

## Estimation of absolute and relative permeabilities in petroleum reservoirs

Tai-yong Lee and John H Seinfeld

Department of Chemical Engineering, California Institute of Technology, Pasadena, CA 91125, USA

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**Abstract.** The estimation of absolute and relative permeabilities for petroleum reservoirs on the basis of noisy data at wells is considered. The spatially varying absolute permeability is estimated by regularisation combined with a bi-cubic spline approximation. Relative permeability is represented by a given function of saturation with unknown coefficients. Numerical results provide an indication of the estimability of the two permeabilities in conventional petroleum production operations.

### 1. Introduction

Once wells have been drilled down into a reservoir containing a recoverable petroleum, the local properties of the reservoir rocks and fluids must be determined. A variety of complex acoustical, electronic and magnetic techniques are available that, when lowered into the well, can be used to determine the local properties of the formation and fluids in the neighbourhood of the well. Estimates of the reservoir properties are needed, however, throughout the entire reservoir, not just at the wells, in order to simulate various production strategies to try to optimise the recovery of the petroleum. To estimate the properties on the reservoir, past production histories are simulated. The properties are determined as those that produce the closest possible match of the observed and predicted histories. This so-called history-matching process has been addressed in the petroleum, hydrology and mathematics literature for some 20 years or so (Jacquard and Jain 1965, Neuman 1973, Carter *et al* 1974, Chen *et al* 1974, Chavent *et al* 1975, Wasserman *et al* 1975, Chen and Seinfeld 1975, Yoon and Yeh 1976, Gavalas *et al* 1976, Van den Bosch and Seinfeld 1977, Shah *et al* 1978, Seinfeld and Chen 1978, Neuman and Yakowitz 1979, Yakowitz and Duckstein 1980, Yeh and Yoon 1981, Yeh *et al* 1983, Tang and Chen 1984, Watson *et al* 1984, Neuman and Carrera 1985, Sun and Yeh 1985, Yeh 1986, Carrera and Neumann 1986, Lee *et al* 1986, Lee and Seinfeld 1986).

In the early stages of production of a petroleum reservoir, it can often be assumed that the reservoir contains only a single fluid, oil. In that case the reservoir behaviour is described by a single linear parabolic PDE for pressure. The reservoir parameters that enter the equation, and are subject to estimation, are the rock porosity  $\phi$  and the absolute permeability  $k$ , both of which vary with location in the reservoir. Most of the above cited references are addressed to the case of a single-phase reservoir. (In case of an aquifer, although the reservoir fluid is water, the pressure is governed by the same PDE as in the

case of an oil reservoir.) Generally one must account for the fact that oil and water are present together in petroleum reservoirs, and the resulting reservoir model consists of two coupled nonlinear PDEs. In addition to the porosity  $\varphi$  and absolute permeability  $k$ , the two-phase case is characterised by the relative permeabilities  $k_{ro}$  and  $k_{rw}$  (subscript o referring to oil and w to water) that are presumed to be functions of the local fluid saturation in the medium. The precise values of the two relative permeabilities are usually not known.

The essential difficulties in the petroleum reservoir inverse problem are twofold. First, the reservoir properties are spatially varying, and the estimation of a spatially varying permeability is well known to be an ill-posed problem (Chavent 1979a, b, Seinfeld and Kravaris 1982, Kravaris and Seinfeld 1985). Second, the oil–water reservoir is a highly nonlinear system, for which rigorous results concerning its inverse problems do not exist.

The ill-posed nature of the single-phase permeability estimation problem has been attacked by Bayesian approaches (Gavalas *et al* 1976, Shah *et al* 1978), regularisation (Neuman and de Marsily 1976, Seinfeld and Kravaris 1982, Tang and Chen 1984, Kravaris and Seinfeld 1985, 1986, Lee *et al* 1986) and spline approximation (Banks and Lamm 1985). While the Bayesian approach requires *a priori* statistical information on the unknown parameters that may not generally be available, and spline approximation, in and of itself, does not guarantee well posedness, the regularisation approach offers both rigorous stability and convenient computational implementation. The first step of the regularisation formulation is to measure the non-smoothness of the parameter by its norm in an appropriate Hilbert space, called the stabilising functional, and then to seek the value of the parameter that minimises the weighted sum of the least-squares discrepancy term and the stabilising functional. In previous applications of regularisation to the petroleum reservoir inverse problem, Lee *et al* (1986) estimated absolute permeability and porosity in a single-phase reservoir and Lee and Seinfeld (1986) estimated the absolute permeability in a two-phase reservoir.

