A Unified Structural Interpretation of Some Well-Known Stability-Test Procedures for Linear Systems

P. P. VAIDYANATHAN, MEMBER, IEEE, AND SANJIT K. MITRA, FELLOW, IEEE

A number of well-known stability-test procedures for continuous- and discrete-time systems are re-examined in a unified manner, leading to well-defined network-theoretic interpretations. The representation and network interpretation are based on the fact that the stability of any linear system (scalar or multivariable) is equivalent to the stability of a related all-pass system, which in turn can always be synthesized as a cascade of (scalar or matrix) two-pair all-pass (lossless) networks. The original system of interest is stable if and only if each all-pass two-pair is stable (and hence "lossless bounded real"). As a result of this interpretation, a number of related issues, such as enumeration of unstable poles, prematured terminations, and singularity situations can all be approached in a unified manner, based only on "two-pair extraction formulas." In addition, the network interpretation also leads to direct test procedures for testing relative stability, and the stability of multi-input, multi-output systems.

I. Introduction

During the last several decades, a number of algorithms and interpretations have evolved [1]–[6], [30]–[35] for testing the stability of both continuous- and discrete-time linear systems. The algorithms do not involve computation of the zeros of the characteristic polynomial. In addition, some of these test procedures lend themselves to network-theoretic interpretations. For example, the Hurwitz stability test for continuous-time systems [1] can be related to the Cauer-type continued-fraction expansion of a reactance function, leading to an *LC* network realization. Similarly, the well-known Schur–Cohn test [5], [6] for checking the stability of a discrete-time system is related closely to the synthesis of an all-pass function, as a cascade of lattice sections [14], [21], [26], [27]. The intriguing paper by Delsarte *et al.* [26], places in evidence some of these relations. On the other hand, for certain other test procedures (such as Routh's test as described in Åström [7]), such a network interpretation is not entirely obvious, and has not been heretofore studied.

Manuscript received March 18, 1985; revised March 14, 1986.
P. P. Vaidyanathan is with the Department of Electrical Engineering, California Institute of Technology, Pasadena, CA 91125, USA.
S. K. Mitra is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93111, USA.
IEEE Log Number 8714144.

The purpose of this paper is to provide a structural interpretation of a number of well-known stability-test procedures, based on the synthesis of an appropriate lossless network, thus leading to a common framework for apparently unrelated test procedures. Our development here is more of a tutorial nature and is not intended to be an exhaustive review of all known test procedures. The framework is based on the observation [6] that the stability of any minimal linear system (continuous-time or discrete-time; scalar or multivariable) is equivalent to the stability of a related all-pass system, and that any all-pass system can be synthesized as a cascade of lossless two-pairs. The original system is stable if and only if the related all-pass system can be realized as a cascade of *stable* all-pass (scalar or matrix) two-pairs. The Routh's test for continuous-time systems and the Schur–Cohn test for discrete-time systems can be interpreted in this manner. In addition, this interpretation places in evidence methods of generalizing these well-known test procedures, in order to avoid certain "singularity" situations.

A unified outlook of some of these test procedures can be found in the work by Delsarte *et al.* [26], and multivariable generalizations can be obtained based on [30]. Generalized interpretations based on Chebyshev functions can be found in [31]. In this paper, we explicitly place in evidence the structural details for several related test procedures based on lossless building blocks, including those in [26]. For example, Routh's test as described in [7] is interpreted in Section V by explicit reformulation in terms of two-pair extraction. Our starting point here is directly a "synthesis approach" rather than the "interpolation approach" taken in [26], [30]. Furthermore, algorithmic means for counting the number of unstable poles are presented for each well-known test procedure, some of these being known for several years [15] in mathematics literature. Finally, a situation known as "singularity" [34] which arises in several test procedures is addressed in a common way, and methods outlined to overcome these.

The above network interpretation also leads to procedures for *directly* testing the *relative* stability of a linear system, without constructing [8] an intermediate polynomial of double the order.

0018-9219/87/0400-0478$01.00 © 1987 IEEE

Notations

The variables s and z denote the transform domain complex variables for continuous- and discrete-time systems, respectively. Boldfaced letters (and upper case calligraphic letters) indicate matrices and vectors, for example $\boldsymbol{H}(z)$, $\boldsymbol{v}(n)$, etc. Superscript t stands for matrix transposition. Superscript * indicates complex conjugation, whereas superscript dagger (†) stands for transposition followed by complex conjugation. For discrete-time systems, the tilde-accent stands for transposition followed by reciprocation of functional argument; for example: $\tilde{\boldsymbol{H}}(z) = \boldsymbol{H}^t(z^{-1})$. For the continuous-time case, $\tilde{\boldsymbol{H}}(s) = \boldsymbol{H}^t(-s)$. The notation $\boldsymbol{A} \leq \boldsymbol{B}$ is an abbreviation for "$\boldsymbol{B} - \boldsymbol{A}$ is positive semi-definite," and $\boldsymbol{A} < \boldsymbol{B}$ is an abbreviation for "$\boldsymbol{B} - \boldsymbol{A}$ is positive-definite." $\boldsymbol{I}_m$ stands for the identity matrix of dimension $m \times m$. The subscript for the identity matrix may be omitted if its order is obvious from the context. The symbol $\boldsymbol{O}$ stands for null vectors and null matrices of appropriate dimensions. For a (real-symmetric) positive-definite matrix $\boldsymbol{P}$, we define its square root $\boldsymbol{P}^{1/2}$ according to the factorization: $\boldsymbol{P} = \boldsymbol{P}^{1/2}\boldsymbol{P}^{t/2}$ where $\boldsymbol{P}^{t/2}$ stands for the transpose of $\boldsymbol{P}^{1/2}$.

Preliminaries

A single-input, single-output discrete-time transfer function $G(z) = N(z)/D(z)$ is a ratio of two polynomials. Assuming that the polynomials $N(z)$ and $D(z)$ are relatively prime, the poles of $G(z)$ are precisely the zeros of $D(z)$. In this paper, the term "stable" implies bounded-input, bounded-output stability [24], that is, the poles are *strictly* inside the unit circle. A transfer function $G(z)$ is said to be *bounded real* (BR) if it is stable, real for real z, and satisfies

$$G^*(e^{j\omega})\, G(e^{j\omega}) \leq 1 \tag{1}$$

for all ω. In addition, if equality holds in (1) for all ω, $G(z)$ is said to be *lossless bounded real* (LBR). If $G(z)$ is such that it is not necessarily stable, but equality holds in (1) for all ω, it is said to be *all-pass*. Thus the *LBR* property is equivalent to *stable all-pass property*. Any all-pass transfer function in fact also satisfies the condition $G(z^{-1})\, G(z) = 1$ for *all* z.

These concepts can be extended to general m-input, p-output systems. Thus a $p \times m$ transfer matrix $\boldsymbol{G}(z)$ is BR if it is stable, real valued for real z, and satisfies

$$\boldsymbol{G}^{\dagger}(e^{j\omega})\, \boldsymbol{G}(e^{j\omega}) \leq \boldsymbol{I}_m \tag{2}$$

for all ω. Moreover, if equality holds in (2) for all ω, then $\boldsymbol{G}(z)$ is LBR. If $\boldsymbol{G}(z)$ is not necessarily stable but still satisfies (2) with equality for all ω, then it is an all-pass transfer matrix. (Note that scattering matrices of continuous-time lossless multiports satisfy the LBR property.) It can be shown that an all-pass transfer matrix actually satisfies the property

$$\boldsymbol{G}^t(z^{-1})\, \boldsymbol{G}(z) = \boldsymbol{I}_m \tag{3a}$$

for *all* z (called paraunitary property). Essentially, a stable all-pass transfer matrix is LBR (assuming it is real for real z).

A statement of the maximum modulus theorem [9] and its matrix version (which can be found in Potapov [29]) are included in Appendix I. Based on these, it can be shown that a BR transfer matrix $\boldsymbol{G}(z)$ satisfies

$$\boldsymbol{G}^{\dagger}(z)\, \boldsymbol{G}(z) \leq \boldsymbol{I}_m, \qquad \text{for } |z| > 1. \tag{3b}$$

Moreover, equality in (3b) implies that $\boldsymbol{G}(z)$ is constant. For scalar BR functions, (3b) clearly implies $|G(z)| \leq 1$ for $|z| \geq 1$.

Digital Two-Pair

A digital two-pair [10], shown in Fig. 1, is a two-input two-output system, described either by the chain parameters

$$\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \begin{bmatrix} Y_2(z) \\ X_2(z) \end{bmatrix} \tag{4}$$

or the transfer parameters

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}. \tag{5}$$

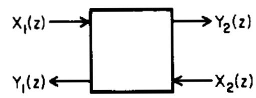

Fig. 1. The digital two-pair.

A "reciprocal" two-pair satisfies the condition $T_{12}(z) = T_{21}(z)$ or equivalently, $AD - BC = 1$, for all z. The descriptions of (4) and (5) are related as indicated in [10]. The chain matrix of (4) and the transfer matrix of (5) are denoted by $\Pi(z)$ and $\mathfrak{T}(z)$, respectively.

A digital two-pair is all-pass if $\mathfrak{T}(z)$ is an all-pass transfer matrix. A digital two-pair is LBR if $\mathfrak{T}(z)$ is LBR. In terms of the chain parameters, paraunitariness is equivalent to

$$1 + \tilde{C}C = \tilde{A}A \qquad 1 + \tilde{B}B = \tilde{D}D \qquad \tilde{C}D = \tilde{A}B. \tag{6}$$

A stable reciprocal two-pair is LBR if and only if

$$A = \tilde{D} \qquad B = \tilde{C} \qquad AD - BC = 1. \tag{7}$$

Given a transfer function $G_m(z)$, the "extraction" of a digital two-pair $\mathfrak{T}(z)$ leads to a remainder $G_{m-1}(z)$ (see Fig. 2) where

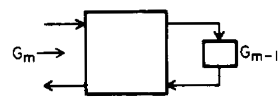

Fig. 2. Digital two-pair extraction.

$G_m(z)$ and $G_{m-1}(z)$ are related by the extraction formula

$$G_{m-1} = (C - AG_m)/(BG_m - D)$$

$$G_m = (C + DG_{m-1})/(A + BG_{m-1}). \tag{8}$$

The subscripts m and $m - 1$ do not necessarily stand for order. Thus unless the two-pair is properly chosen, the order of G_{m-1} is not less than that of G_m.

We can extend the concept of two-pairs to the case where the scalar signals X_1, Y_1, Y_2, X_2 in Fig. 1 are replaced with vector signals $\boldsymbol{X}_1$, $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{X}_2$. Thus

$$\begin{bmatrix} \boldsymbol{X}_1(z) \\ \boldsymbol{Y}_1(z) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}(z) & \boldsymbol{B}(z) \\ \boldsymbol{C}(z) & \boldsymbol{D}(z) \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_2(z) \\ \boldsymbol{X}_2(z) \end{bmatrix} \tag{9}$$

and the transfer parameters $\boldsymbol{T}_{ij}$ are defined accordingly. The parameters are related as

$$\begin{aligned} \boldsymbol{T}_{11} &= \boldsymbol{C}\boldsymbol{A}^{-1} & \boldsymbol{T}_{12} &= \boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \\ \boldsymbol{T}_{21} &= \boldsymbol{A}^{-1} & \boldsymbol{T}_{22} &= -\boldsymbol{A}^{-1}\boldsymbol{B} \end{aligned} \tag{10}$$

and

$$A = T_{21}^{-1} \qquad B = -T_{21}^{-1}T_{22}$$
$$C = T_{11}T_{21}^{-1} \qquad D = T_{12} - T_{11}T_{21}^{-1}T_{22}. \tag{11}$$

Clearly, the above description is meaningful only if A and T_{21} are square, i.e., only if the vectors X_1 and Y_2 have the same number of components. For such cases, the "extraction formula" now becomes

$$G_{m-1} = (D - G_mB)^{-1}(G_mA - C) \tag{12}$$

$$G_m = (C + DG_{m-1})(A + BG_{m-1})^{-1}. \tag{13}$$

Fig. 3 illustrates the matrix two-pair extraction. A matrix LBR two-pair is defined in exactly the same manner as an LBR

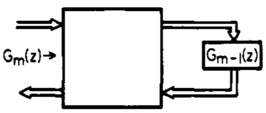

Fig. 3. Matrix two-pair extraction.

two-pair. In particular, the paraunitary property in this case is given by

$$\tilde{C}C + I = \tilde{A}A \qquad \tilde{B}B + I = \tilde{D}D \qquad \tilde{C}D = \tilde{A}B. \tag{14}$$

For continuous-time systems, all above discussions can be extended in an obvious manner, by replacing z with s, z^{-1} with $-s$, and identifying the "steady state" to be $s = j\omega$ rather than $z = e^{j\omega}$.

The Unifying Framework

Given a transfer function to be tested for stability, assume an all-pass function $G(z)$ [or $H(s)$] has been constructed with the same denominator as the given transfer function. A number of ways to synthesize an all-pass function in the form of a cascade of all-pass two-pairs exists [10]–[14]. Each such synthesis procedure leads to a valid stability-test procedure.

The two-pair extraction rules (8), (12), (13) can be used to understand a number of related issues in a unified manner, as outlined below.

1) Any possible common factor between the numerator and denominator of the all-pass function always leads to a premature termination of synthesis, placing in evidence the common factor. This common factor in turn represents poles that are in reciprocal pairs with respect to the unit circle.

2) By employing Rouche's theorem of complex variable theory [15], the general equations (8), (12), (13) can be manipulated to obtain algorithms that enumerate the number of unstable poles.

3) In case of certain unstable transfer functions, a singularity situation [15] sometimes results that prevents continuation of the enumeration procedure. All such situations can be interpreted as singularities of the two-pair being extracted. Moreover, based on the particular two-pair extraction scheme, these singularities can be avoided by changing certain flexible "parameters" of extraction.

In the paper we shall demonstrate these for several instances of all-pass synthesis techniques.

Paper Outline

Section II begins with a review of Jury's description [6] of a test for the stability of discrete-time scalar systems (the Schur–Cohn test). This is followed by the network interpretation, enumeration procedures, and discussion on common-factor propagation. Section III is an extension of the method of Section II, and shows how singularity situations can be avoided by merely manipulating certain two-pair parameters. The relation between this framework and the Nevanlinna–Pick problem can be found in [26]. In Section IV, a discrete-time stability test is presented based on the synthesis of all-pass functions outlined in [11]. Section V deals with the continuous-time counterpart of Section III. In Section VI we obtain an interpretation of Routh's test for continuous-time systems, based on the s-domain all-pass two-pair extraction approach. It can also be shown that this section is the continuous-time counterpart of Section IV. In Section VII, a procedure is outlined for testing a particular kind of relative stability (sector stability [17]) for continuous-time systems. Finally, Section VIII deals with the stability of an m-input p-output system, based on matrix-all-pass synthesis. The work by Delsarte *et al.* [30] for the case of matrix-valued functions is related to our presentation in this section. Numerical examples are included where appropriate.

II. The Schur–Cohn Test

Jury has described [6] a procedure for testing the stability of a discrete-time system based on the Schur–Cohn method. The relation between this test procedure and the procedure for synthesizing an all-pass function as a cascaded lattice structure [14], [21] is known. In fact, the basic mathematical structure underlying this stability-test procedure can be tracked back to the Nevanlinna–Pick interpolation problem, as described by Delsarte *et al.* in [26].

