be condensed. Moreover, this can be done without breaking the Z_2^s symmetry. This is because each box always contains either two E's or two M's. Therefore, even though each E or M transforms nontrivially under Z_2^s, the composite inside each box always transforms trivially. After such condensation, we can check explicitly that all the fractional excitations in the bulk (both quasiparticles and quasistrings) are either condensed or confined. On the surface, there are nontrivial excitations left behind, which are indicated by red circles in Fig. 9. On the top surface, the remaining fractional excitations are E and the composite mM on neighboring layers. These are bosonic quasiparticle and bosonic quasistring excitations and have mutual π braiding statistics. Therefore the top surface has a 3D Z_2^g topological order. The quasiparticle E carries half Z_2^s charge. Moreover, when a mM loop is linked with another mM loop, it has half Z_2^s charge. Therefore we realize the eCm_mC SF pattern on the top surface. Similar arguments show that the same SF pattern is realized on the bottom surface.

Therefore the eCm_mC Z_2Z_2 SET can be realized as the surface state of a 4D system. The bulk of the 4D system should have a nontrivial Z_2^s symmetry protected topological order to cancel the anomaly coming from the surface.

IV. SUMMARY AND DISCUSSION

In this paper, we studied three dimensional topological phases with symmetry and addressed the following questions. (1) How to describe symmetry fractionalization patterns in 3D? (2) How to detect anomalies in the symmetry fractionalization patterns?

In answering the first question, we found that (1) the SF pattern of a quasiparticle is given by a fractional representation of the symmetry, similar to the 2D case. (2) The SF pattern of a quasistring is given by the difference of two 2D SET orders, one on each side of the quasistring in the dimensionally reduced 2D plane containing the quasistring.

Moreover, in combining the SF patterns of quasiparticles and quasistrings into a full description of the SF pattern in 3D topological phases, we need to satisfy the following conditions.

(1) Consistency. The SF pattern of quasiparticles should be the same both in the 3D bulk and in the dimensionally reduced systems used for the description of the SF pattern on quasistrings.

(2) Redundancy. Two SF patterns are the same if they can be related by two kinds of operations: (a) redefining symmetry fluxes when they are linked with nontrivial base loops by attaching quasiparticles to them. Such quasiparticle attachment can be different for different base loops. (b) Redefining symmetry fluxes by attaching quasistrings to them. Such quasistring attachment should be independent of the base loop.

In particular, we find that for 3D Z_2^g gauge theory with Z_2^s symmetry, there are five different SF patterns, as listed in Table I. In answering the second question, we employ the flux fusion method introduced in Ref. [26]. The steps are summarized as follows: (1) use the SF pattern to deduce the fusion rules of symmetry fluxes; (2) consider how symmetry fractionalizes on symmetry fluxes; and (3) determine if *any* SF pattern on symmetry fluxes is consistent with the SF pattern of the fractional excitations and the fusion rules of the symmetry fluxes. An anomaly in SF pattern is detected if no consistent SF pattern on symmetry fluxes exists.

The flux fusion idea applies only to a limited set of SET orders, but it does work in our case of Z_2^g gauge theory with Z_2^s symmetry, and allows us to identify an anomaly in one of the five SF patterns identified above. While this pattern cannot be realized strictly in 3D, it is shown to be realizable as the surface of a 4D system. We find that the other four SF patterns are nonanomalous, and the Z_2^s symmetry can be consistently gauged.

Many questions are still open regarding 3D symmetry enriched topological phases. First, it is possible that there are SF patterns on quasistrings that cannot be captured by the dimensional reduction procedure. This is the case for SF patterns on quasiparticles. For example, consider a quasiparticle carrying fractional charge under Z_2^s symmetry. After dimensional reduction as discussed in Fig. 4, we get a 1D gapped state with Z_2^s symmetry and the quasiparticle exists as a boundary between two parts of the system. Because there are no nontrivial SPT orders in 1D with Z_2^s symmetry, the whole 1D system is in the same phase, and we do not see any nontrivial features of the quasiparticle. The property of fractional charge carried by the quasiparticle is lost. Of course, in the original bulk of the system, such fractional charge can be detected by bringing the quasiparticle around a symmetry flux. However, such braiding process is intrinsic to the bulk dimension and is not well-defined upon dimensional reduction to 1D. Therefore, if we use dimensional reduction to study 2D SET phases with Z_2^s symmetry, we would conclude that there is only one SF pattern (when quasiparticles are not permuted) while in fact there can be a number of them (as discussed in, e.g., Refs. [6,7]).

In studying the SF pattern on quasistrings, we relied on the dimensional reduction procedure. Therefore, similarly, information could have been lost; that is, there could be nontrivial symmetry actions on the quasistrings that become trivial after dimensional reduction. Such symmetry actions would be related to hypothetical loop braiding processes that are intrinsic to 3D, in that they do not survive dimensional reduction. By contrast, the three-loop braiding process can be described in terms of dimensional reduction to 2D quasiparticle braiding processes on a defect plane. It is not known if there are loop braiding processes in 3D beyond three-loop braiding. Therefore our study of 3D SET phases is limited by our understanding of 3D topological order (loop statistics in particular).

Secondly, it is not clear how to detect anomalies in 3D SF patterns in general. The flux fusion method works well for our example, but the method has limitations. For example, it is not known whether it can be usefully extended to handle general non-Abelian groups, or non-Abelian actions of symmetries on fractional excitations when symmetry fluxes change type or topological sectors under other symmetry operations [26]. In 2D, a more general anomaly detection method is known, based on the mathematical framework of G-crossed fusion categories [44]. Does a similar framework exist in 3D? This is a challenging question, given that we do not yet fully understand what topological orders exist in 3D. However, it might be possible to partially answer this question by restricting attention to 3D SET phases with topological orders we already understand, such as untwisted and twisted gauge theories. We leave these questions for future study.

As we were finalizing this manuscript for posting on the arXiv, we noticed a recent preprint [45] in which some of the SET phases discussed here were also studied.