The object of the present paper is to develop an algorithm for the simultaneous estimation of absolute and relative permeabilities in two-phase petroleum reservoirs. In related work on two-phase reservoirs Van den Bosch and Seinfeld (1977) investigated the estimation of constant absolute permeability and porosity and relative permeabilities near a single producing well where radial symmetry can be exploited. Watson *et al* (1984) estimated absolute permeability, porosity and relative permeabilities simultaneously assuming that the absolute permeability and porosity are each a constant independent of location. The present paper addresses the more practical case in which the absolute permeability is spatially varying. (Since the porosity is generally less variable than the permeability and is also better identified, we do not consider its estimation here.)

The next section defines the mathematical model of the oil–water petroleum reservoir. Section 3 then defines the inverse problem associated with estimating absolute and relative permeabilities. In § 4 we present a numerical regularisation algorithm, and § 5 is devoted to a detailed computational example.

## 2. Mathematical model of two-phase petroleum reservoir

Consider a two-dimensional oil–water reservoir that has sufficiently large areal extent so that we can assume that the pressure change and hence the flow in the vertical direction is negligible compared with that in the other two directions (Aziz and Settari 1983, pp 204–243). Assuming that the oil and water phases are immiscible, the equations of

mass conversation for the oil and water phases are

$$\frac{\partial}{\partial t} (\rho_o \varphi S_o) + \nabla \cdot (\rho_o \mathbf{v}_o) = \sum_{\kappa=1}^{N_w} \rho_o q_{o\kappa} \frac{\delta(x-x_\kappa)\delta(y-y_\kappa)}{h} \quad (1)$$

$$\frac{\partial}{\partial t} (\rho_w \varphi S_w) + \nabla \cdot (\rho_w \mathbf{v}_w) = \sum_{\kappa=1}^{N_w} \rho_w q_{w\kappa} \frac{\delta(x-x_\kappa)\delta(y-y_\kappa)}{h} \quad (2)$$

for  $(x, y) \in \Omega$  and  $0 < t < T$  where  $S_o$  and  $S_w$ , the volume fractions of oil and water with respect to the total fluid volume, called oil and water saturations, respectively, satisfy  $S_o \equiv 1 - S_w$ . The oil–water reservoirs that do not include gas phase generally are slightly compressible systems, i.e., the porosity,  $\varphi$ , and the density of oil,  $\rho_o$ , and water,  $\rho_w$ , are weak functions of pressure. It is customary that the functional dependencies are given by  $c_f = (1/\varphi)(d\varphi/dp)$ ,  $c_o = (1/\rho_o)(d\rho_o/dp)$  and  $c_w = (1/\rho_w)(d\rho_w/dp)$  where  $c_f$ ,  $c_o$  and  $c_w$  denote the compressibilities of rock, oil and water and are assumed to be constant over the entire region of pressure change of the reservoir (Aziz and Settari 1983, p 13). The volumetric flow rates of the water and oil phases at the wells located at  $(x_\kappa, y_\kappa)$  are denoted by  $q_{o\kappa}$  and  $q_{w\kappa}$ ,  $\kappa = 1, \dots, N_w$ . For injection wells  $q_o = 0$  and  $q_w > 0$ . For production wells,  $q_o$  and  $q_w$  are negative and the ratio  $q_w/q_o$  is proportional to the ratio of the local flow velocities of water to oil at the bottom of wells. The thickness of the reservoir,  $h$ , is assumed to be constant over the whole reservoir domain†. The linear velocities of the oil and water phases are assumed to be described by Darcy's Law,