In this section, a simple description of this test procedure is given, followed by a circuit-theoretic interpretation, showing the relation to the Gray and Markel lattice structure. Given a discrete-time transfer function $H(z) = P(z)/D(z)$ with no common factors between $P(z)$ and $D(z)$

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1z^{-1} + \cdots + p_Mz^{-M}}{1 + d_1z^{-1} + \cdots + d_Mz^{-M}} \tag{15}$$

we first form an Mth-order all-pass function

$$G_M(z) = \frac{N_M(z)}{D_M(z)} = \frac{z^{-M}D(z^{-1})}{D(z)}$$
$$= \frac{d_{M,M} + d_{M,M-1}z^{-1} + \cdots + z^{-M}}{1 + d_{M,1}z^{-1} + \cdots + d_{M,M}z^{-M}} \tag{16}$$

(where $d_{M,k} \triangleq d_k$) so that $G_M(z)$ is stable *iff* (if and only if) $H(z)$ is stable. In the paper, we assume the coefficients p_k and d_k to be real. From (16) it is clear that if $G_M(z)$ is stable, then the quantity $|d_{M,M}|$, which is the magnitude of the product of all roots of $D(z)$, must be less than unity. Thus defining

$$k_M \triangleq G_M(\infty) = d_{M,M} \tag{17}$$

we have the necessary condition for stability, that

$$k_M^2 < 1 \tag{18}$$

which, however, is far from sufficient. Let us now assume that (18) holds. In order to derive a sufficient condition for stability, we first note that $G_M(z)$ is stable *iff* it is LBR. Now a new transfer function $G_{M-1}(z)$ is formed as follows:

$$z^{-1}G_{M-1}(z) = \frac{G_M(z) - k_M}{1 - k_M G_M(z)}. \tag{19}$$

It is easily verified that $G_{M-1}(z)$ is of order $M-1$. It can further be verified that $|G_{M-1}(e^{j\omega})| = 1$ because $|G_M(e^{j\omega})| = 1$. Thus $G_{M-1}(z)$ is all-pass and has the form

$$G_{M-1}(z) = \frac{d_{M-1,M-1} + \cdots + d_{M-1,1}z^{-(M-2)} + z^{-(M-1)}}{1 + d_{M-1,1}z^{-1} + \cdots + d_{M-1,M-1}z^{-(M-1)}}. \tag{20}$$

Hence, $G_{M-1}(z)$ is stable *iff* it is LBR. From (19) we see that the poles z_0 of $G_{M-1}(z)$ are solutions of

$$G_M(z_0) = \frac{1}{k_M}. \tag{21}$$

By assumption, $k_M^2 < 1$ (otherwise, the test could have been terminated with the conclusion that $G_M(z)$ is unstable), hence (21) implies

$$|G_M(z_0)| > 1. \tag{22}$$

If $G_M(z)$ is LBR, then we know that $|G_M(z)| < 1$ for $|z| > 1$. Moreover, on the unit circle we have $|G_M(z)| = 1$. In conclusion, (22) implies that if G_M is stable then $|z_0| < 1$, hence $G_{M-1}(z)$ is stable. Thus we conclude that

$$G_{M-1}(z) \text{ is LBR (i.e., stable) if } G_M(z) \text{ is LBR (i.e., stable).} \tag{23}$$

Next, to prove the converse, we invert the relation of (19) to arrive at

$$G_M(z) = \frac{k_M + z^{-1}G_{M-1}(z)}{1 + k_M z^{-1}G_{M-1}(z)}. \tag{24}$$

Now, if z_0 is a pole of $G_M(z)$, then

$$z_0^{-1}G_{M-1}(z_0) = -\frac{1}{k_M}. \tag{25}$$

By the assumption that (18) holds, it is clear that $|z_0^{-1}G_{M-1}(z_0)| > 1$. Now, if $G_{m-1}(z_0)$ is LBR, then we know $|G_{M-1}(z)| \leq 1$ for $|z| \geq 1$, hence (25) implies $|z_0| < 1$. As a result, if (18) holds, then we also have the conclusion

$$G_M(z) \text{ is LBR if } G_{M-1}(z) \text{ is LBR.} \tag{26}$$

Summarizing, a necessary and sufficient set of conditions for the all-pass function $G_M(z)$ to be stable are therefore: a) $k_M^2 < 1$ and that b) the all-pass function $G_{M-1}(z)$ be stable.

Thus once we check the condition $k_M^2 < 1$, we merely test for the stability of the lower order all-pass function $G_{M-1}(z)$. The process can now be repeated, generating a set of coefficients[1]

$$k_M, k_{M-1}, \cdots, k_2, k_1 \tag{27}$$

and a set of all-pass functions

$$G_M(z), G_{M-1}(z), \cdots, G_1(z), G_0(z) = 1. \tag{28}$$

The all-pass function $G_M(z)$ is stable *iff* $k_m^2 < 1$ for all m. The most crucial point in the derivation of the test procedure is that, if a coefficient k_m satisfies the condition

$$k_m^2 < 1 \tag{29}$$

then the stability of G_m is equivalent to stability of G_{m-1} given by

$$z^{-1}G_{m-1}(z) = \frac{G_m(z) - k_m}{1 - k_m G_m(z)}. \tag{30}$$

Circuit Interpretation

A circuit interpretation of the test procedure can now be given. Fig. 4 shows an "implementation" of $G_m(z)$, by constraining a digital two-pair with the function $z^{-1}G_{m-1}(z)$.

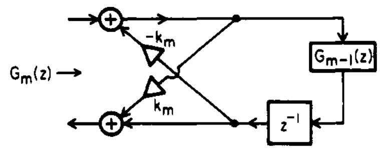

Fig. 4. Circuit interpretation of the Schur-Cohn test.

The two-pair can be described either by the chain matrix

$$\Pi = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & k_m \\ k_m & 1 \end{bmatrix} \tag{31}$$

or by the transfer matrix

$$\mathfrak{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} k_m & 1 - k_m^2 \\ 1 & -k_m \end{bmatrix}. \tag{32}$$

Each iteration in the test, therefore, is an extraction of the above two-pair, from the all-pass function $G_m(z)$, leaving behind an all-pass remainder $z^{-1}G_{m-1}(z)$. The function $G_m(z)$ is stable if and only if k_m satisfies (29), and in addition $G_{m-1}(z)$ is stable. Notice that each two-pair is precisely the Gray and Markel lattice structure [14]. Completion of the test procedure leads to the circuit of Fig. 5, which is a cascaded lattice structure. $G_M(z)$ is stable *iff* all the lattice coefficients k_i in Fig. 5 satisfy (29).

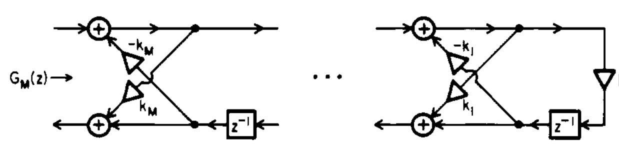

Fig. 5. The cascaded lattice structure associated with the Schur-Cohn test.

Referring to (8), if all the chain parameters of (31) are scaled, this leaves $G_{m-1}(z)$ unchanged (for a given $G_m(z)$). Thus the following scaled lattice (the normalized lattice)

[1] It is necessary for $D(z)$ in (15) to satisfy $D(1) > 0$ and $D(-1) > 0$, for stability. Once this condition is checked, the computation of k_i can be dispensed with; see Jury [6].

$$\Pi = \frac{1}{\sqrt{1-k_m^2}} \begin{bmatrix} 1 & k_m \\ k_m & 1 \end{bmatrix}$$

$$\mathfrak{J} = \begin{bmatrix} k_m & \sqrt{1-k_m^2} \\ \sqrt{1-k_m^2} & -k_m \end{bmatrix} \tag{33}$$

can be obtained if $k_m^2 < 1$, and leads to the equivalent cascaded lattice circuit interpretation of Fig. 6, which is the well-known normalized Gray and Markel structure for an

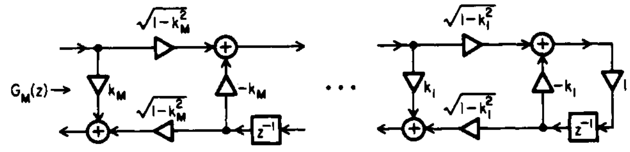

Fig. 6. The normalized cascaded lattice structure associated with the Schur-Cohn test.

all-pass function. In this particular structure, each building block is an LBR two-pair, and the structure of Fig. 6 is an LBR-cascade realization of the given scalar LBR function $G_M(z)$. If $G_M(z)$ is not stable, then the circuit of Fig. 5 can still be obtained, but the circuit of Fig. 6 would have imaginary multipliers.

Propagation of Common Factors: Premature Terminations

Let us assume that the all-pass function $G_m(z)$ given by

$$G_m(z) = \frac{N_m(z)}{D_m(z)} = \frac{d_{m,m} + \cdots + d_{m,1}z^{-(m-1)} + z^{-m}}{1 + d_{m,1}z^{-1} + \cdots + d_{m,m}z^{-m}} \tag{34}$$

has a common factor $W(z)$ of order n between its numerator $N_m(z)$ and denominator $D_m(z)$. From (30) we have

$$z^{-1}N_{m-1}(z) = N_m(z) - k_m D_m(z) \tag{35}$$

$$D_{m-1}(z) = -k_m N_m(z) + D_m(z) \tag{36}$$

which shows that the same factor $W(z)$ is common to both $N_{m-1}(z)$ and $D_{m-1}(z)$. Conversely, $N_m(z)$ and $D_m(z)$ can be written as

$$N_m(z) = z^{-1}N_{m-1}(z) + k_m D_{m-1}(z) \tag{37}$$

$$D_m(z) = k_m z^{-1}N_{m-1}(z) + D_{m-1}(z) \tag{38}$$

which shows that any common factor between $N_{m-1}(z)$ and $D_{m-1}(z)$ is also a common factor of $N_m(z)$ and $D_m(z)$. (An overall constant multiplier in (37), (38) has been ignored for convenience.) In conclusion, the greatest common divisor (*gcd*) of $N_m(z)$ and $D_m(z)$ is the *gcd* of $N_{m-1}(z)$ and $D_{m-1}(z)$ as well. Thus during the stability test, we eventually end up with a remainder function

$$G_n(z) = \frac{N_n(z)}{D_n(z)} = \frac{W(z)}{W(z)} = 1. \tag{39}$$

In other words, even though the given denominator $D_M(z)$ is of Mth degree, the test terminates in $M - n$ steps. This premature termination reveals the *gcd* $W(z)$ between $D_M(z)$ and $N_M(z)$.

Under what conditions does there exist a (nontrivial) *gcd* between $D_M(z)$ and $N_M(z)$? Note that $N_M(z)$ is the mirror image of $D_M(z)$ (see (16)); hence, if $W(z)$ is a common factor, then $W(z)$ itself is a "mirror image polynomial"

$$W(z) = w_0 + w_1 z^{-1} + \cdots + w_1 z^{-(n-1)} + w_0 z^{-n} \tag{40}$$

which means that if z_0 is a zero of $W(z)$, then $1/z_0$ is also a zero. Thus $D_M(z)$ (of which $W(z)$ is a factor) represents the denominator of an unstable system. Note that, as the common factor $W(z)$ automatically reveals itself, it can subsequently be removed from $D_M(z)$ to produce a reduced-order denominator.

For the rest of this section, we assume that any possible *gcd* of the above form has been detected and removed. In particular, therefore, the polynomials involved do not have zeros on the unit circle.

Counting the Number of Unstable Poles

There are applications where, given a polynomial such as $D_M(z)$, one wishes to know the number of zeros inside a circle of a certain radius, say unity. (One such application, for example, is in the estimation of sinusoidal signals immersed in noise.) The above stability test procedure directly lends itself to such an enumeration scheme, based on Rouche's theorem of complex variable theory [15], restated for convenience in Appendix II. Now consider (38). We know that

$$|z^{-1}N_{m-1}(z)| = |D_{m-1}(z)|, \quad \text{whenever } |z| = 1. \tag{41}$$

Moreover, the polynomials in (37), (38) are polynomials in z^{-1}, and hence are analytic in $|z| \geq 1$. Thus by Rouche's theorem:[2]

$$\delta[D_m(z)] = \begin{cases} \delta[D_{m-1}(z)], & k_m^2 < 1 \quad (42) \\ 1 + \delta[N_{m-1}(z)], & k_m^2 > 1 \quad (43) \end{cases}$$

$$\delta[N_m(z)] = \begin{cases} 1 + \delta[N_{m-1}(z)], & k_m^2 < 1 \quad (44) \\ \delta[D_{m-1}(z)], & k_m^2 > 1. \quad (45) \end{cases}$$

(The situation $k_m^2 = 1$ is a "singularity" situation, and will be dealt with shortly.) Now, once the lattice has been successfully generated, the quantities $\delta[N_1(z)]$ and $\delta[D_1(z)]$ can be found by inspection. Then (42)–(45) can be used in order to recursively enumerate the zeros of $D_M(z)$ in the region $|z| > 1$. In other words, assuming that the cascaded lattice of Fig. 5 is given to us, where $k_i^2 \neq 1$ for any i, the parameters k_i reveal exactly how many of the zeros of $D_M(z)$ are in $|z| > 1$. Note also that $N_m(z)$ and $D_m(z)$ are mirror images of each other, hence their zeros are reciprocals of each other. As a result, $N_m(z)$ has as many zeros inside the unit circle as does $D_m(z)$ outside the unit circle, and *vice versa*. In other words, $\delta[N_m(z)] + \delta[D_m(z)] = m$, which can be used instead of (44), (45).

Singularity Situations

Even if $D_M(z)$ and its mirror image $N_M(z)$ do not have any common factors, it is possible during the stability test to arrive at a situation where $k_m^2 = 1$. Under this condition, (35), (36) give

$$z^{-1}G_{m-1} = z^{-1}\frac{N_{m-1}}{D_{m-1}} = \pm 1 \tag{46}$$

i.e., $G_{m-1} = \pm z$ which represents a noncausal system.

[2] $\delta[F(z)]$ is an abbreviation for "the number of zeros of $F(z)$ in the region $|z| > 1$."

Moreover, we cannot proceed further with the test. Also, (37), (38) are not valid any more because the matrix

$$\begin{pmatrix} 1 & -k_m \\ -k_m & 1 \end{pmatrix} \tag{47}$$

is singular. This situation arises when $D_m(z)$ is such that the product of all roots is either 1 or -1. This clearly implies instability of $G_m(z)$ and hence instability of $G_M(z)$. However, if we wish to complete the lattice construction (for example, in order to find the number of unstable poles), then we cannot do so without somehow overcoming the singularity problem. This will be dealt with in the next section. Note also that this singularity implies singularity of the chain matrix of (31). Moreover, $T_{12} = T_{21} = 0$ in (33), which means the two-pair cascade of Fig. 6 is "broken" by the section in question.

Summarizing, given a minimal transfer function $H(z)$, an all-pass function $G_M(z)$ is first formed as in (16) and then synthesized as a cascade of lattice building blocks as in Fig. 5 or 6. In each step of the synthesis, a lower order all-pass function $G_{m-1}(z)$ is generated by extracting a lattice two-pair from an all-pass function $G_m(z)$. The outcomes of the synthesis are listed below.

1) The stability properties of $G_M(z)$, hence that of $H(z)$ is revealed.
2) A network interpretation of the stability test is obtained.
3) The number of unstable poles of $G_M(z)$ (if it is unstable) is determined.
4) Any possible symmetric polynomial factor $W(z)$ in $D_M(z)$ (which in turn is the *gcd* of $D_M(z)$ and $N_M(z)$) is placed in evidence.
5) A possible singularity situation sometimes results for unstable transfer functions which makes it impossible to complete the synthesis of the lattice network.

Stability tests discussed in succeeding sections have a counterpart of items 1) through 5) above. All of them are based on the synthesis of an all-pass function in the form of a cascade of two-pair building blocks. Each of them reveals a possible *gcd* between $N_M(z)$ and $D_M(z)$, and lends itself to a "pole counting" procedure, based only on Rouche's theorem. Finally, each of them has its own "singularity" issues.

III. The Extended Schur-Cohn Test

The first generalization of the stability test of Section II is based on the observation that (30) is not the only means [26] of obtaining a reduced-order all-pass function $G_{m-1}(z)$ from an all-pass function $G_m(z)$. Such a generalization can be related once again to the Nevanlinna-Pick interpolation problem, as observed by Delsarte *et al.* in [26]. Indeed, let us consider the following type of recursion, which is an immediate generalization of (30):

$$G_{m-1}(z) = \left(\frac{1 - a_m z}{z - a_m}\right) \frac{G_m(z) - G_m(a_m)}{1 - G_m(z)\, G_m(a_m)}. \tag{48}$$

Note that, with $a_m = \infty$, this reduces to (30). Assuming that $G_m(z)$ is all-pass, i.e.,

$$G_m^*(e^{j\omega})\, G_m(e^{j\omega}) = 1, \qquad \text{for all } \omega \tag{49}$$

it can be verified from (48) that $G_{m-1}(z)$ is also all-pass. Next, the quantity $G_m(z) - G_m(a_m)$ has a zero at $z = a_m$, which cancels with the factor $(z - a_m)$. Finally, the all-pass nature of $G_m(z)$ ensures that

$$1 - G_m\left(\frac{1}{a_m}\right) G_m(a_m) = 0, \qquad \text{for any } a_m \tag{50}$$

hence the factor $(1 - a_m z)$ in (48) cancels with a factor $(1 - a_m z)$ in $[1 - G_m(z)\, G_m(a_m)]$. Thus $G_{m-1}(z)$ is an all-pass function of reduced order.