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APPENDIX A: DIMENSIONAL REDUCTION OF 3D Z_2^g GAUGE THEORY TO 2D

In this Appendix, we examine in more detail the dimension reduction procedure of 3D Z_2^g gauge theory to 2D, justifying the conclusions made in the main text (Sec. II B 3). We consider a bosonic system in 3D, with Z_2 topological order (as in the deconfined phase of Z_2^g gauge theory). The topologically nontrivial excitations are the point particle e, which we assume to be a boson, and the quasistring m. Here, we do not consider any symmetry. In Appendix B, we give a similar discussion including Z_2^s symmetry and examining in detail the SET orders upon dimensional reduction to 2D.

We start by discussing fusion and braiding properties in 3D. Besides the three-loop braiding parameters $\Phi_{m,m;m}$ and $\Phi_{m;m}$ discussed in the main text, we introduce the following statistical parameters:

$$\phi_e = 0 \text{self-statistics of } e,$$

$$\phi_{e,m} = \pi \text{point-loop mutual statistics of } e \text{ and } m. \tag{A1}$$

Here, ϕ_e is the usual statistical angle for the exchange statistics of two e particles. $\phi_{e,m}$ corresponds to a process where e is brought around a m loop and returned to its original position.

The three-loop braiding parameters are given by

$$\Phi_{m;m} = 0 \text{ or } \pi,$$

$$\Phi_{m,m;m} = 0, \tag{A2}$$

where $\Phi_{m;m}$ denotes the exchange statistics of two m loops linked with a third m base loop and $\Phi_{m,m;m}$ denotes the full braiding between two identical m loops linked with a third base m loop. Note that we can redefine the two linked loops by attaching an e particle to each, $m \to em$, which shifts $\Phi_{m;m} \to \Phi_{m;m} + \pi$. Therefore $\Phi_{m;m}$ is only well-defined modulo π. In addition, $\Phi_{m,m;m} = 2\Phi_{m;m} = 0$, so that $\Phi_{m;m} = 0,\pi$, so we gain no additional information by considering the half-braid three-loop process.

In addition to these braiding processes, we have the fusion rules $e^2 = 1$, and $m^2 = 1$. The meaning of the $e^2 = 1$ fusion rule is familiar. The $m^2 = 1$ fusion rule means that if we fuse two m quasistrings together, we get something that is trivial as a quasistring. However, there could be nontrivial e charge upon fusing two m loops.

Before proceeding, as a brief aside, we would like to establish some consistent notation for different types of braiding processes in 3D and 2D, since below and in the following appendices we will have to consider many different such processes. For the purposes of the present discussion, we label point particles by $a,b,c,\ldots$, and quasistrings by $\alpha,\beta,\gamma,\ldots$. We introduce the following symbols:

ϕ_a: self-statistics of a in 3D,

θ_a: self-statistics of a in 2D,

$\phi_{a,\alpha}$: point-loop statistics of a with α in 3D,

$\theta_{a,b}$: mutual statistics of a and b in 2D,

$\Phi_{\alpha;\beta}$: three-loop braiding (half braid) of two α loops linked with base loop β

$\Phi_{\alpha,\beta;\gamma}$: three-loop braiding (full braid) of α with β, linked with base loop γ. (A3)

We use θ for 2D statistics, and ϕ and Φ for 3D statistics. In addition, we use lowercase letters for particle-particle or particle-loop statistics, and capital letters for three-loop braiding statistics.

Now we describe dimensional reduction to a 2D topologically ordered phase. We always consider periodic boundary conditions in the finite direction. We can reduce either onto vacuum or onto the plane of a m loop, to get two potentially different 2D phases. When we dimensionally reduce onto the plane of m (or of some other quasistring), we refer to this as the *basal plane*. We label the 2D phases by the quasistring type of the basal plane (this is I when we dimensionally reduce onto vacuum). The dependence of 2D statistics on the basal plane is indicated by a superscript; for example, $\theta_{\tilde{m}}^{[m]}$ is the self-statistics of a quasiparticle $\tilde{m}$ on the m basal plane.

The dimensionally reduced version of e is again a point particle, which we still refer to as e. As discussed in Sec. II B 3, dimensional reduction of m quasistrings is more interesting, because these can stretch across the system in the finite direction, effectively becoming point particles in 2D. These point particles are referred to as $\tilde{m}$.

The following fusion and braiding properties hold independent of basal plane:

$$e^2 = 1, \tag{A4}$$

$$\theta_e = 0, \tag{A5}$$

$$\theta_{e,\tilde{m}} = \pi. \tag{A6}$$

In principle, we could have either $\tilde{m}^2 = 1$ or $\tilde{m}^2 = e$. Two $\tilde{m}$'s must fuse to something that is trivial as a quasistring, but we cannot immediately rule out that they could fuse to e, the only nontrivial 2D particle which is trivial as a quasistring. Note that in either case, $\tilde{m}^2$ is a boson, so

$$4\theta_{\tilde{m}} = \theta_{\tilde{m}^2} = 0, \tag{A7}$$

which implies

$$\theta_{\tilde{m}} = \frac{\pi}{2}q, \quad q = 0,\ldots,3. \tag{A8}$$

Suppose $\tilde{m}^2 = e$, then we have

$$\pi = \theta_{\tilde{m},\tilde{m}^2} = 2\theta_{\tilde{m},\tilde{m}} = 4\theta_{\tilde{m}} = 0, \tag{A9}$$

a contradiction. So independent of basal plane, we must have

$$\tilde{m}^2 = 1. \tag{A10}$$

The self-statistics of $\tilde{m}$ may depend not only on the basal plane, but also on the details of the dimensional reduction procedure. The intuition for this is that the phase obtained upon exchanging two $\tilde{m}$'s can quite naturally depend on the "height" of these objects, that is on the system size in the finite direction. However, we have the following relationship with three-loop braiding in 3D:

$$\theta^{[m]}_{\tilde{m},\tilde{m}} - \theta^{[1]}_{\tilde{m},\tilde{m}} = \Phi_{m,m;m} = 0. \tag{A11}$$