$$\mathbf{v}_o = -\frac{kk_{ro}}{\mu_o} \nabla p \quad (3)$$

$$\mathbf{v}_w = -\frac{kk_{rw}}{\mu_w} \nabla p \quad (4)$$

where the absolute permeability  $k$  is a parameter characterising the fluid conductivity of a porous medium,  $\mu_o$  and  $\mu_w$  are the viscosities of oil and water, respectively, and the relative permeabilities of oil and water,  $k_{ro}$  and  $k_{rw}$ , respectively, are assumed to be functions of fluid (water) saturation within the porous medium independent of flow rate and fluid properties. Widely used functional forms of the relative permeabilities, and those employed in this study, are

$$k_{ro}(S_w) = a_o \left( \frac{1 - S_{ro} - S_w}{1 - S_{ro} - S_{iw}} \right)^{b_o} \quad (5)$$

$$k_{rw}(S_w) = a_w \left( \frac{S_w - S_{iw}}{1 - S_{ro} - S_{iw}} \right)^{b_w} \quad (6)$$

for  $S_{iw} \leq S_w \leq 1 - S_{ro}$  where irreducible (or connate) water saturation,  $S_{iw}$ , and residual oil saturation,  $S_{ro}$ , are the lower bounds of  $S_w$  and  $S_o$ , respectively, under which water and oil, respectively, become immobile with reasonable pressure and gradients. The relative permeabilities are each less than unity and typically their sum is also less than unity for  $S_{iw} < S_w < 1 - S_{ro}$  (Collins 1961, pp 53–5). Equations (1)–(6), together with the no-flux boundary condition,

$$\mathbf{n} \cdot \nabla p = 0 \quad (7)$$

† If  $h$  is spatially varying then the integrated properties  $hk$  and  $h\varphi$  are subject to estimation instead of  $k$  and  $\varphi$ , respectively, in the reservoir parameter estimation problem, but it does not change the structure of problems.

for  $(x, y) \in \partial \Omega$  and  $0 < t < T$ , and the given initial conditions

$$p(x, y, 0) = p_0(x, y) \quad (8)$$

$$S_w(x, y, 0) = S_{w0}(x, y) \quad (9)$$

for  $(x, y) \in \Omega$  describe the water-driven oil recovery process for a petroleum reservoir with an impermeable boundary. Equations (1)–(9) are solved numerically using the finite difference approximation. Physically these equations describe the movement of both phases, usually as water is intentionally pumped down certain wells to drive the oil in place toward other wells where it is produced. When the water breaks through at the production wells, the displacement process is considered to be complete.

### 3. The inverse problem

It is desired to estimate simultaneously the absolute permeability  $k$  and the relative permeabilities,  $k_{ro}$  and  $k_{rw}$ , from data normally available at wells that have been drilled into the reservoir. Since  $k_{ro}$  and  $k_{rw}$  are assumed to be given by equations (5) and (6), their estimation reduces to that of the unknown constant parameters  $a_o$ ,  $a_w$ ,  $b_o$  and  $b_w$ . In general,  $a_o$  and  $a_w$  can be determined if the values of  $k_{ro}$  and  $k_{rw}$  are known at two points such as at the connate water or residual oil saturations. Thus,  $b_o$  and  $b_w$  are the more uncertain and will be the subject of estimation here. The measured data consist of the pressure at  $N_o$  wells and at  $N_t$  discrete times over  $0 < t < T$  and of the water fraction of the total flow at each well,

$$f_w = \frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{ro}/\mu_o} \quad (10)$$

The usual least-squares objective function consists of two contributions, one each from the pressure and the water flow observations. We define  $\sigma_p^2$  as the mean-square error between the calculated and measured pressure data

$$\sigma_p^2 = \frac{1}{N_o N_t} \sum_{n=1}^{N_t} \sum_{\nu=1}^{N_o} (p(x_\nu, y_\nu, t_n) - p^{obs n})^2 \quad (11)$$

where  $(x_\nu, y_\nu) \in \Omega$ ,  $\nu = 1, \dots, N_o$  denote the locations of the observations, that is the wells, and  $t_n$ ,  $n = 1, \dots, N_t$  are the observed times. Similarly, we define  $\sigma_f^2$  as the mean-square error in the water flow data,

$$\sigma_f^2 = \frac{1}{N_o N_t} \sum_{n=1}^{N_t} \sum_{\nu=1}^{N_o} (f_w(x_\nu, y_\nu, t_n) - f_w^{obs n})^2. \quad (12)$$

Then the least-squares objective function is given by a weighted sum of the two contributions

$$J_{LS}(k, b_o, b_w) = W_p \sigma_p^2 + W_f \sigma_f^2 \quad (13)$$

where  $W_p$  and  $W_f$  are the weighting coefficients for the pressure and flow rate terms, respectively.