Let us assume we have picked a_m such that $|a_m| > 1$. Then it is clear by maximum modulus theorem that the quantity

$$k_m \triangleq G_m(a_m) \tag{51}$$

satisfies

$$|k_m|^2 < 1 \tag{52}$$

if $G_m(z)$ is LBR. Moreover, assuming that (52) holds, the following is true:

$$G_m(z) \text{ is stable } \textit{iff } G_{m-1}(z) \text{ is stable.} \tag{53}$$

In order to derive a test procedure, it is necessary to "propagate" the stability property of $G_m(z)$ to the reduced-order function $G_{m-1}(z)$, and (53) does precisely this. Remembering that the "unstable pole" $z = a_m$ in (48) cancels, the rest of the proof of (53) follows the same lines as in Section II.

Notice that the parameter a_m can be different for different sections. Thus the generalized stability test procedure is as follows: Form the all-pass function $G_M(z)$ in the usual manner, i.e., as in (16). Then generate the parameters

$$k_m = G_m(a_m), \qquad \text{for } m = M, M-1, \cdots \tag{54}$$

where a_m are constants such that $|a_m| > 1$. If at any stage, k_m violates (52), then $G_M(z)$ is unstable. If all k_m are such that (52) holds, then $G_M(z)$ is stable.

Network Interpretation

Fig. 7 shows a network interpretation of (48), and Fig. 8 shows a cascaded lattice realization of $G_M(z)$ that results

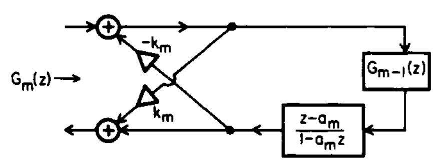

Fig. 7. Circuit interpretation of the extended Schur-Cohn test.

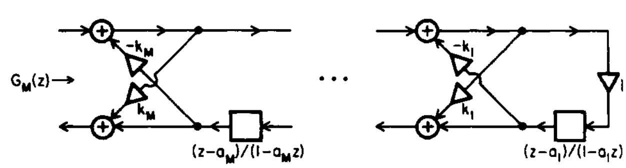

Fig. 8. The cascaded lattice structure associated with the extended Schur-Cohn test.

from the stability test procedure. Fig. 9 shows a normalized realization which is useful when $k_m^2 < 1$ for all m. Moreover, as expected, if $a_m = \infty$, the circuits of Figs. 7–9 reduce to those of Figs. 4–6. Note that the structures of Figs. 8 and 9

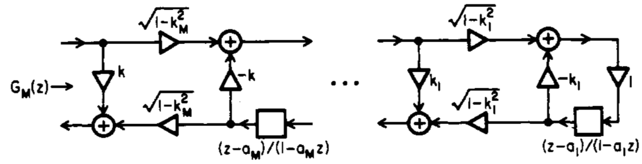

Fig. 9. The normalized cascaded lattice structure associated with the extended Schur-Cohn test.

are not implementable in practice, because of "delay-free loops." However, for the purposes of checking stability of $G_M(z)$, this is not an important issue.

Propagation of the Greatest Common Divisor

Equations (35) and (36) of Section II can be modified for the purposes of this section as follows:

$$(1 - a_m z^{-1}) N_{m-1}(z) = N_m(z) - k_m D_m(z) \tag{55}$$

$$(z^{-1} - a_m) D_{m-1}(z) = -k_m N_m(z) + D_m(z). \tag{56}$$

Conversely

$$N_m(z) = (1 - a_m z^{-1}) N_{m-1}(z) + k_m (z^{-1} - a_m) D_{m-1}(z) \tag{57}$$

$$D_m(z) = k_m (1 - a_m z^{-1}) N_{m-1}(z) + (z^{-1} - a_m) D_{m-1}(z). \tag{58}$$

It is clear from these equations that the *gcd* of $N_m(z)$ and $D_m(z)$ is the same as that of $N_{m-1}(z)$ and $D_{m-1}(z)$. Thus once again, any possible *gcd* in $G_M(z)$ is revealed in the synthesis procedure, in the form of a premature termination. Other related discussions of Section II continue to hold here.

Counting the Number of Unstable Poles

Note that, in (57) and (58), $|a_m| > 1$. Also, $|1 - a_m z^{-1}| = |z^{-1} - a_m|$ on the unit circle. Thus as in Section II, we can apply Rouche's theorem again. The arguments that followed (37), (38) can be repeated with respect to (57), (58) in order to arrive at the following conclusions:

$$\delta[D_m(z)] = \delta[(z^{-1} - a_m) D_{m-1}(z)] = \delta[D_{m-1}(z)], \quad \text{for } |k_m|^2 < 1 \tag{59}$$

$$\delta[D_m(z)] = 1 + \delta[N_{m-1}(z)], \quad \text{if } |k_m|^2 > 1. \tag{60}$$

This set of equations can be iteratively used to obtain the number of unstable poles. Note that $\delta[N_m(z)] = m - \delta[D_m(z)]$ for all m because, by assumption, there are no zeros on the unit circle.

Singularity Issues[3]

A significant advantage of the generalization (48) as compared to (30) of Section II is that the parameter a_m can be any number such that $|a_m| > 1$, with $a_m = \infty$ being only a special case. Moreover "a_m" can be different for each section. Now, recall that if we wish to generate a complete lattice for an unstable system (in order to successfully count the number of unstable poles, for example), then a singularity situation, i.e.,

$$|k_m| = |G_m(a_m)| = 1 \tag{61}$$

might arise again, as in Section II. However, since the choice of a_m is flexible, we can pick a_m such that (61) does not hold, and this enables us to complete the lattice generation, and hence the pole-counting operation. For example, restricting ourselves to real values of a_m, $G_m(a_m)$ is real. Moreover, the equation

$$G_m^2(z) = 1 \tag{62}$$

i.e.,

$$N_m^2(z) - D_m^2(z) = 0 \tag{63}$$

has at most $2m$ real solutions. Thus by trying out the following possible simple integer values of a_m:

$$a_m = 2, 3, 4, 5, \cdots, 2m + 2 \tag{64}$$

we are bound to find one value such that there is no singularity situation. We now proceed with numerical examples to demonstrate these ideas.

Example 3.1:

Let the denominator to be tested be given by $D_2(z) = 1 + 3/2 z^{-1} + 1/2 z^{-2}$. The relevant all-pass function is

$$G_2(z) = \frac{\frac{1}{2} + \frac{3}{2} z^{-1} + z^{-2}}{1 + \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}}. \tag{65}$$

Choosing the free parameter a_2 in (61) as $a_2 = \infty$, we have $k_2 = 1/2$, hence $k_2^2 < 1$. Thus

$$z^{-1} G_1(z) = \frac{G_2(z) - \frac{1}{2}}{1 - \frac{1}{2} G_2(z)} = z^{-1}\left(\frac{1 + z^{-1}}{1 + z^{-1}}\right) \tag{66}$$

which shows that there is a premature termination. The common factor $W(z)$ between $D_2(z)$ and $N_2(z)$ is revealed as $(1 + z^{-1})$, which is a mirror image polynomial as expected.

Example 3.2:

Let the denominator to be tested be given by

$$D_3(z) = 1 + \frac{7}{6} z^{-1} - \frac{17}{6} z^{-2} + z^{-3}. \tag{67}$$

The third-order all-pass function $G_3(z)$ is therefore

$$G_3(z) = \frac{1 - \frac{17}{6} z^{-1} + \frac{7}{6} z^{-2} + z^{-3}}{1 + \frac{7}{6} z^{-1} - \frac{17}{6} z^{-2} + z^{-3}}. \tag{68}$$

If we let $a_3 = \infty$, then $k_3 = G_3(a_3) = 1$, which is a singularity situation. If we take $a_3 = 2$ we get $k_3 = 0$. Even though this does not cause any problems during the procedure, let us avoid $a_3 = 2$, so that the computational aspects are demonstrated well. Assuming $a_3 = 3$, then

$$k_3 = G_3(a_3) = G_3(3) = \frac{1}{5}. \tag{69}$$

Compute $N_3 - k_3 D_3$

$$N_3(z) - k_3 D_3(z) = 4 - \frac{92}{6} z^{-1} + \frac{52}{6} z^{-2} + 4 z^{-3}. \tag{70}$$

The quantity $(1 - a_3 z^{-1}) = (1 - 3z^{-1})$ is a factor of (70), hence by (55)

[3]Often referred to as "critical cases."

$$N_2(z) = \frac{N_3(z) - k_3 D_3(z)}{1 - a_3 z^{-1}} = -\frac{4}{3}\left[-3 + \frac{5}{2} z^{-1} + z^{-2}\right]. \quad (71)$$

Thus the remainder all-pass function is

$$G_2(z) = \frac{-3 + \frac{5}{2} z^{-1} + z^{-2}}{1 + \frac{5}{2} z^{-1} - 3z^{-2}}. \quad (72)$$

This completes one stage of the iteration. For the next stage, let $a_2 = \infty$ and compute

$$k_2 = G_2(a_2) = -3 \quad (73)$$

and obtain

$$N_2(z) - k_2 D_2(z) = (10 - 8z^{-1})z^{-1}. \quad (74)$$

Hence by (35) (since $a_2 = \infty$)

$$N_1(z) = 10 - 8z^{-1} = -8\left[-\frac{5}{4} + z^{-1}\right]. \quad (75)$$

The first-order all-pass remainder is therefore

$$G_1(z) = \frac{-\frac{5}{4} + z^{-1}}{1 - \frac{5}{4} z^{-1}}. \quad (76)$$

By inspection, $G_1(z)$ is unstable having one pole outside the unit circle. Let us, however, compute k_1 for the sake of completion.

$$k_1 = G_1(\infty) = -\frac{5}{4} \quad (77)$$

$$N_1(z) - k_1 D_1(z) = -\frac{9}{16} z^{-1}. \quad (78)$$

Thus $G_0(z) = (-9/16)/(-9/16) = 1$ and the synthesis is complete. The given polynomial $D_3(z)$ obviously represents the denominator of an unstable system because $|k_1| > 1$ and $|k_2| > 1$. The number of unstable poles can now be counted.

$$\delta[D_1(z)] = 1 \qquad \delta[N_1(z)] = 0 \quad \text{by inspection or since } k_1^2 > 1 \quad (79)$$

$$\left.\begin{aligned} \delta[D_2(z)] &= 1 + \delta[N_1(z)] = 1 \\ \delta[N_2(z)] &= \delta[D_1(z)] = 1 \end{aligned}\right\} \text{by (43) and (45) since } k_2^2 > 1 \quad (80)$$

$$\left.\begin{aligned} \delta[D_3(z)] &= \delta[D_2(z)] = 1 \\ \delta[N_3(z)] &= 3 - \delta[D_3(z)] = 2 \end{aligned}\right\} \text{by (59) since } k_3^2 < 1. \quad (81)$$

Thus $\delta[D_3(z)] = 1$, hence the given transfer function has one "unstable" pole.

IV. Further Generalizations of the Discrete-Time Schur-Cohn Test

A main outcome of the discussions in the previous sections is that, if the all-pass function $G_M(z)$ is stable (i.e., $G_M(z)$ is LBR), then it can be realized as a cascade of LBR two-pairs with transfer matrix of the form

$$\mathfrak{Z}_m(z) = \begin{bmatrix} k_m & \sqrt{1 - k_m^2}\,(1 - a_m z^{-1})/(z^{-1} - a_m) \\ \sqrt{1 - k_m^2} & -k_m(1 - a_m z^{-1})/(z^{-1} - a_m) \end{bmatrix},$$

$$|a_m| > 1,\ k_m^2 < 1. \quad (82)$$

Moreover, if the all-pass function $G_M(z)$ is not stable (i.e., not LBR), then it cannot be realized in the above form. Now, given an LBR function $G_M(z)$, there exist *other* ways of realizing it as a cascade of LBR two-pairs of other forms [11]. This then raises the question: Given a procedure for synthesizing an all-pass function as a cascade of all-pass two-pairs, can we always associate a stability test with it? Indeed, this turns out to be the case. The purpose of this section is to outline specific details for some of these methods. The methods not only give rise to test procedures, but also place in evidence any possible *gcd* between the numerator and the denominator of $G_M(z)$. In addition, they lend themselves to counting procedures for enumerating the unstable poles.

Let us begin by considering a digital two-pair with the following chain parameters [11]:

$$A(z) = \frac{1 + \sigma_m z^{-1}}{(1 + z^{-1})}$$

$$B(z) = \frac{(1 - \sigma_m)\, z^{-1}}{(1 + z^{-1})}$$

$$C(z) = \frac{1 - \sigma_m}{(1 + z^{-1})}$$

$$D(z) = \frac{\sigma_m + z^{-1}}{(1 + z^{-1})} \quad (83)$$

or, equivalently, the following transfer parameters:

$$T_{11}(z) = \frac{1 - \sigma_m}{1 + \sigma_m z^{-1}}$$

$$T_{12}(z) = \sigma_m \frac{1 + z^{-1}}{1 + \sigma_m z^{-1}}$$

$$T_{21}(z) = \frac{1 + z^{-1}}{1 + \sigma_m z^{-1}}$$

$$T_{22}(z) = \frac{-(1 - \sigma_m)\, z^{-1}}{1 + \sigma_m z^{-1}}. \quad (84)$$

Let $G_m(z)$ be an mth-order all-pass function such that $G_m(-1) = 1$. Let us extract the two-pair of the above form, with the following value of σ_m:

$$\sigma_m = \left.\frac{G_m'(z)}{G_m'(z) - 1}\right|_{z=-1} \quad (85)$$

where prime denotes derivative with respect to z^{-1}. The remainder G_{m-1} is given by

$$G_{m-1}(z) = \frac{(1 - \sigma_m) - (1 + \sigma_m z^{-1})\, G_m(z)}{(1 - \sigma_m)\, z^{-1} G_m(z) - (\sigma_m + z^{-1})} \quad (86)$$

and it can be verified that

$$|G_{m-1}(e^{j\omega})| = 1, \qquad \text{for all } \omega \quad (87)$$

i.e., $G_{m-1}(z)$ is all-pass. Moreover, with the choice of σ_m as in (85), $G_{m-1}(z)$ is of order $m - 1$, and satisfies $G_{m-1}(-1) = -1$. For proofs, we refer the reader to [11], [16]. Thus the above two-pair extraction can be repeated on the function

$-G_{m-1}(z)$. Given an all-pass function $G_M(z)$, we can therefore obtain a cascade realization as in Fig. 10.

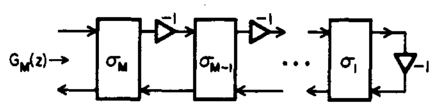

Fig. 10. The LBR two-pair cascade associated with the stability test of Section IV.

In order to derive a stability-test procedure based on this synthesis technique, we make a number of observations. First, if $G_m(z)$ is stable, then it can be shown [16] that σ_m is in the range

$$0 < \sigma_m < 1. \tag{88}$$

(If $\sigma_m = 1$, then $G_{m-1} = G_m$, and there is no progress in the synthesis. On the other hand, $\sigma_m = 0$ corresponds to a singularity situation, to be discussed later.) Next, if σ_m *is* in the range of (88), then $G_m(z)$ is not necessarily stable, but the following is true:

$G_m(z)$ is LBR (i.e., stable) if and only if

$$G_{m-1}(z) \text{ is LBR (i.e., stable).} \tag{89}$$

(This statement will shortly be justified.) We therefore have a situation exactly analogous to that in Section II. The procedure for testing the stablility of $G_M(z)$ is therefore to generate all the σ_m's. Then $G_M(z)$ is stable *iff* each σ_m is in the range of (88).