This implies

$$\theta^{[m]}_{\tilde{m}} - \theta^{[1]}_{\tilde{m}} = 0,\pi. \tag{A12}$$

Next, note that $\tilde{m}$ can be redefined by $\tilde{m} \to e\tilde{m}$, and this can be done differently for different basal planes. This redefinition shifts $\theta_{\tilde{m}} \to \theta_{\tilde{m}} + \pi$, so we can always choose

$$\theta_{\tilde{m}} \equiv \theta^{[m]}_{\tilde{m}} = \theta^{[1]}_{\tilde{m}}. \tag{A13}$$

In addition, $\theta_{\tilde{m}}$ itself can be shifted by π, and up to such shifts the most general choice consistent with $\tilde{m}^2 = 1$ is

$$\theta_{\tilde{m}} = 0,\pi/2. \tag{A14}$$

These choices are distinct, with the former corresponding to the toric code, and the latter to the double semion theory.

To understand the different possible outcomes of the dimensional reduction of the 3D Z_2 gauge theory, it helps to think about the reverse process of coupling layers of 2D Z_2 gauge theories. Suppose that we take two layers of 2D Z_2 gauge theories (both toric code or both double semion) and couple them by condensing pairs of gauge charges. After such condensation, the gauge flux in each layer is confined but the pairs of gauge fluxes in both layers remain deconfined and become the new gauge flux of the resulting Z_2 gauge theory. The resulting Z_2 gauge theory is of the toric code type because the gauge flux is either bosonic or fermionic. We can continue this layer coupling process until we reach the 3D limit where the topological order becomes that of the 3D toric code. Now imagine the reverse process of dimensional reduction. Exactly what 2D topological order is resulted depends on the details of the dimensional reduction process including, for example, how many 2D layers remains (even or odd) and whether each layer is of toric code or double semion type. If each layer is a double semion and an odd number of layers remain, we obtain a double semion type topological order. Otherwise, we obtain a toric code type topological order. Without specifying these details, both results are possible.

APPENDIX B: DIMENSIONAL REDUCTION OF 3D Z_2^g GAUGE THEORY WITH Z_2^s GLOBAL SYMMETRY TO 2D

Following the approach of Appendix A, we now add Z_2^s on-site unitary symmetry, to consider Z_2Z_2 SET phases in 3D. We give a detailed description of the possible SET orders upon reduction to 2D. We proceed by gauging Z_2^s, which introduces two new objects into our description. These are the symmetry charge Q, which is a point particle, and the symmetry flux Ω, which is a quasistring.

We first study fusion rules, point-point, and point-loop statistics in the gauged theory. The fusion rules are

$$Q^2 = 1, \tag{B1}$$

$$m^2 = 1, \tag{B2}$$

$$e^2 = 1 \quad \text{or } Q, \tag{B3}$$

$$\Omega^2 = 1 \quad \text{or } m. \tag{B4}$$

The first two fusion rules are fixed, while the last two depend on the properties of the Z_2Z_2 SET phase. The case $e^2 = 1$ corresponds to “integer” Z_2^s charge of e, which we denote by $e0$. Similarly, the case $e^2 = Q$ corresponds to “half-integer” Z_2^s charge of e, and this is denoted eC.

Turning to statistics, the point particles have exchange statistics

$$\phi_e = \phi_Q = 0. \tag{B5}$$

$\phi_e = 0$ by assumption, and $\phi_Q = 0$ because Q is a trivial, local excitation before gauging, so it must have bosonic self-statistics after gauging.

For point-loop statistics, we have

$$\phi_{e,m} = \pi, \tag{B6}$$
$$\phi_{Q,m} = 0, \tag{B7}$$
$$\phi_{Q,\Omega} = \pi. \tag{B8}$$

$\phi_{Q,m} = 0$ because any other choice would contradict Q being a trivial excitation before gauging. Moreover, $\phi_{Q,\Omega} = \pi$ is a defining property of the flux line Ω. There is also the point-loop statistical angle $\phi_{e,\Omega}$, which we now relate to the fusion rules.

First, we consider the $e0$ case, that is assume $e^2 = 1$. Then

$$0 = \phi_{e^2,\Omega} = 2\phi_{e,\Omega}, \tag{B9}$$

which implies

$$\phi_{e,\Omega} = 0,\pi. \tag{B10}$$

If $\phi_{e,\Omega} = \pi$, we can redefine $e \to Qe$, which sets $\phi_{e,\Omega} = 0$ while leaving the other statistics angles and fusion rules unchanged. Then we have

$$\phi_{e,\Omega^2} = 2\phi_{e,\Omega} = 0, \tag{B11}$$

which fixes the $\Omega^2 = 1$ fusion rule.

Next, we consider eC, that is $e^2 = Q$. Then

$$\pi = \phi_{e^2,\Omega} = 2\phi_{e,\Omega}, \tag{B12}$$

which implies

$$\phi_{e,\Omega} = \pi/2, 3\pi/2. \tag{B13}$$

Again, by redefining $e \to Qe$ if needed, we can choose $\phi_{e,\Omega} = \pi/2$. Then we have

$$\phi_{e,\Omega^2} = 2\phi_{e,\Omega} = \pi, \tag{B14}$$

which fixes the $\Omega^2 = m$ fusion rule.