The conventional least-squares identification problem is to estimate  $k(x, y)$ ,  $b_o$  and  $b_w$  to minimise  $J_{LS}$ . The spatial variation of  $k$  leads to an ill-posed inverse problem, and hence we turn to a regularisation formulation. Kravaris and Seinfeld (1985, 1986) introduced the

concept of regularisation for the estimation of coefficients in PDEs. Regularisation of a problem refers to solving a problem related to the original problem, called the regularised problem, the solution of which is both ‘regular’ and approximates the solution of the original problem. In Tikhonov’s regularisation formulation (Tikhonov and Arsenin 1977), the measure of non-smoothness of the parameter being estimated, called the stabilising functional, is represented by a norm of the parameter in an appropriate Hilbert space, for example

$$J_{ST}(k) = \|k\|_{H^3(\Omega)}^2 \quad (14)$$

where the Sobolev  $H^3(\Omega)$  is the set of functions that are square integrable over  $\Omega$  and have square-integrable derivatives up to order 3. More precisely Tikhonov’s stabilising functional is given by

$$J_{ST}(k) = \sum_{m=0}^3 \zeta_m \int \int_{\Omega} \sum_{\nu=0}^m \binom{m}{\nu} \left( \frac{\partial^m k(\xi, \eta)}{\partial \xi^\nu \partial \eta^{m-\nu}} \right)^2 d\xi d\eta \quad (15)$$

where convenient dimensionless variables are  $\xi = N_x x / x_L$  and  $\eta = N_y y / y_L$ , where  $x_L$  and  $y_L$  are the lateral reservoir dimensions, and  $N_x$  and  $N_y$  are the number of PDE grid cells employed along the  $x$  and  $y$  directions, respectively. The conditions for the coefficients  $\zeta_m$  are  $\zeta_0 > 0$ ,  $\zeta_1 > 0$ ,  $\zeta_2 > 0$  and  $\zeta_3 > 0$  (Tikhonov 1963); or  $\zeta_0 \geq 0$ ,  $\zeta_1 \geq 0$ ,  $\zeta_2 \geq 0$  and  $\zeta_3 > 0$  (Tikhonov and Arsenin 1977, pp 69–70). As Trummer (1984) has pointed out, using the stabilising functional that includes the Euclidean norm of the parameter itself leads to the underestimation of the parameter. Locker and Prenter (1980) suggested the use of a stabilising functional with a differential operator. Lee and Seinfeld (1986) used the stabilising functional with the gradient operator ( $\nabla$ ) so that it does not include the Euclidean norm term ( $\zeta_0 \equiv 0$  in equation (15)) for the estimation of absolute permeability.

The regularisation formulation of the inverse problem seeks the minimum of the smoothing functional,

$$J_{SM}(k, b_o, b_w; \beta) = J_{LS}(k, b_o, b_w) + \beta J_{ST}(k) \quad (16)$$

where  $\beta$  is the regularisation parameter that represents the relative importance given to  $J_{ST}$ . In the present problem,  $J_{LS}$  is composed of the two terms as shown in equation (13), hence  $J_{SM}$  includes three quantities,  $W_p \sigma_p^2$ ,  $W_f \sigma_f^2$  and  $\beta J_{ST}$  where two of the three weighting coefficients  $W_p$ ,  $W_f$  and  $\beta$  must be determined independently.  $W_f/W_p$  can be chosen as the ratio  $\tilde{\sigma}_p^2/\tilde{\sigma}_f^2$  where  $\tilde{\sigma}_p^2$  and  $\tilde{\sigma}_f^2$  denote the variances associated with the pressure and production data measurements, respectively (Watson *et al* 1980). In the present study,  $\tilde{\sigma}_p^2/\tilde{\sigma}_f^2$  is assumed to be known and  $W_f/W_p$  is chosen as that value. An important question regarding the regularisation method is determining a suitable value of  $\beta$  for the given noisy data especially where the noise level may or may not be known. The value of  $\beta$  is chosen in several different ways (Miller 1970, Tikhonov and Arsenin 1977, pp 87–94, Craven and Wahba 1979). Miller suggest that  $\beta$  be determined from the ratio of an upper bound of the measurement error to an upper bound of the measure of non-smoothness. Craven and Wahba (1979) used the method of generalised cross validation (GCV) to determine the regularisation parameter. Since GCV requires parametric sensitivity information, this method is not practical for such a large scale problem like reservoir parameter estimation. Lee and Seinfeld (1986) developed an algorithm based on Miller’s idea that determines the regularisation parameters automatically during the estimation process without requiring a *a priori* information.