Circuit Interpretation

If each chain parameter in (83) is divided by $\sqrt{\sigma_m}$, this does not affect $G_{m-1}(z)$ for a given $G_m(z)$ (see (86)). The resulting two-pair has transfer parameters

$$T_{11}(z) = \frac{1-\sigma_m}{1+\sigma_m z^{-1}}$$

$$T_{12}(z) = T_{21}(z) = \frac{\sqrt{\sigma_m}\,(1+z^{-1})}{1+\sigma_m z^{-1}}$$

$$T_{22}(z) = -\frac{(1-\sigma_m)\,z^{-1}}{1+\sigma_m z^{-1}}. \tag{90}$$

It can be shown [16] that this two-pair is LBR *iff* each σ_m lies in the range of (88). In other words, the all-pass function $G_M(z)$ is stable *iff* it can be realized as a cascade of LBR two-pairs of the form (90) as shown in Fig. 10. In Fig. 11 we show the internal details of an orthogonal implementation [13] of a building block with a transfer matrix as in (90).

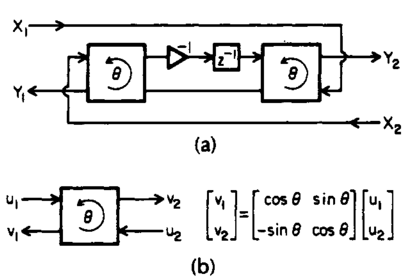

Fig. 11. (a) An orthogonal implementation of a typical two-pair in Fig. 10. (b) Definition of the planar rotator in Fig. 11(a).

Premature Terminations and Propagation of gcd

It can be shown that if $W(z)$ is a polynomial factor in common with $N_m(z)$ and $D_m(z)$, then it is also a common factor of $N_{m-1}(z)$ and $D_{m-1}(z)$. The converse is also true. As a result, the *gcd* of $N_M(z)$ and $D_M(z)$ propagates during the synthesis, leading to a premature termination, unless N_M and D_M are relatively prime. The *gcd* $W(z)$ is placed in evidence at the termination. It is easily verified that the *gcd* is a mirror-image polynomial.

As the *gcd* gets revealed in the synthesis process, it is easy to cancel it in $G_M(z)$. We shall accordingly assume for the rest of the section that, for each m, $N_m(z)$ and $D_m(z)$ are relatively prime; in particular, $G_m(z)$ has *no poles or zeros on the unit circle.*

Counting the Number of Unstable Poles

In a manner somewhat analogous to that in the previous sections, we can invoke Rouche's theorem in order to enumerate the number of unstable poles of $G_M(z)$. In order to see this, let us invert (86) and rewrite $G_m(z)$ in terms of $G_{m-1}(z)$

$$G_m(z) = \frac{N_m(z)}{D_m(z)} = \frac{(1-\sigma_m) + (\sigma_m + z^{-1})\,G_{m-1}(z)}{1+\sigma_m z^{-1} + (1-\sigma_m)\,z^{-1}G_{m-1}(z)}. \tag{91}$$

The poles of $G_m(z)$ are zeros of

$$1 + \sigma_m z^{-1} + (1-\sigma_m)\,z^{-1}G_{m-1}(z). \tag{92}$$

Let k denote the number of zeros of $D_{m-1}(z)$ inside the unit circle. Then $m-1-k$ denotes the number of zeros outside (since, by assumption, there are no zeros on the unit circle). $G_{m-1}(z)$ can now be written as

$$G_{m-1}(z) = P(z)\,Q(z) \tag{93}$$

where $P(z)$ is a stable all-pass function of order k and $1/Q(z)$ is a stable all-pass function of order $m-1-k$. Thus the poles of $G_m(z)$ are *precisely* the zeros of

$$F(z) \triangleq \frac{(1+\sigma_m z^{-1})}{Q(z)} + (1-\sigma_m)\,z^{-1}P(z). \tag{94}$$

Note that $F(z) \neq 0$ for $|z| = 1$, and that the functions

$$\frac{1+\sigma_m z^{-1}}{Q(z)} \qquad (1-\sigma_m)\,z^{-1}P(z) \tag{95}$$

are analytic in $|z| \geq 1$. Moreover, $|P(z)| = |Q(z)| = 1$ on the unit circle. Finally, note that for $|z| = 1$

$$|(1-\sigma_m)\,z^{-1}| \leq |1+\sigma_m z^{-1}|, \quad \text{if } \sigma_m > 0$$

$$|(1-\sigma_m)\,z^{-1}| \geq |1+\sigma_m z^{-1}|, \quad \text{if } \sigma_m < 0. \tag{96}$$

Invoking Rouche's theorem (Appendix II) therefore leads to the following conclusions:

$$\delta[D_m] = \begin{cases} m-1-k = \delta[D_{m-1}], & \text{if } 0 < \sigma_m < 1 \\ m-k = \delta[D_{m-1}]+1, & \text{if } \sigma_m > 1 \\ k+1 = m-\delta[D_{m-1}], & \text{if } \sigma_m < 0 \end{cases} \tag{97}$$

where $\delta[\cdot]$ denotes the number of zeros outside the unit circle. Equation (97) directly enables us to count the number of unstable poles in $G_M(z)$ by recursively evaluating $\delta[D_1]$,

$\delta[D_2], \cdots$, etc. This can be done, once the circuit of Fig. 10 is generated.

Note that, if $0 < \sigma_m < 1$, then $\delta[D_m] = 0$ if and only if $\delta[D_{m-1}] = 0$. This then proves the claim made by (89) earlier.

Singularity Situations

If the parameter σ_m turns out to be equal to zero at a certain stage, this leads to the remainder

$$G_{m-1}(z) = -z. \tag{98}$$

If we try to re-evaluate $G_m(z)$ based on (91), we would get

$$G_m(z) = \frac{0}{0} \tag{99}$$

which shows that the inverse relation of (91) is not valid any more. This is a singularity situation, analogous to the condition $k_m^2 = 1$ in Sections II and III.

V. Continuous-Time Stability Test Procedures

In Sections II, III, and IV, we used the fact that any discrete-time all-pass function $G(z)$ is stable if and only if there exists a cascade of LBR two-pairs that realize the $G(z)$. Depending upon the exact nature of the two-pair, a number of stability-test procedures could then be placed in evidence. The situation in the continuous-time domain is quite analogous. Given a transfer function

$$H(s) = \frac{P(s)}{D(s)} = \frac{p_0 s^M + p_1 s^{M-1} + \cdots + p_M}{s^M + d_1 s^{M-1} + \cdots + d_M} \tag{100}$$

with no common factor between $P(s)$ and $D(s)$, an all-pass function can be formed as follows:

$$\begin{aligned} G_M(s) &= \frac{N_M(s)}{D_M(s)} = \frac{D(-s)}{D(s)} \\ &= \frac{(-1)^M s^M + \cdots - d_{M-1}s + d_M}{s^M + \cdots + d_{M-1}s + d_M}. \end{aligned} \tag{101}$$

The overall idea once again is the same as before: If $G_M(s)$ is stable (i.e., LBR), then it can be realized as a cascade of analog LBR two-pairs. The converse statement is also true. Thus the stability-test procedure is essentially a procedure to synthesize $G_M(s)$.

Perhaps the simplest way to begin this section is to refer back to Section III and obtain an analog of (48). Indeed, given an mth-order all-pass function $G_m(s)$, let us define

$$G_{m-1}(s) = \left(\frac{a_m + s}{a_m - s}\right) \frac{G_m(s) - k_m}{1 - k_m G_m(s)} \tag{102}$$

where

$$k_m = G_m(a_m) \tag{103}$$

and a_m is any real *positive* constant. First, it can be verified that $|G_{m-1}(j\omega)|^2 = 1$, i.e., $G_{m-1}(s)$ is also all-pass. Second, it can be verified that $G_{m-1}(s)$ is of order $(m-1)$. In order to see this, note that $G_m(s) - k_m$ has a zero at $s = a_m$ leading to a cancellation of $(a_m - s)$. Also, since $G_m(s)$ is all-pass, $G_m(a_m)G_m(-a_m) = 1$, hence $(a_m + s)$ is a factor of $1 - k_m G_m(s)$ and cancels in (102). Next, by application of the maximum-modulus principle, it is clear that, if $G_m(s)$ is stable (i.e., LBR), then $|k_m| < 1$. Furthermore, if $|k_m| < 1$, then the following statement is true:

$$G_m(s) \text{ is stable } \textit{iff } G_{m-1}(s) \text{ is stable.} \tag{104}$$

The developments are therefore analogous to those in Section III, and so the details are omitted. Given the all-pass function $G_M(s)$ as in (101), we generate the sequence

$$G_{M-1}(s), G_{M-2}(s), \cdots, G_0(s) = \pm 1 \tag{105}$$

$$k_M, k_{M-1}, \cdots, k_1 \tag{106}$$

by using (102) and (103). Then $G_M(s)$ is stable if and only if $|k_m| < 1$ for all m in (106).

The circuit interpretation is shown in Fig. 12 and is self-explanatory. In the case that $G_M(s)$ is stable, each lattice section in Fig. 12 can be redrawn as in Fig. 13, leading to a normalized realization. This normalized realization is an LBR two-pair cascade.

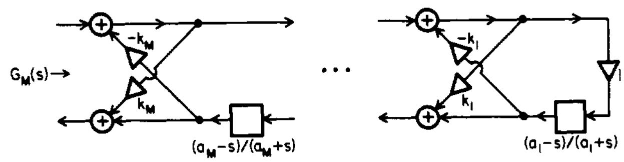

Fig. 12. Circuit interpretation of the continuous-time stability test of Section V.

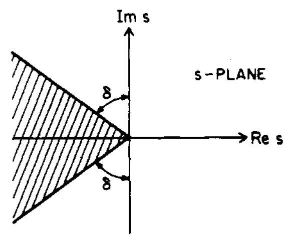

Fig. 13. Normalized cascaded lattice corresponding to the test in Section V.

In a manner analogous to Sections II, III, and IV, a possible *gcd* between $D_M(s)$ and $N_M(s)$ can be extracted during the synthesis process, because of a premature termination. Such a *gcd* can then be canceled. Let us therefore assume for the rest of the section that $D_M(s)$ and $N_M(s)$ are relatively prime. In particular, none of the polynomials $D_m(s)$, $N_m(s)$ can then have zeros *on the* $j\omega$ axis. Also, if $G_M(s)$ is unstable, it is possible to encounter a singularity situation where $k_m^2 = 1$. This can be avoided by changing the value of a_m to any other (positive) value as in Section III. Thus we can *always* successfully generate the cascaded lattice of Fig. 12 and then obtain a procedure for counting the number of unstable poles of $H(s)$.

Example 5.1:

Let $D(s) = s^3 + 2s^2 - s + 3$ be the denominator to be tested. This is clearly unstable because of the negative coefficient, but serves to demonstrate the ideas. The relevant all-pass function is

$$G_3(s) = \frac{-s^3 + 2s^2 + s + 3}{s^3 + 2s^2 - s + 3}. \tag{107}$$

Let us choose $a_3 = 1$, then $k_3 = G_3(1) = 1$ which is a singularity situation. This can trivially be avoided by picking $a_3 = 2$. Then

$$k_3 = G_3(2) = \frac{5}{17}. \tag{108}$$

The remainder function $G_2(s)$ computed according to (102) is

$$G_2(s) = \frac{11s^2 + 10s + 9}{11s^2 - 10s + 9}. \tag{109}$$

Now let $a_2 = 1$, then

$$k_2 = G_2(1) = 3. \tag{110}$$

The reduced-order remainder $G_1(s)$ is then

$$G_1(s) = \frac{-11s + 9}{11s + 9}. \tag{111}$$

Once again let $a_1 = 1$, then

$$k_1 = G_1(1) = -\frac{1}{10}. \tag{112}$$

The final remainder $G_0(s)$ is 1, and the test is complete. As $k_2^2 > 1$, the test confirms that $G_3(s)$ is indeed unstable. Moreover, as there is no premature termination, $D_3(s)$ and $D_3(-s)$ are relatively prime.

From the set of values $\{k_M, k_{M+1}, \cdots, k_1\}$, it is possible to obtain an exact count of the number of unstable poles. The results are as follows:

$$\delta[D_m(s)] = \delta[D_{m-1}(s)], \qquad \text{if } k_m^2 < 1 \tag{113}$$

$$\delta[D_m(s)] = 1 + \delta[N_{m-1}(s)], \qquad \text{if } k_m^2 > 1 \tag{114}$$

where $\delta[F(s)]$ denotes the number of zeros in Re $s > 0$. Also, $\delta[D_m(s)] + \delta[N_m(s)] = m$ for all m, by assumption that there are no zeros on the imaginary axis.

For the above example

$$\delta[D_1(s)] = 0$$

$$\delta[N_1(s)] = 1 \tag{115}$$

$$\delta[D_2(s)] = 1 + \delta[N_1(s)] = 2, \qquad \text{as } k_2^2 > 1 \tag{116}$$

$$\delta[D_3(s)] = \delta[D_2(s)] = 2, \qquad \text{as } k_3^2 < 1. \tag{117}$$

Thus there are two unstable poles, i.e., $D(s)$ has 2 zeros in Re $s > 0$. Explicitly evaluating the zeros of $D(s)$, we get the following solutions:

$$s_1 = -2 \cdot 7572 \qquad s_{2,3} = 1 \cdot 0431 e^{\pm j(0.38175\pi)} \tag{118}$$

which verifies the theory.

Example 5.2:

Let $D(s) = s^4 + 3s^3 + 3s^2 + 3s + 2$. The relevant all-pass function is

$$G_4(s) = \frac{s^4 - 3s^3 + 3s^2 - 3s + 2}{s^4 + 3s^3 + 3s^2 + 3s + 2}. \tag{119}$$

Let $a_4 = 1$, then $k_4 = G_4(1) = 0$. (The situation $k_4 = 0$ is quite normal, and simply implies that there is a pole of $G_4(s)$ at $s = -a_4 = -1$.) Then the remainder is

$$G_3(s) = \frac{1+s}{1-s} G_4(s) = \frac{-s^3 + 2s^2 - s + 2}{s^3 + 2s^2 + s + 2}. \tag{120}$$

Next, let $a_3 = 1$, then $k_3 = G_3(1) = \frac{1}{3}$. This leads to

$$G_2(s) = \frac{1+s}{1-s} \frac{G_3(s) - \frac{1}{3}}{1 - \frac{1}{3} G_3(s)} = \frac{4s^2 + 4}{4s^2 + 4}. \tag{121}$$

We now have a premature termination, and this reveals that $s^2 + 1$ is the greatest common factor between $D_4(s)$ and $N_4(s)$. Since $|k_4| < 1$ and $|k_3| < 1$, all the zeros of $D(s)$ are in Re $s < 0$, except for the factor $(s^2 + 1)$ which represents a conjugate pair of zeros at $s = \pm j$.