Therefore, choosing $e0$ or eC completely fixes the fusion rules, point-point, and point-loop statistics. To summarize the two cases, for $e0$ we have

$$\begin{aligned} Q^2 &= 1 \quad \phi_e = \phi_Q = 0,\\ e^2 &= 1 \quad \phi_{e,m} = \phi_{Q,\Omega} = \pi,\\ \Omega^2 &= 1 \quad \phi_{Q,m} = 0,\\ m^2 &= 1 \quad \phi_{e,\Omega} = 0; \end{aligned} \tag{B15}$$

while for eC:

$$\begin{aligned} Q^2 &= 1 \quad \phi_e = \phi_Q = 0,\\ e^2 &= Q \quad \phi_{e,m} = \phi_{Q,\Omega} = \pi,\\ \Omega^2 &= m \quad \phi_{Q,m} = 0,\\ m^2 &= 1 \quad \phi_{e,\Omega} = \pi/2. \end{aligned} \tag{B16}$$

So far, we have not said anything about the action of symmetry on m, or about three-loop braiding in the gauged theory. Our goal is to describe the symmetry action on m, and we will do this by dimensional reduction to a 2D SET phase with Z_2^s symmetry. Those properties not having to do with symmetry have already been discussed in Appendix A.

To describe the symmetry action on m, we will need to consider dimensional reduction onto the vacuum, and onto the m basal plane. Let us first describe the general properties of the dimensionally reduced theory for some fixed basal plane (we thus do not worry about basal plane labels for the moment). The particles of the theory are $e,Q,\tilde{m},\tilde{\Omega}$. The properties of the latter two particles can depend on the choice of basal plane, so if we want to include basal plane labels, we should write $\tilde{\Omega}_I$, $\tilde{m}_m$, and so on.

Let us describe what we know about the fusion rules and statistics of the dimensionally reduced theory, first for the $e0$ case. The fusion rules are

$$e^2 = 1, \tag{B17}$$

$$Q^2 = 1, \tag{B18}$$

$$\tilde{m}^2 = 1,Q, \tag{B19}$$

$$\tilde{\Omega}^2 = 1,e,Q,eQ. \tag{B20}$$

The last two fusion rules are partially undetermined so far. We know that in 3D, $m^2 = 1$ and $\Omega^2 = 1$, which means, for example, that two $\tilde{m}$'s cannot fuse to the dimensional reduction of a nontrivial quasistring. We also already showed in Appendix A that that two $\tilde{m}$'s cannot fuse to e (or to eQ in the gauged theory). The statistics are

$$\theta_e = \theta_Q = \theta_{e,Q} = \theta_{Q,\tilde{m}} = 0, \tag{B21}$$

$$\theta_{e,\tilde{m}} = \theta_{Q,\tilde{\Omega}} = \pi, \tag{B22}$$

$$\theta_{e,\tilde{\Omega}} = 0, \tag{B23}$$

$$\theta_{\tilde{m}} = 0,\pi/2 = ? \tag{B24}$$

$$\theta_{\tilde{\Omega}} = ? \tag{B25}$$

$$\theta_{\tilde{m},\tilde{\Omega}} = ?. \tag{B26}$$

We recall from Appendix A that $\theta_{\tilde{m}} = \theta_{\tilde{m}_I} = \theta_{\tilde{m}_m} = 0,\pi/2$, while the other unknown parameters may depend on basal plane.

In the eC case, we have for the fusion rules:

$$e^2 = Q, \tag{B27}$$

$$Q^2 = 1, \tag{B28}$$

$$\tilde{m}^2 = 1,Q, \tag{B29}$$

$$\tilde{\Omega}^2 = \tilde{m},e\tilde{m},Q\tilde{m},eQ\tilde{m}. \tag{B30}$$

Again, the last two fusion rules are partially undetermined for now, but now have a different structure because $\Omega^2 = m$ in 3D. The statistics are

$$\theta_e = \theta_Q = \theta_{e,Q} = \theta_{Q,\tilde{m}} = 0 \tag{B31}$$

$$\theta_{e,\tilde{m}} = \theta_{Q,\tilde{\Omega}} = \pi \tag{B32}$$

$$\theta_{e,\tilde{\Omega}} = \pi/2 \tag{B33}$$

$$\theta_{\tilde{m}} = 0, \quad \pi/2 = ? \tag{B34}$$

$$\theta_{\tilde{\Omega}} = ? \tag{B35}$$

$$\theta_{\tilde{m},\tilde{\Omega}} = ?. \tag{B36}$$

Again, $\theta_{\tilde{m}}$ is the same for all basal planes.

We see that the properties of the dimensionally reduced theory can be specified by four pieces of information. (There are also fusion rules to specify, but these are determined by the braiding statistics.) First, we can either have $e0$ or eC. Then the three remaining pieces of information are $\theta_{\tilde{m}}$, $\theta_{\tilde{\Omega}}$ and $\theta_{\tilde{m},\tilde{\Omega}}$. We will see that the last of these can always be chosen as either 0 or $\pi/2$, which correspond respectively to integer and half-integer Z_2^s charge of $\tilde{m}$. Following the same notation for e, we denote these two possibilities using 0 and C, respectively. We organize the four pieces of information into the 4-tuple $(e0/C,\tilde{m}0/C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$. For example, we would write $(C,0,\pi/2,-\pi/8)$ to describe a dimensionally reduced theory where e has half-charge, $\tilde{m}$ has integer charge and statistics $\theta_{\tilde{m}} = \pi/2$, and $\theta_{\tilde{\Omega}} = -\pi/8$.

A very important issue is which redefinitions of the various particles are allowed. The symmetry charge Q is fixed and cannot be redefined. The gauge charge e is also fixed by our conventional choice of $\Phi_{e,\Omega} = \theta_{e,\tilde{\Omega}}$. The following redefinitions are allowed:

$$\tilde{\Omega} \to e\tilde{\Omega} \tag{B37}$$

$$\tilde{\Omega} \to Q\tilde{\Omega} \tag{B38}$$

$$\tilde{m} \to Q\tilde{m}. \tag{B39}$$

These redefinitions do not affect the fusion rules and statistics angles that we have already fully determined. Looking at the above redefinitions from the point of view of the 3D theory, they all involve binding particles to quasistrings, and therefore can be done differently for different basal planes. We thus refer to these as *local* redefinitions. The redefinition $\tilde{m} \to e\tilde{m}$ is *not* allowed, because we have already made the conventional choice $\theta_{\tilde{m}} = \theta_{\tilde{m}_I} = \theta_{\tilde{m}_m} = 0,\pi/2$.