The absolute permeability in a two-phase reservoir is primarily estimated from the pressure data (Van den Bosch and Seinfeld 1977, Watson *et al* 1984). Thus we can

determine  $\beta/W_p$  from the ratio of an upper bound of  $\sigma_p^2$  to an upper bound of  $J_{ST}$ . In practice, these values are usually not known and Lee and Seinfeld (1986) used the values of  $J_{ST}$  and the pressure discrepancy of the results of the non-regularised ( $\beta=0$ ) estimation to determine  $\beta$ . Without loss of generality  $W_p$  will be specified as  $1/\bar{\sigma}_p^2$ .

Spline approximation of spatially-varying parameters has several merits including a built-in smoothing and computational convenience (Banks and Lamm 1985, Kravaris and Seinfeld 1986). The spline approximation of the spatially varying absolute permeability is given by

$$k(x, y) = \sum_{l_y=1}^{N_{ys}} \sum_{l_x=1}^{N_{xs}} \chi^{*4} \left( 4 - l_x + \frac{x}{\Delta x_s} \right) \chi^{*4} \left( 4 - l_y + \frac{y}{\Delta y_s} \right) W_l \quad (17)$$

where  $\chi^{*4}(\theta)$  is a cubic B-spline function,

$$\chi^{*4}(\theta) = \begin{cases} \frac{1}{6}\theta^3 & \theta \in [0, 1] \\ \frac{1}{6} + \frac{1}{2}(\theta-1) + \frac{1}{2}(\theta-1)^2 + \frac{1}{2}(\theta-1)^3 & \theta \in [1, 2] \\ \frac{2}{3} - (\theta-2)^2 + \frac{1}{2}(\theta-2)^3 & \theta \in [2, 3] \\ \frac{1}{6} - \frac{1}{2}(\theta-3) + \frac{1}{2}(\theta-3)^2 - \frac{1}{6}(\theta-3)^3 & \theta \in [3, 4] \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

$\Delta x_s$  and  $\Delta y_s$  are the grid spacings for the spline approximation and  $l = l_x + N_{xs}(l_y - 1)$  for  $l_x = 1, \dots, N_{xs}$  and  $l_y = 1, \dots, N_{ys}$ . In applying the spline approximation to the parameter estimation problem, if the number of spline coefficients,  $N_{xs} \times N_{ys} = N_s$ , is too few, then the spline approximation cannot represent the spatial details properly. On the other hand, the number of spline coefficients should not exceed the number of grid cells for the solution of the PDES. When the spline approximation is used together with regularisation, the smoothing power of the spline approximation becomes less important than in its absence and  $N_{xs}$  and  $N_{ys}$  can be chosen as large as the numbers of grid cells along the  $x$  and  $y$  directions for the solution of the PDES (Lee and Seinfeld 1986). The unknown parameters characterising  $k(x, y)$  are now  $W_l$ ,  $l = 1, \dots, N_s$ .

The mathematical theory of regularisation does not suggest any guidelines about the highest order of the derivative term that is included in equation (15). It is clear that in the case of discrete regularisation with the spline approximation the choice of Sobolev space is closely related to the choice of spline function. We choose the Sobolev space  $H^3(\Omega)$  so that all nontrivial derivatives of cubic B-spline functions contribute to the evaluation of the stabilising functional.