VI. Reinterpretation of Routh's Test in Terms of LBR Two-Pair Extraction

The well-known Routh's test for continuous-time systems [7] can be interpreted in terms of the synthesis of an LBR function as a cascade of LBR two-pairs. In order to do this, let us begin with a polynomial

$$D_m(s) = d_{0,m}s^m + d_{1,m}s^{m-1} + \cdots + d_{m,m} \tag{122}$$

and construct the all-pass function

$$G_m(s) = \frac{N_m(s)}{D_m(s)} = \frac{D_m(-s)}{D_m(s)}$$
$$= \frac{(-1)^m d_{0,m}s^m + \cdots + d_{m,m}}{d_{0,m}s^m + d_{1,m}s^{m-1} + \cdots + d_{m,m}}. \tag{123}$$

We wish to extract a first-order two-pair with chain parameters ($A(s)$, $B(s)$, $C(s)$, $D(s)$) such that the remainder is a reduced-order all-pass function. In order to ensure that the remainder is all-pass, we impose paraunitary property (6) on the two-pairs. In addition, let us restrict ourselves to reciprocal two-pairs (i.e., $AD - BC = 1$). Thus let

$$A(s) = 1 + \sigma_m s \qquad B(s) = \beta_m + \lambda_m s$$
$$C(s) = \beta_m - \lambda_m s \qquad D(s) = 1 - \sigma_m s. \tag{124a}$$

The reciprocity conditon constrains β_m, λ_m, and σ_m to be

$$\beta_m = 0 \quad \text{and} \quad \lambda_m = \pm\sigma_m. \tag{124b}$$

The two-pair parameters are therefore

$$A(s) = 1 + \sigma_m s \qquad B(s) = \lambda_m s$$
$$C(s) = -\lambda_m s \qquad D(s) = 1 - \sigma_m s \tag{125}$$

$$\lambda_m = \pm\sigma_m. \tag{126}$$

Now the remainder all-pass function, given by

$$G_{m-1}(s) = \frac{A(s)\, G_m(s) - C(s)}{D(s) - B(s)\, G_m(s)} \tag{127}$$

has a denominator

$$\begin{aligned} D_{m-1}(s) &= D(s)\, D_m(s) - B(s)\, N_m(s) \\ &= -s^{m+1}[(-1)^m \lambda_m + \sigma_m]\, d_{0,m} \\ &\quad - s^m[(-1)^{m-1} \lambda_m d_{1,m} - d_{0,m} + \sigma_m d_{1,m}] \\ &\quad - s^{m-1}[(-1)^m \lambda_m d_{2,m} - d_{1,m} + \sigma_m d_{2,m}] \\ &\quad - \\ &\quad \vdots \end{aligned} \tag{128}$$

A reduced-order remainder can be obtained by setting

$$\lambda_m = (-1)^{m-1} \sigma_m \tag{129}$$

$$\sigma_m = d_{0,m}/2d_{1,m}. \tag{130}$$

The chain parameters are therefore

$$A(s) = 1 + \sigma_m s \qquad B(s) = -(-1)^m \sigma_m s$$
$$C(s) = (-1)^m \sigma_m s \qquad D(s) = 1 - \sigma_m s. \tag{131}$$

Note that the reduced-order denominator has leading coefficient $d_{0,m-1}$ precisely equal to $d_{1,m}$. By construction, the extracted two-pair is paraunitary. It is therefore LBR provided it is stable, i.e., if $\sigma_m > 0$.

For the time being, let us assume that there are no common factors between $N_m(s)$ and $D_m(s)$. If $G_m(s)$ is stable, then all the coefficients in $D_m(s)$ should have the same sign, with none missing. Thus σ_m is in the range $0 < \sigma_m < \infty$. In other words, if $G_m(s)$ is stable, then the extracted two-pair is LBR. It can further be shown [12] that, if $\sigma_m > 0$, then the stability of $G_m(s)$ is equivalent to the stability of $G_{m-1}(s)$. Thus the stability test procedure is as follows: Evaluate the test parameter set $\sigma_M, \sigma_{M-1}, \cdots, \sigma_1$. The given polynomial $D_M(s)$ represents a stable denominator if and only if all σ_m are strictly positive.

Now, the Routh's test as described in Aström [7] obtains the lower order polynomial $D_{m-1}(s)$ from $D_m(s)$ according to the following recursion:

$$D_{m-1}(s) = D_m(s) - \frac{\alpha_m s}{2}[D_m(s) - (-1)^m D_m(-s)]$$

$$= \frac{\alpha_m s}{2}(-1)^m D_m(-s) + \left(1 - \frac{\alpha_m s}{2}\right) D_m(s). \quad (132)$$

Thus we can identify

$$B(s) = -\frac{\alpha_m s}{2}(-1)^m \qquad D(s) = 1 - \frac{\alpha_m s}{2} \quad (133)$$

where α_m is defined to be $d_{0,m}/d_{1,m}$. In other words, the polynomials generated in the recursion procedure of Routh's test are the denominator polynomials in the successive allpass functions, in the above two-pair extraction approach. The parameter set tested in the Routh's procedure is

$$d_{0,M}, d_{0,M-1}, d_{0,M-2}, \cdots, d_{0,0} \quad (134)$$

where $d_{0,m}$ is the leading coefficient in successive denominators. If all these coefficients have the same sign, then $G_M(s)$ is stable. The number of changes in sign in the sequence of (134) is known to be equal to the number of unstable poles of $G_M(s)$. It is clear from the above description that the σ-parameters are

$$\sigma_M = \frac{d_{0,M}}{2d_{0,M-1}} \qquad \sigma_{M-1} = \frac{d_{0,M-1}}{2d_{0,M-2}}, \cdots. \quad (135)$$

Accordingly, the number of σ-parameters with a negative sign is equal to the number of unstable poles of $G_M(s)$. Notice that, unlike in all the test procedures discussed earlier in this paper, the enumeration of unstable poles is now exceptionally simple, and is obvious by inspection of the resulting two-pair cascade.

Finally, if there is a common factor $W(s)$ between $N_M(s)$ and $D_M(s)$, it leads to a premature termination, and $W(s)$ is placed in evidence at the termination step. As the details are quite similar to those in the test procedure based on (102), we omit these.

VII. Relative Stability Test Procedures

In a number of physical problems, it is important to restrict the poles of the linear system to a certain subregion of the left-half complex s-plane. Depending upon the nature of the subregion, various types of relative stability can be defined. (For example, see the authoritative survey by Gutman and Jury [17].) In this section, we consider one particular type of relative stability. Referring to Fig. 14, if all the poles of the (continuous-time) system are confined to the shaded area, we call the system "relatively stable." Procedures for testing this kind of "relative stability" have already been reported in the literature [8], [18].

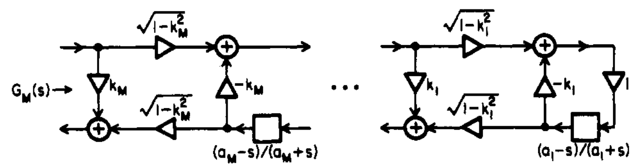

Fig. 14. Pertaining to the relative-stability test.

Given a polynomial $D(s) = D_M(s) = s^M + d_{1,M}s^{M-1} + d_{2,M}s^{M-2} + \cdots + d_{M,M}$, a typical test procedure for this kind of relative stability is to form a new polynomial of twice the degree

$$P(s) = D(se^{j\delta})\, D(se^{-j\delta}). \quad (136)$$

If $D(s)$ has real coefficients, then so does $P(s)$, because of the definition of (136). Moreover, if $1/D(s)$ is relatively stable, then $1/P(s)$ is stable in the conventional sense (i.e., all poles in Re $s < 0$). The converse is also true. Thus conventional test procedures can be applied to $P(s)$.

Even though this procedure is quite simple and useful, it involves testing a polynomial whose degree is two times higher. Moreover, even though a circuit interpretation can be obtained as in earlier sections, the poles of the resulting circuit are *not* the poles of the desired original transfer function.

In this section, we outline a test procedure for this type of relative stability, based on the synthesis of a lossless function as a cascaded lossless two-pair structure. The poles of the resulting network are precisely the poles of the original transfer function (i.e., zeros of $D_M(s)$). However in this procedure, unlike in the procedures based on (136), we require complex network elements.[4] As a first step, we make the observation that, as long as the coefficients $d_{k,M}$ of $D_M(s)$ are real, the zeros of $D_M(s)$ are conjugate pairs and so our problem is equivalent to testing whether the zeros of $D_M(s)$ are confined to the extended shaded region of Fig. 15. We now proceed to do this.

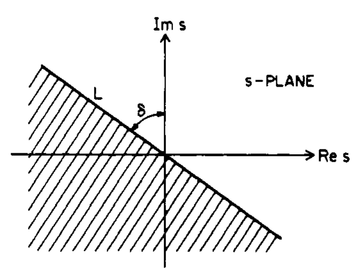

Fig. 15. Redefinition of relative-stability region.

Given a polynomial

$$D_m(s) = \sum_{k=0}^{m} d_{m-k}s^k, \qquad d_0 = 1 \quad (137)$$

we constructed a "modified" mth-order all-pass function

[4]In this context, note that Routh-like algorithms for complex polynomials have been reported in the past. See, for example, [38].

$$G_m(s) = \frac{\sum_{k=0}^{m} d^*_{m-k}(-se^{-j2\delta})^k}{D_m(s)}. \quad (138)$$

Note that $G_m(s)$ is the conventional all-pass function if $\delta = 0$. The above function $G_m(s)$ has the following properties:

P1. $|G_m(s)| = 1$ for all s on the boundary line L, where $s = re^{j(\delta + \pi/2)}$ in Fig. 15.

P2. If s_0 is a pole, i.e., $D_m(s_0) = 0$, then $s_1 \triangleq -s_0^* e^{j2\delta}$ is a zero of $G_m(s)$. In other words, the poles and zeros are symmetric with respect to the boundary line L in Fig. 15.

P3. If we define a new function

$$\hat{G}_{m-1}(s) = \frac{G_m(s) - k_m}{1 - k_m^* G_m(s)} \quad (139)$$

then $\hat{G}_{m-1}(s)$ satisfies properties P1 and P2.

These properties are easily verified by making direct use of the definition of (138). Next, based on maximum-modulus theorem (Appendix I), we have the following property:

P4. If $G_m(s)$ is relatively stable, i.e., has all poles to the left of the boundary line L, then $|G_m(s)| < 1$ at all points that are strictly to the right of L. (This, of course, assumes that $G_m(s)$ is not constant.)

Once again consider (139). If $|k_m| < 1$, then we have at a pole s_0 of $\hat{G}_{m-1}(s)$

$$|G_m(s_0)| = \left|\frac{1}{k_m}\right| > 1. \quad (140)$$

Thus if $G_m(s)$ is relatively stable, then s_0 must lie to the left of L (by property P4), i.e., $\hat{G}_{m-1}(s)$ is also relatively stable. By rewriting (139) as

$$G_m(s) = \frac{k_m + \hat{G}_{m-1}(s)}{1 + k_m^* \hat{G}_{m-1}(s)} \quad (141)$$

we immediately see that if $|k_m| < 1$, then relative stability of $\hat{G}_{m-1}(s)$ implies that of $G_m(s)$. This leads us to conclude:

If $|k_m| < 1$ then $G_m(s)$ is relatively stable if and only if $\hat{G}_{m-1}(s)$ is relatively stable, where $\hat{G}_{m-1}(s)$ is defined by (139).

This conclusion is valid *regardless* of how k_m itself is defined. In order to arrive at a stability-test procedure, it only remains to define k_m appropriately, and modify (139) in order to bring about an order reduction.

Thus let k_m be defined as

$$k_m = G_m(a_m) \quad (142)$$

where $s = a_m$ lies strictly to the right of the line L. Now $(a_m - s)$ is a factor of the quantity $[G_m(s) - k_m]$. Next, because of property P2, it can be verified that $(a_m^* e^{j2\delta} + s)$ is a factor of the quantity $[1 - k_m^* G_m(s)]$. Consequently, the rational function

$$G_{m-1}(s) = \frac{a_m^* e^{2j\delta} + s}{a_m - s} \cdot \frac{G_m(s) - k_m}{1 - k_m^* G_m(s)} \quad (143)$$

is of order $(m - 1)$ where m is the order of $G_m(s)$. Moreover, $G_{m-1}(s)$ satisfies properties similar to P1 and P2. Thus given a modified all-pass function G_m, (143) generates a lower order modified all-pass function G_{m-1}. Moreover, if $G_m(s)$ is relatively stable, k_m satisfies $|k_m|^2 < 1$. Finally, if k_m satisfies this condition, then relative stability of $G_m(s)$ is equivalent to that of $G_{m-1}(s)$.

The stability test procedure is therefore to generate the sequence of modified all-pass functions

$$G_M(s), G_{M-1}(s), \cdots, G_1(s), G_0(s) = \text{constant} \quad (144)$$

and the sequence of constants

$$k_M, k_{M-1}, \cdots, k_1. \quad (145)$$

The system is relatively stable if and only if $|k_m| < 1$ for all m. The choice of a_m in (142) is entirely arbitrary, as long as it lies to the right of the line L. This flexibility in the choice of a_m can be used in order to avoid possible singularity situations such as $|k_m|^2 = 1$, when one is attempting to count the number of poles outside the shaded region of Fig. 15.

Fig. 16 shows a circuit interpretation of (143), and Fig. 17 shows the cascaded lattice structure that is obtained as a by-product of the stability test. Finally, Fig. 18 shows a normalized circuit. The quantity δ appears in the circuit in an

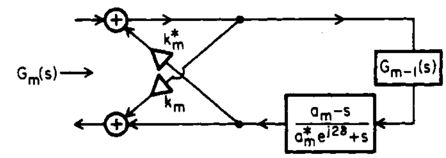

Fig. 16. Circuit interpretation for the relative-stability test.

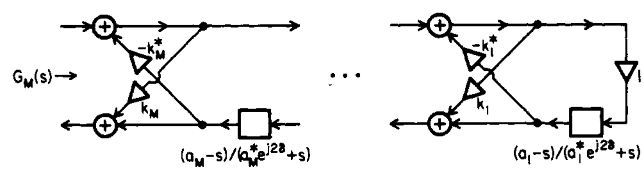

Fig. 17. The cascaded lattice associated with the relative-stability test.

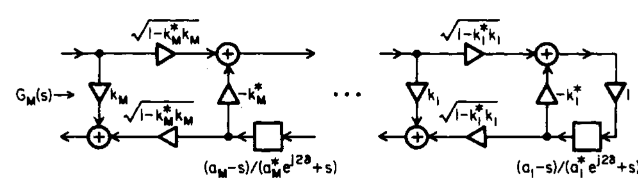

Fig. 18. The normalized version of the cascaded lattice associated with the relative-stability test.

easily controllable manner. Thus "δ" can be "tuned" in order to restrict the poles of the circuit of Fig. 17 to the shaded region of Fig. 14. As long as the parameters "k_m" in this figure have magnitudes bounded by 1, relative stability is *structurally* ensured.

If the modified all-pass of (138) is relatively stable, then it is called modified LBR. If $G_M(s)$ is modified LBR, then each two-pair in Fig. 18 is a modified LBR two-pair.

The propagation of a possible common factor between $D_M(s)$ and $N_M(s)$ can be handled in a manner analogous to that in earlier sections. Similarly, procedures can be established for counting the number of poles that fall outside the relative stability region.

VIII. Stability Tests for Multi-Input Multi-Output System

The stability test procedures described in earlier sections can be extended to the case of discrete- and continuous-time systems with several inputs and outputs. In this section, let us confine our attention only to discrete-time systems and obtain test procedures for an m-input p-output system characterized by a $p \times m$ transfer matrix $\boldsymbol{H}(z)$. (Some results for continuous-time systems can be found in [36].) Assume that $\boldsymbol{H}(z)$ is given in the form of a right matrix fraction description (MFD) [19]

$$\boldsymbol{H}(z) = \boldsymbol{A}(z)\, \boldsymbol{D}^{-1}(z) \tag{146}$$

where $\boldsymbol{A}(z)$ and $\boldsymbol{D}(z)$ are matrix polynomials of degree M. Thus

$$\boldsymbol{D}(z) = \boldsymbol{D}_M(z) = \mathfrak{D}_0 + \mathfrak{D}_1 z^{-1} + \cdots + \mathfrak{D}_M z^{-M} \tag{147}$$

$$\boldsymbol{A}(z) = \mathfrak{A}_0 + \mathfrak{A}_1 z^{-1} + \cdots + \mathfrak{A}_M z^{-M} \tag{148}$$

where $\mathfrak{D}_k$ and $\mathfrak{A}_k$ are $m \times m$ and $p \times m$ matrices, respectively. Assume that the MFD of (146) is irreducible, i.e., $\boldsymbol{A}(z)$ and $\boldsymbol{D}(z)$ are right coprime [19], so that the determinantal zeros of $\boldsymbol{D}(z)$ are the *poles* of the system.