The following redefinition is also allowed:

$$\tilde{\Omega} \to \tilde{m}\tilde{\Omega} \quad \text{and} \quad e \to Qe. \tag{B40}$$

However, this redefinition has to be made the same way for all basal planes, because it involves binding together two quasistrings; that is, it actually is associated with a redefinition $\Omega \to m\Omega$ in the 3D theory. Therefore we refer to it as a *global* redefinition. At the same time we redefine $\tilde{\Omega}$, we also have to send $e \to Qe$, in order to keep $\theta_{e,\tilde{\Omega}}$ fixed. This redefinition also leaves invariant all the fusion rules and statistics angles that have already been fully determined.

To proceed, we now classify the different possible dimensionally reduced theories. We do this only up to local redefinitions, taking the global redefinition Eq. (B40) into account later. We first consider the $e0$ case, then move on to eC.

1. Dimensionally reduced SET phases with *e*0

Recall that for $e0$, we have the following fusion rules:

$$e^2 = 1, \tag{B41}$$

$$Q^2 = 1, \tag{B42}$$

$$\tilde{m}^2 = 1, Q, \tag{B43}$$

$$\tilde{\Omega}^2 = 1, e, Q, eQ. \tag{B44}$$

The statistics are given by

$$\theta_e = \theta_Q = \theta_{e,Q} = \theta_{Q,\tilde{m}} = 0, \tag{B45}$$

$$\theta_{e,\tilde{m}} = \theta_{Q,\tilde{\Omega}} = \pi, \tag{B46}$$

$$\theta_{e,\tilde{\Omega}} = 0, \tag{B47}$$

$$\theta_{\tilde{m}} = 0, \quad \pi/2 = ? \tag{B48}$$

$$\theta_{\tilde{\Omega}} = ? \tag{B49}$$

$$\theta_{\tilde{m},\tilde{\Omega}} = ?. \tag{B50}$$

$\tilde{m}^2 = 1$ corresponds to $\tilde{m}0$, and $\tilde{m}^2 = Q$ corresponds to $\tilde{m}C$. If $\tilde{m}^2 = 1$, we have

$$0 = \theta_{\tilde{m}^2,\tilde{\Omega}} = 2\theta_{\tilde{m},\tilde{\Omega}}, \tag{B51}$$

which implies $\theta_{\tilde{m},\tilde{\Omega}} = 0,\pi$. We can then redefine $\tilde{m} \to Q\tilde{m}$ as needed to set $\theta_{\tilde{m},\tilde{\Omega}} = 0$. Similarly, if $\tilde{m}^2 = Q$, we have

$$\pi = \theta_{\tilde{m}^2,\tilde{\Omega}} = 2\theta_{\tilde{m},\tilde{\Omega}}, \tag{B52}$$

implying $\theta_{\tilde{m},\tilde{\Omega}} = \pi/2, 3\pi/2$, and we can redefine $\tilde{m} \to Q\tilde{m}$ as needed to set $\theta_{\tilde{m},\tilde{\Omega}} = \pi/2$. We have thus fixed $\theta_{\tilde{m},\tilde{\Omega}}$ (depending on the $\tilde{m}^2$ fusion rule), and in doing so we have used up our freedom to redefine $\tilde{m}$. Now, we consider the remaining undetermined information, treating the $\tilde{m}0$ and $\tilde{m}C$ cases in turn.

a. Dimensionally reduced theories with *e*0$\tilde{m}$0

First, we take the $\tilde{m}0$ case, considering 2D theories with data $(0,0,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$. We have

$$\theta_{\tilde{\Omega}^2,\tilde{m}} = 2\theta_{\tilde{m},\tilde{\Omega}} = 0, \tag{B53}$$

which implies that only $\tilde{\Omega}^2 = 1$ or $\tilde{\Omega}^2 = Q$ are consistent fusion rules. Also, since $\tilde{\Omega}^2$ is a boson, we have $\theta_{\tilde{\Omega}} = \pi q/2$ for $q = 0,\ldots,3$. Therefore $\theta_{\tilde{\Omega},\tilde{\Omega}} = 2\theta_{\tilde{\Omega}} = \pi q$, so $\theta_{\tilde{\Omega}^2,\tilde{\Omega}} = 0$. However, this means only

$$\tilde{\Omega}^2 = 1 \tag{B54}$$

is a consistent fusion rule. The fusion rules are now completely fixed in this case.

We can now redefine $\tilde{\Omega} \to Q\tilde{\Omega}$ as needed to shift $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi$, which does not affect any of the fusion rules or other statistics angles. This allows us to choose

$$\theta_{\tilde{\Omega}} = 0, \pi/2. \tag{B55}$$

The only local redefinition left is $\tilde{\Omega} \to e\tilde{\Omega}$. If we do this alone it modifies $\theta_{\tilde{m},\tilde{\Omega}}$, so at the same time we can redefine $\tilde{m} \to Q\tilde{m}$. The resulting redefinition does not affect any of the fusion rules or statistics angles. Therefore we have found four possibilities, labeled by $(0,0,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$, where $\theta_{\tilde{m}},\theta_{\tilde{\Omega}} = 0,\pi/2$.

b. Dimensionally reduced theories with *e*0$\tilde{m}C$

Next, we take the $\tilde{m}C$ case, considering 2D theories with data $(0,C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$. We have

$$\theta_{\tilde{\Omega}^2,\tilde{m}} = 2\theta_{\tilde{m},\tilde{\Omega}} = \pi, \tag{B56}$$

which implies that only $\tilde{\Omega}^2 = e$ or $\tilde{\Omega}^2 = eQ$ are consistent fusion rules.