#### 4. Numerical algorithm

The reservoir parameter estimation problem is a large nonlinear least-squares problem. The number of unknown parameters to be estimated is the same magnitude as the number of grid cells for the discretisation of the PDES. That number is of the order of at least one hundred in the field applications. In general, because of the size of the estimation problem a minimisation that requires the first derivative of the objective function is preferred over one that requires the second derivatives. The first-order derivatives of the least-squares performance index can be derived using optimal control theory (Chen *et al* 1974, Chavent

et al 1975). The functional derivative of  $J_{LS}$  with respect to  $k(x, y)$  is

$$\frac{\delta J_{LS}}{\delta k} = - \int_0^T \left( \frac{k_{rw}}{\mu_w} \nabla \psi_w \cdot \nabla p + \frac{k_{ro}}{\mu_o} \nabla \psi_o \cdot \nabla p \right) dt \quad (19)$$

and the partial derivatives of  $J_{LS}$  with respect to  $b_o$  and  $b_w$  are

$$\begin{aligned} \frac{\partial J_{LS}}{\partial b_o} = & - \int_0^T \int \int_{\Omega} dx dy dt \left[ \frac{k}{\mu_o} \frac{\partial k_{ro}}{\partial b_o} \nabla \psi_o \cdot \nabla p \right. \\ & + \sum_{\kappa=1}^{N_w} \left( \psi_w \frac{\partial q_{w\kappa}}{\partial b_o} + \psi_o \frac{\partial q_{o\kappa}}{\partial b_o} \right) \frac{\delta(x-x_{\kappa})\delta(y-y_{\kappa})}{h} \\ & \left. + \frac{2W_f}{N_o N_t} \sum_{n=1}^{N_t} \sum_{\nu=1}^{N_o} (f_w - f_w^{\text{obs } n}) \frac{\partial f_w}{\partial b_o} \delta(x-x_{\nu})\delta(y-y_{\nu})\delta(t-t_n) \right] \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{\partial J_{LS}}{\partial b_w} = & - \int_0^T \int \int_{\Omega} dx dy dt \left[ \frac{k}{\mu_w} \frac{\partial k_{rw}}{\partial b_w} \nabla \psi_w \cdot \nabla p \right. \\ & + \sum_{\kappa=1}^{N_w} \left( \psi_w \frac{\partial q_{w\kappa}}{\partial b_w} + \psi_o \frac{\partial q_{o\kappa}}{\partial b_w} \right) \frac{\delta(x-x_{\kappa})\delta(y-y_{\kappa})}{h} \\ & \left. + \frac{2W_f}{N_o N_t} \sum_{n=1}^{N_t} \sum_{\nu=1}^{N_o} (f_w - f_w^{\text{obs } n}) \frac{\partial f_w}{\partial b_w} \delta(x-x_{\nu})\delta(y-y_{\nu})\delta(t-t_n) \right]. \end{aligned} \quad (21)$$

The adjoint variables  $\psi_o$  and  $\psi_w$  satisfy the following adjoint equations

$$\begin{aligned} & -(c_w + c_f) \frac{\partial}{\partial t} (\varphi S_w \psi_w) - (c_o + c_f) \frac{\partial}{\partial t} (\varphi S_o \psi_o) \\ & - \nabla \cdot \left( \frac{kk_{rw}}{\mu_w} \nabla \psi_w + \frac{kk_{ro}}{\mu_o} \nabla \psi_o \right) \\ & = \frac{2W_p}{N_o N_t} \sum_{n=1}^{N_t} \sum_{\nu=1}^{N_o} (p - p^{\text{obs } n}) \delta(x-x_{\nu})\delta(y-y_{\nu})\delta(t-t_n) \end{aligned} \quad (22)$$