Let us construct a new $m \times m$ transfer matrix $\boldsymbol{G}_M(z)$ with denominator equal to $\boldsymbol{D}_M(z)$

$$\boldsymbol{G}_M(z) = \boldsymbol{N}_M(z)\, \boldsymbol{D}_M^{-1}(z) \tag{149}$$

such that

$$\boldsymbol{G}_M^{\dagger}(e^{j\omega})\, \boldsymbol{G}_M(e^{j\omega}) = \boldsymbol{I}_m \tag{150}$$

i.e., $\boldsymbol{G}_M(z)$ is an all-pass transfer matrix (see Section I). Thus $\boldsymbol{G}_M(z)$ is LBR if and only if it is stable. Moreover, testing the stability of $\boldsymbol{G}_M(z)$ is equivalent to testing the stability of $\boldsymbol{H}(z)$. It can be shown that (150) implies

$$\boldsymbol{G}_M^{t}(z^{-1})\, \boldsymbol{G}_M(z) = \boldsymbol{I}_m \tag{151}$$

for all z.

In the scalar case, given the polynomial $D(z)$, we can construct the all-pass function $G_M(z)$ as in (16), essentially by *inspection* because the numerator of $G_M(z)$ is the mirror image of $D(z)$. Moreover, unless $D(z)$ has a factor $W(z)$ which is a mirror image polynomial by itself, the numerator and denominator of $G_M(z)$ are relatively prime. In the case of multi-input multi-output systems, the construction of the all-pass matrix $\boldsymbol{G}_M(z)$ from a given polynomial $\boldsymbol{D}_M(z)$ is nontrivial, and we are not aware of any *inspection-based* procedures analogous to the scalar case. However, one possible method is to compute $\tilde{\boldsymbol{D}}(z)\, \boldsymbol{D}(z)$ and then obtain a special factor $\boldsymbol{N}_M(z)$ such that $\boldsymbol{N}_M(z)$ is distinct from $\boldsymbol{D}(z)$. If we succeed in computing a spectral factor $\boldsymbol{N}_M(z)$ such that it is relatively right coprime with respect to $\boldsymbol{D}(z)$, then the test procedure to be described next is meaningful, as it does not terminate in a premature manner. The computation of such spectral factors is itself a nontrivial issue [23] and will not be discussed further. Assuming that the function $\boldsymbol{G}_M(z)$ has been formed, it now remains to obtain an iterative procedure for generating a sequence of all-pass transfer matrices

$$\boldsymbol{G}_M(z), \boldsymbol{G}_{M-1}(z), \cdots, \boldsymbol{G}_1(z), \boldsymbol{G}_0(z) = \text{constant matrix} \tag{152}$$

and a sequence of appropriately defined coefficient matrices (analogous to "lattice coefficients" of Sec. II and III)

$$\mathcal{K}_M, \mathcal{K}_{M-1}, \cdots, \mathcal{K}_k, \cdots, \mathcal{K}_1 \tag{153}$$

such that

1) if $\boldsymbol{G}_k(z)$ is stable, then $\mathcal{K}_k^{\dagger}\mathcal{K}_k < \boldsymbol{I}_m$;
2) if $\mathcal{K}_k^{\dagger}\mathcal{K}_k < \boldsymbol{I}_m$, then "$\boldsymbol{G}_k(z)$ is stable if and only if $\boldsymbol{G}_{k-1}(z)$ is stable."

Let the all-pass $\boldsymbol{G}_k(z)$ have the irreducible MFD form

$$\boldsymbol{G}_k(z) = \boldsymbol{N}_k(z)\, \boldsymbol{D}_k^{-1}(z) \tag{154}$$

where

$$\boldsymbol{N}_k(z) = \sum_{i=0}^{k} \mathfrak{N}_{ki} z^{-1}$$

$$\boldsymbol{D}_k(z) = \sum_{i=0}^{k} \mathfrak{D}_{ki} z^{-1}. \tag{155}$$

Assume that $\boldsymbol{G}_k(z)$ is not constant, in order to avoid trivialities. Here $\mathfrak{N}_{ki}$ and $\mathfrak{D}_{ki}$ are $m \times m$ real matrices. $\mathfrak{D}_{k0}$ is assumed to be nonsingular. It can be shown (Appendix I) that, if $\boldsymbol{G}_k(z)$ is stable (i.e, LBR), then it satisfies

$$\boldsymbol{G}_k^{\dagger}(z_0)\, \boldsymbol{G}_k(z_0) \leq \boldsymbol{I}_m \tag{156}$$

for all z_0 such that $|z_0| > 1$. Thus in particular if the quantity $\mathcal{K}_k$ is defined as

$$\mathcal{K}_k = \boldsymbol{G}_k(\infty) \tag{157}$$

then the following is a necessary condition for stability:

$$\mathcal{K}_k^{\dagger}\mathcal{K}_k \leq \boldsymbol{I}_m. \tag{158}$$

Now there are three possible situations:

Case 1: Equation (158) is violated, in which case we simply discontinue the testing, with the conclusion that $\boldsymbol{G}_k(z)$ is unstable.

Case 2: Equation (158) is satisfied with *strict* inequality, in which case we extract a matrix two-pair from $\boldsymbol{G}_k(z)$ with suitable chain parameters, and obtain a lower order all-pass matrix $\boldsymbol{G}_{k-1}(z)$ such that $\boldsymbol{G}_k(z)$ is stable *iff* $\boldsymbol{G}_{k-1}(z)$ is stable. Details of this will shortly be presented.

Case 3: Equation (158) is satisfied but not with strict inequality.

If $G_k(z)$ is a scalar all-pass function, then Case 3 implies that $G_k(z)$ is unstable. However, in the matrix case, $\boldsymbol{G}_k(z)$ may be stable even in Case 3, and hence this case requires careful handling. Indeed, a nonconstant all-pass matrix $\boldsymbol{G}_k(z)$ can be stable (hence LBR) even if the inequalities of (156) and (158) are not strict. For example, consider the following 2×2 transfer matrix:

$$\boldsymbol{F}(z) = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1}{2} + \sqrt{3}z^{-1} & -1 + \frac{\sqrt{3}}{2} z^{-1} \\ \frac{\sqrt{3}}{2} - z^{-1} & -\sqrt{3} - \frac{1}{2} z^{-1} \end{bmatrix}. \tag{159}$$

It is easily verified that $\boldsymbol{F}(z)$ is LBR, with

$$\mathcal{K} = \boldsymbol{F}(\infty) = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{\sqrt{3}}{2} & -\sqrt{3} \end{bmatrix}. \tag{160}$$

Moreover, letting $V = (1 \; -2)^t$, we get

$$V^t \mathcal{K}^t \mathcal{K} V = V^t V \tag{161}$$

which implies that $\mathcal{K}^t\mathcal{K}$ does not satisfy *strict* inequality in (158). Moreover, it can be shown that

$$F(z)\, V = \frac{\sqrt{5}}{2} \begin{matrix} 1 \\ \sqrt{3} \end{matrix} = \mathcal{K} V \tag{162}$$

for all z, for this particular V. In other words, for input sequences of the form

$$x(n) = Vs(n) \tag{163}$$

where $s(n)$ is scalar (i.e., for sequences $x(n)$ in the direction V), the system $F(z)$ is memoryless. We abbreviate this phenomenon by saying that "$F(z)$ is memoryless in the direction V."

We now proceed to generalize these observations before continuing with our description of the stability test: let $F(z)$ be an $m \times m$ LBR transfer matrix. Let $\mathcal{K}$ be defined as $\mathcal{K} = F(\infty)$. If there exists a vector $V \neq O$ such that $V^t\mathcal{K}^t\mathcal{K}V = V^tV$, then $F(z)$ is memoryless in the direction V. In other words, $F(z)V$ is constant and hence for inputs of the form of (163) the output $y(n)$ depends only on $x(n)$ and not on $x(n-k)$, $k > 0$. In fact, if $f(0), f(1), \cdots$ represents the impulse response (matrix) sequence of $F(z)$, i.e.,

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

then $y(n) = f(0)\, x(n)$ for such inputs. We omit a formal justification of these statements here for brevity, and refer the reader to [20] for details.

Let us now return to Case 3. If $G_k(z)$ is such that (158) holds but not with strict inequality, then $\mathcal{K}_k^t\mathcal{K}_k$ has a maximum eigenvalue equal to unity. Let V be the corresponding eigenvector. If $G_k(z)$ is LBR, then we know $G_k(z)V$ should be constant. If we find that $G_k(z)V$ is not constant, then we can terminate the stability test, with the conclusion that $G_k(z)$ (and hence $G_M(z)$) is unstable. On the other hand, if $G_k(z)V$ is constant, then $G_k(z)$ is memoryless in the direction V, and we can obtain an all-pass function $G_k^{(1)}(z)$ of dimension $(m-1) \times (m-1)$ such that $G_k^{(1)}(z)$ is stable *iff* $G_k(z)$ is stable. Appendix III describes this procedure. If $G_k^{(1)}(z)$ is also memoryless in a certain direction, then we can repeat the above process until we obtain an $(m-r) \times (m-r)$ all-pass function $G_k^{(r)}(z)$ which is not memoryless in any direction. The test procedure is now continued on the function $G_k^{(r)}(z)$, because $G_k^{(r)}(z)$ is now bound to fall under Case 1 or Case 2. As Case 3 can be taken care of in the manner described above, it only remains to describe how to handle Case 2.

When Case 2 is satisfied, we extract from $G_k(z)$ a matrix two-pair with the following chain parameters:

$$\mathcal{A} = (I_m - \mathcal{K}_k^t\mathcal{K}_k)^{-t/2} \qquad \mathcal{B} = \mathcal{K}_k^t(I_p - \mathcal{K}_k\mathcal{K}_k^t)^{-t/2} \tag{164}$$

$$\mathcal{C} = \mathcal{K}_k(I_m - \mathcal{K}_k^t\mathcal{K}_k)^{-t/2} \qquad \mathcal{D} = (I_p - \mathcal{K}_k\mathcal{K}_k^t)^{-t/2} \tag{165}$$

where the parameter $\mathcal{K}_k$ is chosen to be

$$\mathcal{K}_k = G_k(\infty). \tag{166}$$

Notice that, in view of the strict inequality in (158), the inverses appearing in (164), (165) are well defined.

It can then be shown that the remainder function is of the form $z^{-1}G_{k-1}(z)$ where the $m \times m$ matrix $G_{k-1}(z)$ is given by

$$G_{k-1}(z) = N_{k-1}(z)\, D_{k-1}^{-1}(z) \tag{167}$$

with

$$N_{k-1} = \sum_{i=0}^{k-1} \mathfrak{N}_{k-1,i} z^{-i}$$

$$D_{k-1} = \sum_{i=0}^{k-1} \mathfrak{D}_{k-1,i} z^{-1}. \tag{168}$$

Moreover, the chain parameters of (164), (165) satisfy [20] the paraunitary property (Section I), and therefore the extracted two-pair is LBR, with a transfer matrix

$$\mathfrak{T}_k = \begin{bmatrix} \mathcal{K}_k & (I_p - \mathcal{K}_k\mathcal{K}_k^t)^{1/2} \\ (I_m - \mathcal{K}_k^t\mathcal{K}_k)^{t/2} & -(I_m - \mathcal{K}_k^t\mathcal{K}_k)^{t/2}\, \mathcal{K}_k^t(I_p - \mathcal{K}_k\mathcal{K}_k^t)^{-t/2} \end{bmatrix}. \tag{169}$$

Thus the remainder $G_{k-1}(z)$ is a lower order all-pass function. It can be shown that the determinantal zeros of $D_{k-1}(z)$ are given by the solutions of

$$\det\,[D_k(z) - \mathcal{K}_k^t N_k(z)] = 0. \tag{170}$$

If $G_k(z)$ is stable, then (158) holds and (156) holds for $|z_0| > 1$. Moreover, for Case 2, these are strict inequalities. As a result, it can be shown that the solutions of (170) lie strictly inside the unit circle. Hence, if $G_k(z)$ is stable (i.e., LBR), then $G_{k-1}(z)$ is LBR. The converse statement can also be established, provided $\mathcal{K}_k$ satisfies (158) with strict inequality.

In summary, if $G_k(z)$ falls under Case 1, we terminate the test with the conclusion that it is unstable. If $G_k(z)$ falls under Case 2, we extract a two-pair as described above and obtain a lower order all-pass matrix $G_{k-1}(z)$. If, however, $G_k(z)$ falls under Case 3, then we attempt to construct an $(m-r) \times (m-r)$ all-pass matrix $G_k^{(r)}(z)$, satisfying the properties described earlier. If such a construction is not successful for reasons outlined earlier, the conclusion is that $G_k(z)$ and hence $G_M(z)$ are unstable. If the construction succeeds, then we proceed to extract an LBR matrix two-pair with chain parameters as in (164), (165) where $\mathcal{K}_k$ now stands for $G_k^{(r)}(\infty)$, and where the matrices I_m and I_p should be replaced with I_{m-r}. Clearly, the entire process terminates in a finite number of iterations.

Circuit Interpretation

For simplicity, let us first assume that, at each stage of the LBR two-pair extraction, the inequality of (158) is strict. The test procedure then leads to the synthesis of the $m \times m$ all-pass matrix $G_M(z)$ in the form of an LBR (matrix) two-pair cascade separated by delay units as shown in Fig. 19. Each two-pair is LBR, i.e., characterized by a $2m \times 2m$ orthogonal transfer matrix. If the test procedure fails, i.e., if some $\mathcal{K}_k$ violates (158), then the shown circuit cannot be constructed

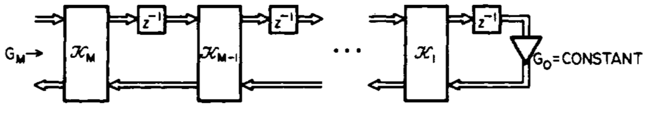

Fig. 19. Circuit interpretation of the multivariable-stability test procedure.

because the matrix square roots and the matrix inverses involved in (169) may not be meaningful.

In summary once again, $G_M(z)$ has been synthesized as a cascade of LBR matrix two-pairs. Note the resemblance of the transfer matrix of (169) to the normalized transfer matrix of (33), Section II. In fact, (169) reduces to (33) when $m = 1$. Thus the results of this section represent an extension of Jury's test [6] and simultaneously an extension of the normalized Gray and Markel lattice structures [21].

If at any stage of LBR extraction, $G_k(z)$ falls under Case 3, i.e., (158) holds without strict inequality, then we define the matrix $G_k^{(1)}(z)$ as described earlier, and proceed with LBR extraction from $G_k^{(1)}(z)$. Fig. 20 shows the circuit interpretation for such an example. This figure demonstrates the

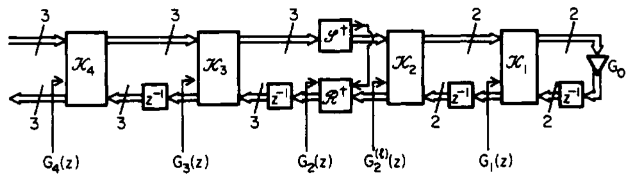

Fig. 20. Demonstration of cascaded lattice synthesis under Case 3 situations.

case where $G_4(z)$ and $G_3(z)$ fall under Case 2 whereas $G_2(z)$ falls under Case 3. As a result, $G_2(z)$ (which is assumed to be a 3×3 matrix) is implemented in terms of another 2×2 all-pass matrix $G_2^{(1)}(z)$. The succeeding matrix two-pair extractions are done on 2×2 all-pass functions.