Again $\tilde{\Omega}^2$ is a boson, so $\theta_{\tilde{\Omega}} = \pi q/2$ for $q = 0,\ldots,3$, and $\theta_{\tilde{\Omega}^2,\tilde{\Omega}} = 0$. Therefore we have

$$\tilde{\Omega}^2 = e, \tag{B57}$$

and the fusion rules are fixed.

Again, we redefine $\tilde{\Omega} \to Q\tilde{\Omega}$ as needed to shift $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi$, which does not affect any of the fusion rules or other statistics angles. This allows us to choose

$$\theta_{\tilde{\Omega}} = 0, \pi/2. \tag{B58}$$

Again, the only local redefinition left is $\tilde{\Omega} \to e\tilde{\Omega}$. If we do this alone it modifies $\theta_{\tilde{m},\tilde{\Omega}}$, so at the same time we can redefine $\tilde{m} \to Q\tilde{m}$. The resulting redefinition does not affect any of the fusion rules or statistics angles. We have thus found four dimensionally reduced theories, labeled by $(0,C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$, where $\theta_{\tilde{m}},\theta_{\tilde{\Omega}} = 0,\pi/2$.

c. Behavior under global redefinitions

In total, then, there are eight distinct dimensionally reduced theories with $e0$, with data $(0,0/C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$, where $\theta_{\tilde{m}},\theta_{\tilde{\Omega}} = 0,\pi/2$. These theories are distinct under local redefinitions. We now would like to consider how they map into one another under the global redefinition Eq. (B40). $\tilde{\Omega} \to \tilde{\Omega}' = \tilde{m}\tilde{\Omega}$, $e \to e' = Qe$, which must be made in the same way for all basal planes.

For the $e0\tilde{m}0$ theories, we find under the global redefinition that $\theta'_{\tilde{\Omega}} = \theta_{\tilde{\Omega}} + \theta_{\tilde{m}}$, while the other statistics angles, and all the fusion rules, are invariant. We can thus write

$$(0,0,\theta_{\tilde{m}},\theta_{\tilde{\Omega}}) \to (0,0,\theta_{\tilde{m}},\theta_{\tilde{\Omega}} + \theta_{\tilde{m}}). \tag{B59}$$

Note that $\theta_{\tilde{\Omega}} + \theta_{\tilde{m}}$ can be chosen to be $0,\pi/2$, as it can be shifted by π if needed by making another redefinition $\tilde{\Omega} \to Q\tilde{\Omega}$. So the two $e0\tilde{m}0$ theories with $\theta_{\tilde{m}} = 0$ are invariant under the global redefinition, while the other two, with $\theta_{\tilde{m}} = \pi/2$, are exchanged under the redefinition.

For the $e0\tilde{m}C$ theories, we find under the global redefinition that $\theta'_{\tilde{\Omega}} = \theta_{\tilde{\Omega}} + \theta_{\tilde{m}} + \pi/2$, while the other statistics angles, and all the fusion rules, are invariant. We can thus write

$$(0,C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}}) \to (0,C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}} + \theta_{\tilde{m}} + \pi/2). \tag{B60}$$

Similar to the case above, the two $e0\tilde{m}C$ theories with $\theta_{\tilde{m}} = \pi/2$ are left invariant, while the two theories with $\theta_{\tilde{m}} = 0$ are exchanged.

2. Dimensionally reduced SET phases with *eC*

Recall that in the eC case the fusion rules are

$$e^2 = Q, \tag{B61}$$

$$Q^2 = 1, \tag{B62}$$

$$\tilde{m}^2 = 1, Q, \tag{B63}$$

$$\tilde{\Omega}^2 = \tilde{m}, e\tilde{m}, Q\tilde{m}, eQ\tilde{m}. \tag{B64}$$

The statistics are

$$\theta_e = \theta_Q = \theta_{e,Q} = \theta_{Q,\tilde{m}} = 0, \tag{B65}$$

$$\theta_{e,\tilde{m}} = \theta_{Q,\tilde{\Omega}} = \pi, \tag{B66}$$

$$\theta_{e,\tilde{\Omega}} = \pi/2, \tag{B67}$$

$$\theta_{\tilde{m}} = 0, \quad \pi/2 = ? \tag{B68}$$

$$\theta_{\tilde{\Omega}} = ? \tag{B69}$$

$$\theta_{\tilde{m},\tilde{\Omega}} = ?, \tag{B70}$$

with $\theta_{\tilde{m}}$ the same for all basal planes. As before in the $e0$ case, we can use the $\tilde{m} \to Q\tilde{m}$ redefinition to set $\theta_{\tilde{m},\tilde{\Omega}} = 0$ when $\tilde{m}^2 = 1$, and $\theta_{\tilde{m},\tilde{\Omega}} = \pi/2$ when $\tilde{m}^2 = Q$.

a. Dimensionally reduced theories with eC$\tilde{m}$0

First, we consider $\tilde{m}^2 = 1$, or $\tilde{m}0$. In this case, we have

$$\theta_{\tilde{\Omega}^2,\tilde{m}} = 2\theta_{\tilde{\Omega},\tilde{m}} = 0. \tag{B71}$$

If $\theta_{\tilde{m}} = 0$, this implies $\tilde{\Omega}^2 = \tilde{m}, Q\tilde{m}$. On the other hand, if $\theta_{\tilde{m}} = \pi/2$, this implies $\tilde{\Omega}^2 = e\tilde{m}, eQ\tilde{m}$.

First, we consider the case $\theta_{\tilde{m}} = 0$. Then, because $\tilde{\Omega}^2 = \tilde{m}, Q\tilde{m}$, $\tilde{\Omega}^2$ is a boson, so $\theta_{\tilde{\Omega}} = \pi q/2$ for $q = 0,\ldots,3$. Then

$$\theta_{\tilde{\Omega}^2,\tilde{\Omega}} = 2\theta_{\tilde{\Omega},\tilde{\Omega}} = 4\theta_{\tilde{\Omega}} = 0, \tag{B72}$$

and we must have

$$\tilde{\Omega}^2 = \tilde{m}. \tag{B73}$$

Now, we consider the redefinition $\tilde{\Omega} \to \tilde{\Omega}' = e\tilde{\Omega}$, combined with $\tilde{m} \to \tilde{m}' = Q\tilde{m}$. This redefinition shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi/2$, while preserving all other statistics angles and all the fusion rules. Therefore we can use this to set $\theta_{\tilde{\Omega}} = 0$. The remaining local redefinition, $\tilde{\Omega} \to Q\tilde{\Omega}$, only shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi$, and is thus superfluous. We have thus found a 2D theory with data $(C,0,0,0)$.