from the terms including the variation of  $p$  and

$$\begin{aligned} & -\frac{\partial}{\partial t} [\varphi(\psi_w - \psi_o)] + \varphi((c_w + c_f)\psi_w - (c_o + c_f)\psi_o) \frac{\partial p}{\partial t} \\ & + \frac{k}{\mu_w} \frac{\partial k_{rw}}{\partial S_w} \nabla \psi_w \cdot \nabla p + \frac{k}{\mu_o} \frac{\partial k_{ro}}{\partial S_w} \nabla \psi_o \cdot \nabla p \\ & - \sum_{\kappa=1}^{N_w} \left( \psi_w \frac{\partial q_{w\kappa}}{\partial S_w} + \psi_o \frac{\partial q_{o\kappa}}{\partial S_w} \right) \frac{\delta(x-x_{\kappa})\delta(y-y_{\kappa})}{h} \\ & = \frac{2W_f}{N_o N_t} \sum_{n=1}^{N_t} \sum_{\nu=1}^{N_o} (f_w - f_w^{\text{obs } n}) \frac{\partial f_w}{\partial S_w} \delta(x-x_{\nu})\delta(y-y_{\nu})\delta(t-t_n) \end{aligned} \quad (23)$$

from the terms including the variation of  $S_w$  for  $(x, y) \in \Omega$  and  $0 < t < T$  with the terminal constraints

$$\psi_o(x, y, T) = 0 \quad (24)$$

$$\psi_w(x, y, T) = 0 \quad (25)$$

for  $(x, y) \in \Omega$  and the boundary condition

$$\mathbf{n} \cdot \left( \frac{kk_{rw}}{\mu_w} \nabla \psi_w + \frac{kk_{ro}}{\mu_o} \nabla \psi_o \right) = 0 \quad (26)$$

for  $(x, y) \in \partial\Omega$  and  $0 < t < T$ . The discrete adjoint equation can be derived from the discrete reservoir equations (1)–(9), which is used in the computations.

Shah *et al* (1978) have evaluated the sensitivities of the reservoir state variables ( $p$  and  $f_w$ ) to the parameters. It is considerably more difficult to simultaneously estimate  $k$ ,  $b_o$  and  $b_w$  than to estimate  $k$  only or to estimate  $k_{ro}$  and  $k_{rw}$  only. Since the quantities appear as  $kk_{ro}$  and  $kk_{rw}$ ,  $p$  and  $f_w$  are especially insensitive to changes in  $k$ ,  $b_o$  and  $b_w$ . This observation suggests that  $k$  and  $(b_o, b_w)$  should be estimated separately during the minimisation process.

The problem is to estimate the spline coefficients,  $W_l$ ,  $l=1, \dots, N_s$ , and the dimensionless exponents,  $b_o$  and  $b_w$ , in the relative permeability expressions, that minimise the smoothing functional  $J_{SM}$ . Consider the minimisation of  $J_{SM}$  by a steepest descent technique. The gradient of  $J_{SM}$  with respect to  $W_l$ ,  $l=1, \dots, N_s$  is

$$\mathbf{g}_w = \begin{pmatrix} \partial J_{SM}/\partial W_1 \\ \vdots \\ \partial J_{SM}/\partial W_{N_s} \end{pmatrix}. \quad (27)$$

The partial derivatives  $\partial J_{SM}/\partial W_l$ ,  $l=1, \dots, N_s$  can be directly calculated from the functional derivative  $\delta J_{LS}/\delta k$  and the partial derivatives  $\partial J_{ST}/\partial W_l$ . The gradient of  $J_{SM}$  with respect to  $(b_o, b_w)$  is simply

$$\mathbf{g}_b = \begin{pmatrix} \partial J_{LS}/\partial b_o \\ \partial J_{LS}/\partial b_w \end{pmatrix}. \quad (28)$$

In the following discussion we will refer to  $(W_1, \dots, W_{N_s})$  and  $(b_o, b_w)$  as  $\mathbf{W}$  and  $\mathbf{b}$ , respectively. The line search step in the steepest descent method starting at  $(\mathbf{W}, \mathbf{b})$  along the descent direction  $\mathbf{d}_w = -\mathbf{g}_w(\mathbf{W}, \mathbf{b})$  and  $\mathbf{d}_b = -\mathbf{g}_b(\mathbf{W}, \mathbf{b})$  is to find the step length  $s$  such that

$$\mathbf{g}_w(\mathbf{W} + s\mathbf{d}_w, \mathbf{b} + s\mathbf{d}_b) \cdot \mathbf{d}_w + \mathbf{g}_b(\mathbf{W} + s\mathbf{d}_w, \mathbf{b} + s\mathbf{d}_b) \cdot \mathbf{d}_b = 0. \quad (29)$$