Premature Termination and gcrd Propagation

Earlier in this section we assumed that the matrix all-pass functions involved are given by *irreducible* MFDs, i.e., $N_k(z)$ and $D_k(z)$ in (154) are relatively right coprime. Let us remove this assumption. In other words, let the $m \times m$ matrix polynomial $R(z)$ denote the greatest common right divisor (gcrd), i.e.,

$$N_k(z) = \overline{N}_k(z)\, R(z) \qquad D_k(z) = \overline{D}_k(z)\, R(z) \tag{171}$$

where $\overline{N}_k(z)$ and $\overline{D}_k(z)$ are right coprime. Next, it can be shown that the transfer matrices $G_k(z) = G_{k-1}(z)$ (154), (167) are related as follows:

$$z^{-1}N_{k-1}(z) = (I_p - \mathcal{K}_k\mathcal{K}_k^{\dagger})^{-1/2}\,[N_k(z) - \mathcal{K}_k D_k(z)] \tag{172}$$

$$D_{k-1}(z) = (I_m - \mathcal{K}_k^{\dagger}\mathcal{K}_k)^{-1/2}\,[D_k(z) - \mathcal{K}_k^{\dagger} N_k(z)] \tag{173}$$

and conversely

$$N_k(z) = \mathcal{K}_k(I_m - \mathcal{K}_k^{\dagger}\mathcal{K}_k)^{-t/2}\, D_{k-1}(z) + (I_p - \mathcal{K}_k\mathcal{K}_k^{\dagger})^{-t/2}\, z^{-1}N_{k-1}(z) \tag{174}$$

$$D_k(z) = (I_m - \mathcal{K}_k^{\dagger}\mathcal{K}_k)^{-t/2}\, D_{k-1}(z) + \mathcal{K}_k^{\dagger}(I_p - \mathcal{K}_k\mathcal{K}_k^{\dagger})^{-t/2}\, z^{-1}N_{k-1}(z). \tag{175}$$

It can further be shown, based on (171)–(175) that $R(z)$ is a gcrd of $D_k(z)$ and $N_k(z)$ if and only if it is a gcrd of $D_{k-1}(z)$ and $N_{k-1}(z)$. As a result, the gcrd propagates down the line during the stability test procedure, and leads to a "premature termination," eventually yielding a remainder function $G_n(z)$ such that

$$N_n(z) = \mathcal{R}R(z) \qquad D_n(z) = R(z) \tag{176}$$

where $\mathcal{R}$ is an $m \times m$ constant orthogonal matrix. The premature termination therefore places the gcrd $R(z)$ in evidence.

From the discussions of this subsection, it is clear that if the MFD of (154) is in irreducible form, then the MFD in (167) is also in irreducible form. Thus if the MFD for $G_M(z)$ in (149) is irreducible, then the intermediate MFDs for each $G_k(z)$ are irreducible as well.

Counting the Number of Unstable Poles

Monden and Arimoto [22] have advanced a procedure for enumerating the determinantal zeros of polynomial matrices. Their procedure has been applied for investigating the stability of fitted and multivariate autoregressions. The procedure in [22] is based on an extension of Rouche's theorem for polynomial matrices (Appendix II).

For the recursion procedure described earlier in this section, a similar enumeration scheme can be formulated. The enumeration procedure works under the condition that the matrices $\mathcal{K}_k$ are such that for each k, the quantity $(I_m - \mathcal{K}_k^{\dagger}\mathcal{K}_k)$ is *definite*, i.e., either positive definite or negative definite, but not indefinite. If, for a given k, $(I_m - \mathcal{K}_k^{\dagger}\mathcal{K}_k)$ is positive definite, then we define $N_{k-1}(z)$ and $D_{k-1}(z)$ as per (172), (173), whereas if it is negative definite, we define

$$z^{-1}N_{k-1}(z) = (\mathcal{K}_k\mathcal{K}_k^{\dagger} - I_p)^{-1/2}\,[N_k(z) - \mathcal{K}_k D_k(z)] \tag{177}$$

$$D_{k-1}(z) = (\mathcal{K}_k^{\dagger}\mathcal{K}_k - I_m)^{-1/2}\,[D_k(z) - \mathcal{K}_k^{\dagger} N_k(z)]. \tag{178}$$

Note that, in (177), (178), the polynomials $N_{k-1}(z)$ and $D_{k-1}(z)$ still have lower order (as in (172), (173)), and $G_{k-1}(z) = N_{k-1}(z)\, D_{k-1}^{-1}(z)$ is still all-pass. Let us now consider the following matrix polynomials:

$$F_1(z) = N_k(z) - \mathcal{K}_k D_k(z) \tag{179}$$

$$F_2(z) = D_k(z) - \mathcal{K}_k^{\dagger} N_k(z). \tag{180}$$

For any $m \times m$ polynomial matrix $P(z)$, let $\delta[P(z)]$ denote the number of determinantal zeros (i.e., zeros of $\det P(z)$) in the region $|z| > 1$. Clearly, the determinantal zeros of $F_1(z)$ (and $F_2(z)$) are the same as those of $z^{-1}N_{k-1}(z)$ (and $D_{k-1}(z)$) regardless of whether (172), (173) or (177), (178) are used.

We now draw some conclusions about the quantities $\delta[D_k(z)]$ and $\delta[N_k(z)]$ knowing the quantities $\delta[D_{k-1}(z)]$ and $\delta[N_{k-1}(z)]$. Let us begin by observing that, on the unit circle of the z-plane we have, by all-pass property,

$$N_k^{\dagger}(e^{j\omega})\, N_k(e^{j\omega}) = D_k^{\dagger}(e^{j\omega})\, D_k(e^{j\omega}). \tag{181}$$

Thus if $I_m - \mathcal{K}_k^{\dagger}\mathcal{K}_k > 0$, then we have

$$\mathcal{K}_k^{\dagger}\mathcal{K}_k < I_m \tag{182}$$

$$N_k^{\dagger}(e^{j\omega})\, N_k(e^{j\omega}) \geq [\mathcal{K}_k D_k(e^{j\omega})]^{\dagger}\, [\mathcal{K}_k D_k(e^{j\omega})] \tag{183}$$

$$D_k^{\dagger}(e^{j\omega})\, D_k(e^{j\omega}) \geq [\mathcal{K}_k^{\dagger} N_k(e^{j\omega})]^{\dagger}\, [\mathcal{K}_k^{\dagger} N_k(e^{j\omega})] \tag{184}$$

whereas if $I_m - \mathcal{K}_k^{\dagger}\mathcal{K}_k < 0$, then we have the above inequalities reversed. In order to be able to apply Rouche's theorem, we need to have strict inequality in (183), (184), under the condition of (182). Now let us examine the implication of a possible equality at frequency ω_0 in (183). This would imply the existence of a non-null vector y such that

$$y^{\dagger}N_k^{\dagger}(e^{j\omega_0})\, N_k(e^{j\omega_0})\, y = y^{\dagger}D_k^{\dagger}(e^{j\omega_0})\, \mathcal{K}_k^{\dagger}\mathcal{K}_k D_k(e^{j\omega_0})\, y. \tag{185}$$

But in view of the all-pass property (181), this implies

$$V^\dagger V = V^\dagger \mathcal{K}_k^\dagger \mathcal{K}_k V \quad (186)$$

where

$$V = D_k(e^{j\omega_0})\, y. \quad (187)$$

If $\mathcal{K}_k$ satisfies (182), then (186) implies $V = 0$. Thus from (181) we have

$$N_k(e^{j\omega_0})\, y = 0 = D_k(e^{j\omega_0})\, y. \quad (188)$$

Thus $N_k(z)$ and $D_k(z)$ have an eigenvalue of zero at $z = e^{j\omega_0}$, and the eigenvectors are the *same*. We now show that this situation is not possible, and hence establish that we have a strict inequality in (183). For this, recall that we assumed the MFD for $G_k(z)$ to be irreducible. As a result, there exists [19] a unimodular matrix $U(z)$

$$U(z) = \begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix} \quad (189)$$

such that, for all z,

$$\begin{bmatrix} U_{11}(z) & U_{12}(z) \\ U_{21}(z) & U_{22}(z) \end{bmatrix} \begin{bmatrix} D_k(z) \\ N_k(z) \end{bmatrix} = \begin{bmatrix} R(z) \\ 0 \end{bmatrix} \quad (190)$$

where $R(z)$ is also unimodular. But (188) and (190) imply that

$$R(e^{j\omega_0})\, y = 0 \quad (191)$$

which is not possible because $R(z)$, being unimodular, has a constant nonzero determinant.

In conclusion, therefore, we have the following strict inequalities on the unit circle, for $\mathcal{K}_k^\dagger \mathcal{K}_k < I_m$:

$$N_k^\dagger(z)\, N_k(z) > [\mathcal{K}_k D_k(z)]^\dagger\, [\mathcal{K}_k D_k(z)] \quad (192)$$

$$D_k^\dagger(z)\, D_k(z) > [\mathcal{K}_k^\dagger N_k(z)]^\dagger\, [\mathcal{K}_k^\dagger N_k(z)]. \quad (193)$$

If $\mathcal{K}_k^\dagger \mathcal{K}_k > I_m$, these (strict) inequalities are reversed. Next, notice that $N_k(z)$ and $D_k(z)$ are polynomials in z^{-1}, hence analytic in the region $|z| \geq 1$. Consequently, by the matrix version of Rouche's theorem (Appendix II), we have the following conclusion: *Let $G_M(z)$ be an $m \times m$ all-pass matrix with an irreducible MFD as in (149). Generate the sequence of $m \times m$ all-pass functions as in (152) and the sequence of $m \times m$ lattice coefficients as in (153). Then, if $I_m - \mathcal{K}_k^\dagger \mathcal{K}_k$ is definite for all k*

$$\left.\begin{aligned} \delta[z^{-1}N_{k-1}(z)] &= \delta[D_k(z)], \\ \delta[D_{k-1}(z)] &= \delta[N_k(z)], \end{aligned}\right\} \quad \text{for } \mathcal{K}_k^\dagger \mathcal{K}_k > I_m \quad (194)$$

and

$$\left.\begin{aligned} \delta[z^{-1}N_{k-1}(z)] &= \delta[N_k(z)], \\ \delta[D_{k-1}(z)] &= \delta[D_k(z)], \end{aligned}\right\} \quad \text{for } \mathcal{K}_k^\dagger \mathcal{K}_k < I_m. \quad (195)$$

In order to start the iterative count described by the above equations, we observe that $\delta[N_0(z)] = 0$ and $\delta[D_0(z)] = 0$. We can now work upwards with (194) and (195) to eventually obtain $\delta[D_M(z)]$, which represents the number of unstable poles of $G_M(z)$.

IX. Concluding Remarks

A number of stability test procedures for continuous- and discrete-time linear systems have been presented in a unified manner, based on lossless network synthesis. Some of the results presented are elaborations of the presentations based on the Nevanlinna–Pick problem [26], while others are based on different types of structure synthesis recently reported for discrete-time systems. Extensions to the multivariable case have also been included. We feel that some of these approaches can be extended to the case of two-dimensional systems [37] and to delay-differential systems [39]. The computational aspects of these problems have not been discussed. However, it is possible that, by using ad hoc specialized tests, one might be able to obtain simpler and more efficient numerical test procedures.

Appendix I

For a scalar complex-valued function $F(z)$, the maximum modulus theorem [9] can be stated as follows:

Maximum Modulus Theorem: Let $F(z)$ be analytic on and inside a simple closed contour C in the z-plane, and let M denote the maximum value attained by $|F(z)|$ on C. Then $|F(z)| < M$ everywhere inside C, unless $F(z)$ is a constant.

By identifying C with the unit circle, and making the change of variables $z \to z^{-1}$, we obtain the following conclusion:

If the discrete time transfer function $G(z)$ is BR, then $|G(z)| \leq 1$ for all $|z| \geq 1$. Moreover, unless $G(z)$ is a constant, $|G(z)| < 1$ for $|z| > 1$.

Based on this observation, we can derive the following conclusions if $G(z)$ is LBR (i.e., stable all-pass):

$$\begin{aligned} |G(z)| &> 1, &&\text{if } |z| < 1 \\ &< 1, &&\text{if } |z| > 1 \\ &= 1, &&\text{if } |z| = 1. \end{aligned} \quad (A1)$$

Next, for continuous-time systems, similar conclusions can be drawn by identifying C as the $j\omega$-axis. Thus if $H(s)$ is BR then $|H(s)| \leq 1$ for Re $s \geq 0$, and unless $H(s)$ is constant, $|H(s)| < 1$ for Re $s > 0$. Moreover, if $H(s)$ is LBR, then

$$\begin{aligned} |H(s)| &> 1, &&\text{if Re } s < 0 \\ &< 1, &&\text{if Re } s > 0 \\ &= 1, &&\text{if Re } s = 0. \end{aligned} \quad (A2)$$

A matrix version [29] of the above theorem is also useful. We next state it in a form suitable for our applications in this paper. The proof can be found in Potapov [29]. An alternate proof based on linear-system concepts can be found in [20].

Matrix Version of Maximum Modulus Theorem: Let $F(z)$ be analytic on and outside the unit circle of the z-plane and let

$$F^\dagger(e^{j\omega})\, F(e^{j\omega}) \leq M \quad (A3)$$

for all ω. Then $F(z)$ satisfies

$$F^\dagger(z)\, F(z) \leq M \quad (A4)$$

for all z outside the unit circle. In particular, if $F(z)$ is LBR, then $F^\dagger(z)\, F(z) \leq I$ for all z such that $|z| > 1$.

Notice that, even if $F(z)$ is not constant, (A4) does not have to be a strict inequality. This is unlike the case of the scalar version of this theorem.

Appendix II

Rouche's theorem of complex variable theory [15] can be stated as follows:

Rouche's Theorem: If $P(z)$ and $Q(z)$ are analytic interior to a simple closed contour C, and if they are continuous on C and if

$$|P(z)| < |Q(z)| \tag{A5}$$

on the contour C, then the function $F(z) = P(z) + Q(z)$ has the same number of zeros inside C as does $Q(z)$.

An extended version ([15, p. 5]) is more useful in certain situations; the extended version says that Rouche's theorem is valid even if $|P(z)| \le |Q(z)|$ on C provided that $F(z)$ defined above is not zero anywhere on C.

A matrix version of Rouche's theorem is also known [22]. We now state this in a form suitable for our application in this paper.

Matrix Version of Rouche's Theorem: Let $\boldsymbol{P}(z)$ and $\boldsymbol{Q}(z)$ be $m \times m$ matrix functions of z, analytic on and inside a simple closed curve C. Assume that each entry in the matrices is a ratio of two polynomials in z. Moreover, let

$$\boldsymbol{P}^{\dagger}(z)\,\boldsymbol{P}(z) < \boldsymbol{Q}^{\dagger}(z)\,\boldsymbol{Q}(z), \quad \text{on } C. \tag{A6}$$

Now consider the matrix function

$$\boldsymbol{F}(z) = \boldsymbol{P}(z) + \boldsymbol{Q}(z). \tag{A7}$$

Then the number of zeros of det $[\boldsymbol{F}(z)]$ inside C is the same as the number of zeros of det $[\boldsymbol{Q}(z)]$ inside C.

(In particular, one can also identify C to be the unit circle. A mapping of the form $z \to 1/z$ then enables us to replace the phrase "inside the unit circle" with the phrase "outside the unit circle.")

Proof of the Matrix Version: We include a proof because the matrix version does not seem to be well known. Our proof is a simplification of the one presented in [22].

Equation (A6) implies

$$\boldsymbol{y}^{\dagger}\boldsymbol{P}^{\dagger}(z)\,\boldsymbol{P}(z)\,\boldsymbol{y} < \boldsymbol{y}^{\dagger}\boldsymbol{Q}^{\dagger}(z)\,\boldsymbol{Q}(z)\,\boldsymbol{y}, \quad \text{on } C \tag{A8}$$

for all non-null $\boldsymbol{y}$. In particular

$$|\lambda|^2\boldsymbol{y}^{\dagger}\boldsymbol{P}^{\dagger}(z)\,\boldsymbol{P}(z)\,\boldsymbol{y} < \boldsymbol{y}^{\dagger}\boldsymbol{Q}^{\dagger}(z)\,\boldsymbol{Q}(z)\,\boldsymbol{y}, \quad \text{on } C \tag{A9}$$

for every λ such that $|\lambda| \le 1$ and for $\boldsymbol{y} \ne 0$. Let us now define a scalar function

$$h(z, \lambda) = \det[\lambda\boldsymbol{P}(z) + \boldsymbol{Q}(z)], \quad 0 \le \lambda \le 1. \tag{A10}$$

Note that $h(z, \lambda)$ is a polynomial in λ. We next claim that

$$h(z, \lambda) \ne 0, \quad \text{on } C. \tag{A11}$$

Indeed, if $h(z, \lambda) = 0$ for some z_0 on C, this would imply

$$\lambda\boldsymbol{P}(z_0)\,\boldsymbol{y} + \boldsymbol{Q}(z_0)\,\boldsymbol{y} = 0 \tag{A12}$$

for some $\boldsymbol{y} \ne 0$. This in turn means

$$|\lambda|^2\boldsymbol{y}^{\dagger}\boldsymbol{P}^{\dagger}(z_0)\,\boldsymbol{P}(z_0)\,\boldsymbol{y} = \boldsymbol{y}^{\dagger}\boldsymbol{Q}^{\dagger}(z_0)\,\boldsymbol{Q}(z_0)\,\boldsymbol{y}, \quad \text{on } C \tag{A13}$$

violating (A9).