Second, we need to consider the case $\theta_{\tilde{m}} = \pi/2$. Then, because $\tilde{\Omega}^2 = e\tilde{m}, eQ\tilde{m}$, we have

$$\theta_{\tilde{\Omega}^2} = \frac{3\pi}{2} = -\frac{\pi}{2}. \tag{B74}$$

Therefore we have for the statistics of $\tilde{\Omega}$,

$$\theta_{\tilde{\Omega}} = \frac{\pi}{2}q - \frac{\pi}{8}, \quad q = 0,\ldots,3. \tag{B75}$$

Then we have

$$\theta_{\tilde{\Omega}^2,\tilde{\Omega}} = 2\theta_{\tilde{\Omega},\tilde{\Omega}} = 4\theta_{\tilde{\Omega}} = -\frac{\pi}{2}. \tag{B76}$$

This implies we must have the fusion rule

$$\tilde{\Omega}^2 = eQ\tilde{m}. \tag{B77}$$

Once again, we consider $\tilde{\Omega} \to \tilde{\Omega}' = e\tilde{\Omega}$, combined with $\tilde{m} \to \tilde{m}' = Q\tilde{m}$. Again this shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi/2$, while preserving all other statistics angles and all the fusion rules, and we can set $\theta_{\tilde{\Omega}} = -\pi/8$. The remaining local redefinition, $\tilde{\Omega} \to Q\tilde{\Omega}$, only shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi$, and is thus superfluous. We have thus found a 2D theory with data $(C,0,\pi/2,-\pi/8)$.

b. Dimensionally reduced theories with eC$\tilde{m}$C

Now we consider $\tilde{m}^2 = Q$, or $\tilde{m}C$. In this case, we have

$$\theta_{\tilde{\Omega}^2,\tilde{m}} = 2\theta_{\tilde{\Omega},\tilde{m}} = \pi. \tag{B78}$$

If $\theta_{\tilde{m}} = 0$, this implies $\tilde{\Omega}^2 = e\tilde{m}, eQ\tilde{m}$. On the other hand, if $\theta_{\tilde{m}} = \pi/2$, this implies $\tilde{\Omega}^2 = \tilde{m}, Q\tilde{m}$.

First, we consider the case $\theta_{\tilde{m}} = 0$. Then, because $\tilde{\Omega}^2 = e\tilde{m}, eQ\tilde{m}$, we have $\theta_{\tilde{\Omega}^2} = \pi$, implying

$$\theta_{\tilde{\Omega}} = \frac{\pi}{2}q + \frac{\pi}{4}, \quad q = 0,\ldots,3. \tag{B79}$$

Then we have

$$\theta_{\tilde{\Omega}^2,\tilde{\Omega}} = 2\theta_{\tilde{\Omega},\tilde{\Omega}} = 4\theta_{\tilde{\Omega}} = \pi, \tag{B80}$$

which implies we have the fusion rule

$$\tilde{\Omega}^2 = e\tilde{m}. \tag{B81}$$

Again, we consider the redefinition $\tilde{\Omega} \to \tilde{\Omega}' = e\tilde{\Omega}$, combined with $\tilde{m} \to \tilde{m}' = Q\tilde{m}$. Again this shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi/2$, while preserving all other statistics angles and all the fusion rules, and we can set $\theta_{\tilde{\Omega}} = \pi/4$. The remaining local redefinition, $\tilde{\Omega} \to Q\tilde{\Omega}$, only shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi$, and is thus superfluous. We have thus found a 2D theory with data $(C,C,0,\pi/4)$.

Second, we need to consider the case $\theta_{\tilde{m}} = \pi/2$. Then because $\tilde{\Omega}^2 = \tilde{m}, Q\tilde{m}$, we have $\theta_{\tilde{\Omega}^2} = \pi/2$, implying

$$\theta_{\tilde{\Omega}} = \frac{\pi}{2}q + \frac{\pi}{8}, \quad q = 0,\ldots,3. \tag{B82}$$

Then we have

$$\theta_{\tilde{\Omega}^2,\tilde{\Omega}} = 2\theta_{\tilde{\Omega},\tilde{\Omega}} = 4\theta_{\tilde{\Omega}} = \frac{\pi}{2}, \tag{B83}$$

which implies we have the fusion rule

$$\tilde{\Omega}^2 = \tilde{m}. \tag{B84}$$

Again, we consider the redefinition $\tilde{\Omega} \to \tilde{\Omega}' = e\tilde{\Omega}$, combined with $\tilde{m} \to \tilde{m}' = Q\tilde{m}$. Again this shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi/2$, while preserving all other statistics angles and all the fusion rules, and we can set $\theta_{\tilde{\Omega}} = \pi/8$. The remaining local redefinition, $\tilde{\Omega} \to Q\tilde{\Omega}$, only shifts $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi$, and is thus superfluous. We have thus found a 2D theory with data $(C,C,\pi/2,\pi/8)$.

c. Behavior under global redefinitions

In total, there are four distinct 2D theories with eC, with data $(C,0,0,0)$, $(C,0,\pi/2,-\pi/8)$, $(C,C,0,\pi/4)$, and $(C,C,\pi/2,\pi/8)$. Let us consider the behavior of these under the global redefinition $\tilde{\Omega} \to \tilde{\Omega}' = \tilde{m}\tilde{\Omega}$. In order to preserve the parameter $\theta_{e,\tilde{\Omega}}$, we must also redefine $e \to e' = Qe$. In addition, if $\theta_{\tilde{m}} = \pi/2$, to preserve $\theta_{\tilde{m},\tilde{\Omega}}$, we must redefine $\tilde{m} \to \tilde{m}' = Q\tilde{m}$. We find that all four 2D theories are invariant under the global redefinition.