Next, since $\boldsymbol{P}(z)$ and $\boldsymbol{Q}(z)$ are analytic on and inside C, the function $h(z, \lambda)$ is an analytic function of z on and inside C, for fixed λ. Thus by the argument theorem [9]

$$\frac{1}{2\pi j}\int_C \frac{h'(z, \lambda)}{h(z, \lambda)}\,dz = N_z - N_p \tag{A14}$$

where N_z and N_p are the number of zeros and poles of $h(z, \lambda)$ inside C. Clearly, $N_p = 0$, hence

$$N_z(\lambda) = \frac{1}{2\pi j}\int_C \frac{h'(z, \lambda)}{h(z, \lambda)}\,dz \tag{A15}$$

represents the number of zeros of $h(z, \lambda)$ inside C, as a function of λ. Note that the prime in (A15) denotes derivative with respect to z. Next consider the quantity

$$g(z, \lambda, \lambda_1) = \frac{h'(z, \lambda)}{h(z, \lambda)} - \frac{h'(z, \lambda_1)}{h(z, \lambda_1)}. \tag{A16}$$

Since $h(z, \lambda)$ is a polynomial in λ, and since (A16) is zero for $\lambda = \lambda_1$, the quantity $(\lambda - \lambda_1)$ is a factor of $g(z, \lambda, \lambda_1)$. Moreover, for all z on the contour C, $g(z, \lambda, \lambda_1)$ is finite for $0 \le \lambda, \lambda_1 \le 1$ in view of (A11). Consequently

$$N_z(\lambda) - N_z(\lambda_1) = \frac{(\lambda - \lambda_1)}{2\pi j}\int_C g_1(z, \lambda, \lambda_1)\,dz \tag{A17}$$

where $|g_1(z, \lambda, \lambda_1)| \le M < \infty$ for some fixed M, for all z on C. In other words, $N_z(\lambda)$ is continuous in λ for $0 \le \lambda \le 1$. But $N_z(\lambda)$ is integer valued, hence

$$N_z(0) = N_z(1). \tag{A18}$$

Consequently, the number of zeros of $h(z, 0)$ inside C is equal to the number of zeros of $h(z, 1)$ inside C. This completes the proof, in view of the definition of $h(z, \lambda)$ as in (A10).

Appendix III

In this Appendix we deal with all-pass functions that are memoryless in a certain direction. Specifically, let $\boldsymbol{F}(z)$ be an $m \times m$ all-pass transfer matrix, that is memoryless in the direction $\boldsymbol{V}$, i.e., $\boldsymbol{F}(z)\boldsymbol{V}$ = constant for all z. (We assume $\boldsymbol{V}^{\dagger}\boldsymbol{V} = 1$ without loss of generality.) We indicate here how we can obtain an $(m-1) \times (m-1)$ all-pass transfer matrix $\boldsymbol{F}^{(1)}(z)$ such that $\boldsymbol{F}(z)$ is stable *iff* $\boldsymbol{F}^{(1)}(z)$ is stable. For this, first consider the $m \times m$ transfer matrix

$$\boldsymbol{G}(z) = \boldsymbol{F}(z)\,\boldsymbol{S} \tag{A19}$$

where $\boldsymbol{S}$ is an $m \times m$ unitary matrix whose first column is $\boldsymbol{V}$. Clearly, $\boldsymbol{G}(z)$ is all-pass. Moreover

$$\boldsymbol{G}(z) = [\boldsymbol{c}\;\; \boldsymbol{G}_1(z)] \tag{A20}$$

where $\boldsymbol{c}\,(=\boldsymbol{F}(z)\,\boldsymbol{V})$ is a constant vector such that $\boldsymbol{c}^{\dagger}\boldsymbol{c} = 1$. Let us define another unitary matrix whose first row is $\boldsymbol{c}^{\dagger}$

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{c}^{\dagger} \\ \boldsymbol{C} \end{bmatrix}. \tag{A21}$$

Clearly, $\boldsymbol{R}\,\boldsymbol{G}(z)$ is also all-pass, and we have

$$\boldsymbol{R}\,\boldsymbol{G}(z) = \boldsymbol{R}\,\boldsymbol{F}(z)\,\boldsymbol{S} = \begin{bmatrix} 1 & 0 \\ 0 & \boldsymbol{F}^{(1)}(z) \end{bmatrix} \tag{A22}$$

where $\boldsymbol{F}^{(1)}(z)$ is $(m-1) \times (m-1)$ all-pass. Moreover, from (A19) and (A22), it is clear that $\boldsymbol{F}(z)$ is stable *iff* $\boldsymbol{F}^{(1)}(z)$ is stable. Fig. 21 is a circuit interpretation of this decoupling process and shows how $\boldsymbol{F}(z)$ can be implemented in terms of $\boldsymbol{F}^{(1)}(z)$.

If $\boldsymbol{F}^{(1)}(z)$ is also memoryless in a certain direction, we repeat the process and obtain an $(m-2) \times (m-2)$ all-pass function $\boldsymbol{F}^{(2)}(z)$. In this way we can eventually obtain an $(m-r) \times$

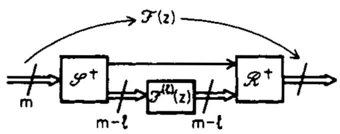

Fig. 21. Circuit interpretation of the decoupling process.

$(m - r)$ all-pass function $F^{(r)}(z)$ that it is not memoryless in any direction. Moreover, $F^{(r)}(z)$ is stable *iff* $F(z)$ is stable. It therefore suffices to test only the stability of $F^{(r)}(z)$.

Acknowledgment

The authors wish to thank Prof. E. I. Jury of the University of Miami, Coral Gables, FL, and Prof. Y. Neuvo of the Tampere University of Technology, Tampere, Finland, for useful comments and encouragements. The comments and suggestions by the anonymous reviewers are also gratefully acknowledged. This work was supported in part by the National Science Foundation under Grants DCI-850817 and ECS-8404245.

References

[1] E. A. Guillemin, *Synthesis of Passive Networks.* New York, NY: Wiley, 1975.
[2] A. Hurwitz, "On the conditions under which an equation has only roots with negative real parts," *Mathematische Annalen*, vol. 46, pp. 273–284, 1895.
[3] E. J. Routh, *Dynamics of System of Rigid Bodies*, 3rd ed. London, England: MacMillan, 1892.
[4] I. Schur, "Uber potenzreihen, die in innern des einheitskreises basehränkt sind," *J. für Mathematik*, vol. 147, pp. 205–232, 1917.
[5] A. Cohn, "Uber die anzahl der wurzeln einer algebraischen gleichung in einem kreise," *Math. Z.*, vol. 14, pp. 110–148, 1922.
[6] E. I. Jury, *Theory and Applications of the Z-Transform Method.* New York, NY: Wiley, 1974.
[7] K. J. Åström, *Introduction to Stochastic Control Theory.* New York, NY: Academic Press, 1970.
[8] B. D. O. Anderson, N. K. Bose, and E. I. Jury, "A simple test for zeros of a complex polynomial in a sector," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 437–438, Aug. 1974.
[9] E. C. Titchmarsh, *The Theory of Functions.* London, England: Oxford Univ. Press, 1968.
[10] S. K. Mitra, P. S. Kamat, and D. C. Huey, "Cascaded lattice realization of digital filters," *Int. J. Circuit Theory Appl.*, vol. 5, pp. 3–11, 1977.
[11] P. P. Vaidyanathan and S. K. Mitra, "Low passband sensitivity digital filters: A generalized viewpoint and synthesis procedures," *Proc. IEEE*, vol. 72, pp. 404–423, Apr. 1984.
[12] S. K. Mitra, P. P. Vaidyanathan, and B. D. O. Anderson, "A general theory and synthesis procedure for low-sensitivity active *RC* filters," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 687–699, July 1985.
[13] P. P. Vaidyanathan, "A unified approach to orthogonal digital filters and wave digital filters based on LBR two-pair extraction," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 673–686, July 1985.
[14] A. H. Gray, Jr. and J. D. Markel, "Digital lattice and ladder filter synthesis," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, no. 6, pp. 491–500, Dec. 1973.
[15] M. Marden, "Geometry of polynomials," Amer. Math. Soc., Providence, RI, 1966.
[16] P. P. Vaidyanathan and S. K. Mitra, "A general theory and synthesis procedure for low sensitivity digital filter structures," Tech. Rep. 82-09, Dep. Elec. Comput. Eng., Univ. California, Santa Barbara, 1982.
[17] S. Gutman and E. I. Jury, "A general theory for matrix root-clustering in subregions of the complex plane," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 853–863, Aug. 1981.
[18] G. H. Hostetter, "An improved test for the zeros of a polynomial in a sector," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 433–434, June 1975.
[19] T. Kailath, *Linear Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1980.
[20] P. P. Vaidyanathan and S. K. Mitra, "A general family of multivariable digital lattice filters," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 1234–1245, Dec. 1985.
[21] A. H. Gray, Jr. and J. D. Markel, "A normalized digital filter structure," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, no. 3, pp. 268–277, June 1975.
[22] Y. Monden and S. Arimoto, "Generalized Rouche's theorem and its applications to multivariate autoregressions," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 733–738, Dec. 1980.
[23] D. C. Youla, "On the factorization of rational matrices," *IRE Trans. Informat. Theory*, vol. IT-7, pp. 172–189, July 1961.
[24] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing.* Englewood Cliffs, NJ: Prentice-Hall, 1975.
[25] G. C. Temes and J. W. LaPatra, *Circuit Synthesis and Design.* New York, NY: McGraw-Hill, 1977.
[26] Ph. Delsarte, Y. Genin, and Y. Kamp, "On the role of the Nevanlinna-Pick problem in circuit and system theory," *Int. J. Circuit Theory Appl.*, vol. 9, pp. 177–187, 1981.
[27] D. Williamson, "A note on stability and lattice filter realizations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 770–772, June 1983.
[28] D. C. Youla, and M. Saito, "Interpolation with positive-real functions," *J. Franklin Inst.*, vol. 284, pp. 77–108, Aug. 1967.
[29] V. P. Potapov, "The multiplicative structure of *J*-contractive matrix functions," in *Amer. Math. Soc., Translation Ser. 2*, vol. 15, pp. 131–243, 1960.
[30] Ph. Delsarte, Y. Genin, and Y. Kamp, "The Nevanlinna-Pick problem for matrix-valued functions," *SIAM J. Appl. Math.*, vol. 36, no. 1, pp. 47–61, Feb. 1979.
[31] M. R. Stojic and D. D. Siljak, "Generalization of Hurwitz, Nyquist, and Mikhailov stability criteria," *IEEE Trans. Automat. Contr.*, vol. AC-10, pp. 250–254, Apr. 1965.
[32] D. D. Siljak, "Algebraic criteria for positive realness relative to the unit circle," *J. Franklin Inst.*, vol. 296, pp. 116–122, Aug. 1973.
[33] S. Barnett and D. D. Siljak, "Routh's algorithm: A centennial survey," *SIAM Rev.*, vol. 19, no. 3, pp. 472–489, July 1977.
[34] F. R. Gantmacher, *The Theory of Matrices.* New York, NY: Chelsa, 1974.
[35] S. M. Ahn, "Stability of a matrix polynomial in discrete systems," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 1122–1124, Oct. 1982.
[36] B. D. O. Anderson and R. E. Bitmead, "Stability of matrix polynomials," *Int. J. Contr.*, vol. 26, pp. 235–247, 1977.
[37] E. I. Jury, "Stability of multidimensional scalar and matrix polynomials," *Proc. IEEE*, vol. 66, pp. 1018–1047, Sept. 1978.
[38] S. D. Agashe, "A new general Routh-like algorithm to determine the number of RHP roots of a real or complex polynomial," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 406–409, Apr. 1985.
[39] D. Hertz, E. I. Jury, and E. Zeheb, "Stability independent and dependent delay for delay differential systems," *J. Franklin Inst.*, vol. 318, pp. 143–150, Sept. 1984.

P. P. Vaidyanathan (Member, IEEE) was born in Calcutta, India, on October 16, 1954. He received the B.Sc. (Hons.) degree in physics, and the B. Tech. and M. Tech. degrees in radiophysics and electronics from the University of Calcutta, India, in 1974, 1977, and 1979, respectively, and the Ph.D. degree in electrical and computer engineering from the University of California, Santa Barbara, in 1982.

He was a Postdoctoral Fellow at the University of California, Santa Barbara, from September 1982 to February 1983. Since March 1983 he has been with the California Institute of Technology, Pasadena, as an Assistant Professor of Electrical Engineering. His main research interests are in digital signal processing, linear systems, and filter design. He was the recipient of the Award for Excellence in Teaching at the California Institute of Technology for 1983–1984, and a recipient of NSF's Presidential Young Investigator Award, starting from the year 1986.

Dr. Vaidyanathan served as the Vice Chairman of the Technical Program Committee for the 1983 IEEE International Symposium on Circuits and Systems. He currently serves as an Associate Editor of the IEEE Transactions on Circuits and Systems.

Sanjit K. Mitra (Fellow, IEEE) received the B.S. (Hons.) degree in physics from the Utkal University, India, in 1953; the M.Sc. (Tech.) degree in radio physics and electronics from the University of Calcutta, India, in 1956; and the M.S. and Ph.D. degrees in electrical engineering from the University of California, Berkeley, in 1960 and 1962, respectively.

He was a member of the faculty of the Cornell University, Ithaca, NY, from 1962 to 1965, and a member of Technical Staff of the Bell Laboratories from 1965 to 1967. He joined the faculty of the University of California, Davis, in 1967 and transferred to the Santa Barbara (UCSB) campus in 1977 as a professor of Electrical and Computer Engineering. From July 1979 to June 1982 he served as the Chairman of the Department of Electrical and Computer Engineering at UCSB. He is a consultant to the Lawrence Livermore National Laboratory, Livermore, CA, and the Los Alamos National Laboratory, Los Alamos, NM, and is the Consulting Editor for the *Electrical/Computer Science and Engineering Series* of the Van Nostrand Reinhold Company, New York. He has also served as a consultant to Ampex Corporation, Fairchild Semiconductor, Siliconix, HDR Sciences, the U.S. Army, General Motors Delco Division, and the World Bank. He has held visiting appointments at the Indian Institute of Technology, New Delhi, India; Kobe University, Japan; University of Erlangen-Nuernberg, West Germany, the Australian National University, Canberra, Australia; Tampere University of Technology, Tampere, Finland; the Federal University of Rio de Janeiro, Brazil, and the University of Zagreb, Yugoslavia. He has published a number of papers in active and passive networks, and digital filters, and is the author of *Analysis and Synthesis of Linear Active Networks* and *An Introduction to Digital and Analog Integrated Circuits and Applications* and Editor of *Active Inductorless Filters*, Co-Editor of *Modern Filter Theory and Design* (with G. C. Temes), Co-Editor of *Two Dimensional Digital Signal Processing* (with M. P. Ekstrom). He holds two patents in active filters. He is the recipient of the 1973 F. E. Terman Award and 1985 AT&T Foundation Award of the American Society of Engineering Education, a Visiting Professorship from the Japan Society for Promotion of Science in 1972, and Fulbright Distinguished Lecturer Award for Brazil in 1984, and for Yugoslavia in 1986.

Dr. Mitra is a Fellow of the American Association for the Advancement of Science and a member of the American Society for Engineering Education, Sigma Xi and Eta Kappa Nu. He is a member of the Advisory Board of the George R. Brown School of Engineering of the Rice University, Houston, TX, and an Advisory Professor of the Northern Jiaotong University, Beijing, China. He was an Associate Editor of the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS, and served on the editorial boards of the PROCEEDINGS OF THE IEEE and IEEE PRESS, and presently is on the editorial board of *Circuits, Systems and Signal Processing*. He has been a member of the Administrative Committee of the IEEE Circuits and Systems Society, Local Arrangements Chairman (1969), General Chairman (1974), and Program Chairman (1983) of the Annual IEEE International Symposium on Circuits and Systems and is presently the Past President of the IEEE Circuits and Systems Society.