APPENDIX C: FRACTIONALIZATION PATTERNS

Here, we use the results of Appendix B to describe the possible fractionalization patterns in Z_2Z_2 SET phases. The description here is equivalent to that given in the main text, but is more detailed in accounting for all the properties of the 2D SET orders upon dimensional reduction.

Obviously, one piece of information in the fractionalization pattern is the symmetry charge of e, so we have either $e0$ or eC. The remaining information has to do with symmetry action on m quasistrings. Using the dimensional reduction approach, a crucial point is that the symmetry action on m is encoded in differences of the properties from one basal plane to another. Because $\theta_{\tilde{m}}$ is the same in all basal planes, we then have two pieces of information. One is written m_m0/C, which expresses whether the difference in symmetry charge of $\tilde{m}$ between the two basal planes is integer or fractional. The other is written Ω_m0/s, which has to do with the difference $\theta^{[m]}_{\tilde{\Omega}} - \theta^{[I]}_{\tilde{\Omega}}$. If this difference is $0 \bmod \pi$, we write Ω_m0, and if it is $\pi/2 \bmod \pi$, we write $\Omega_m s$, where the "s" stands for semion. Putting this information together, we would write, e.g., $e0m_m0\Omega_m0$ to specify the entire fractionalization pattern.

Each fractionalization pattern corresponds to several different choices of dimensionally reduced theories on the I and m basal planes. Therefore it is important to be sure that the information given in the fractionalization pattern is always well-defined under global redefinitions, for all possible choices of dimensionally reduced theories. Or, if some information is not well-defined, it should be ill-defined for *all* possible choices of dimensionally reduced theories corresponding to a given fractionalization pattern.

Let's first check this for the $e0m_m0\Omega_m0$ fractionalization pattern. We first note that it does not matter which dimensionally reduced theory occurs on which basal plane. For $e0$, the global redefinition does not affect the charge of $\tilde{m}$, so m_m0 is certainly well-defined. However, the statistics of $\tilde{\Omega}$ can change under global redefinition, so we have to be more careful. One possibility is that both basal planes have the same theory with data $(0,0,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$. While this data changes under global redefinition if $\theta_{\tilde{m}} = \pi/2$, the difference in $\theta_{\tilde{\Omega}}$ between the two planes is unchanged, and Ω_m0 is thus well-defined. The other possibility is that both basal planes have the same theory with data $(0,C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$, where again the difference of $\theta_{\tilde{\Omega}}$ between the planes is unchanged by global redefinition.

Next, we consider $e0m_m0\Omega_m s$. Again, m_m0 is well-defined. Here, one possibility is that one basal plane has data $(0,0,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$, while the other has data $(0,0,\theta_{\tilde{m}},\theta_{\tilde{\Omega}}+\pi/2)$. The difference in $\theta_{\tilde{\Omega}}$ between the planes is unchanged under global redefinition. This is also clearly true for the other possibility, which is that one plane has data $(0,C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}})$, while the other has data $(0,C,\theta_{\tilde{m}},\theta_{\tilde{\Omega}}+\pi/2)$.

Next, we consider $e0m_mC$. We do not specify Ω_m here, because we will see it is not well-defined. The most general possibility is that one plane has data $(0,0,\theta_{\tilde{m}},\theta^{[a]}_{\tilde{\Omega}})$, while the other has $(0,C,\theta_{\tilde{m}},\theta^{[b]}_{\tilde{\Omega}})$. The difference m_mC is clearly well-defined. Under global redefinition we have

$$\left(0,0,\theta_{\tilde{m}},\theta^{[a]}_{\tilde{\Omega}}\right) \to \left(0,0,\theta_{\tilde{m}},\theta^{[a]}_{\tilde{\Omega}}+\theta_{\tilde{m}}\right), \tag{C1}$$

$$\left(0,C,\theta_{\tilde{m}},\theta^{[b]}_{\tilde{\Omega}}\right) \to \left(0,C,\theta_{\tilde{m}},\theta^{[b]}_{\tilde{\Omega}}+\theta_{\tilde{m}}+\pi/2\right). \tag{C2}$$

Therefore the difference $\theta^{[b]}_{\tilde{\Omega}} - \theta^{[a]}_{\tilde{\Omega}}$ shifts by $\pi/2$ under global redefinition, and is thus not well-defined. (Note that it is also consistently ill-defined, for all possible choices of dimensionally reduced theories.)

Now we consider fractionalization patterns with eC. In this case, for a fixed $\theta_{\tilde{m}}$, there are only two possible dimensionally reduced theories, which have $\tilde{m}0$ and $\tilde{m}C$. Global redefinition leaves all these 2D theories invariant, so it plays no role.

First we consider the eCm_m0 fractionalization pattern. In this case, both I and m basal planes have to be in the same dimensionally reduced theory—any one of the four choices is fine. The statistics of $\tilde{\Omega}$ is thus the same in both basal planes, so we do not need to specify anything about Ω_m.

Finally, we consider eCm_mC. In this case, for a given $\theta_{\tilde{m}}$, the I basal plane is in one of the two possible theories, and the m plane is in the other. We see that the difference $\theta^{[m]}_{\tilde{\Omega}} - \theta^{[I]}_{\tilde{\Omega}} = \pm\pi/4$. At first glance, it looks like there might be two possibilities here, having to do with the plus or minus sign, but $\pm\pi/4$ are equivalent under local redefinitions that shift $\theta_{\tilde{\Omega}} \to \theta_{\tilde{\Omega}} + \pi/2$ in one of the basal planes. Therefore there is a single $eC\Omega_mC$ fractionalization pattern.

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