In its practical application, the line search step represented by equation (29) is dependent on the linear scale (unit) of absolute permeability and thus is not unique. To see this let  $\hat{k} = ck$  where  $c$  is an arbitrary positive constant. Then  $\hat{\mathbf{W}} = c\mathbf{W}$  and the line search step is to find  $\hat{s}$  such that

$$\frac{1}{c^2} \mathbf{g}_w \left( \frac{\hat{\mathbf{W}}}{c} + \hat{s} \frac{\mathbf{d}_w}{c}, \mathbf{b} + \hat{s}\mathbf{d}_b \right) \cdot \mathbf{d}_w + \mathbf{g}_b \left( \frac{\hat{\mathbf{W}}}{c} + \hat{s} \frac{\mathbf{d}_w}{c}, \mathbf{b} + \hat{s}\mathbf{d}_b \right) \cdot \mathbf{d}_b = 0. \quad (30)$$

The arbitrariness of  $c$  suggests a modification of the line search step that finds the set of step lengths  $(r, s)$  such that

$$\mathbf{g}_w(\mathbf{W} + r\mathbf{d}_w, \mathbf{b} + s\mathbf{d}_b) \cdot \mathbf{d}_w = 0 \quad (31)$$

$$\mathbf{g}_b(\mathbf{W} + r\mathbf{d}_w, \mathbf{b} + s\mathbf{d}_b) \cdot \mathbf{d}_b = 0 \quad (32)$$

independent of the linear scale  $c$ . Thus, the one-dimensional line search is replaced by the two-dimensional minimisation of  $J_{SM}(\mathbf{W} + r\mathbf{d}_w, \mathbf{b} + s\mathbf{d}_b)$  with respect to  $r$  and  $s$ .

To estimate  $(k, b_o, b_w)$  simultaneously, the following three-step algorithm will be used assuming that no *a priori* information is available for the spatial variation of  $k(x, y)$  and  $\beta$ .

*Step 1.* Assuming that  $k(x, y) = \bar{k}$  over the whole domain, find  $(\bar{k}, b_o, b_w)$  that minimise  $J_{LS}$ .

*Step 2.* Starting from  $W_l = \bar{k}$ ,  $l = 1, \dots, N_s$ , calculated from step 1, minimise  $J_{LS}$  with respect to  $(W, b_o, b_w)$ . Compute  $\beta = W_p \sigma_p^2 / J_{LS}$  at convergence.

*Step 3.* Using  $\beta$  and starting from  $(W, b_o, b_w)$  determined in step 2, minimise  $J_{SM}$  with respect to  $W$ ,  $b_o$  and  $b_w$ .

In each step, the minimisation of  $J_{LS}$  or  $J_{SM}$  will be carried out by the steepest descent method using equations (31)–(32).

Step 2 of the algorithm is the conventional least-squares estimation of  $k$  by spline approximation, and of  $b_o$  and  $b_w$  that give the best fit of observed pressure and flow data. As a consequence of spline approximation, step 2 converges to a certain minimum although that minimum may not be physically acceptable. The major contribution of step 3 in the algorithm is to alleviate the ill-conditioning of the estimated  $k$  by a regularisation. Generally the exponents of the relative permeabilities,  $b_o$  and  $b_w$ , will not change significantly in step 3. In practice, therefore, step 3 can usually be replaced by

*Step 3'.* Using  $\beta$ ,  $b_o$  and  $b_w$  and starting from  $W$  determined in step 2, minimise  $J_{SM}$  with respect to  $W$ .

In step 3' the smoothing functional  $J_{SM}$  is minimised with respect to the single set of parameters,  $W$ , and the minimisation can be carried out by a general multivariate gradient algorithm. The partial conjugate gradient method of Nazareth (1977) is chosen as it is suitable for a large scale minimisation.

For the numerical implementation of the stabilising functional with the gradient operator,  $J_{ST}$  with  $\zeta_0 = 0$  in equation (15), the weighting coefficients  $\zeta_m$ ,  $m = 1, 2$  and  $3$ , need to be specified. Since the integration in equation (15) is based on the length scale of discretisation of the PDES,  $x_L/N_x$  and  $y_L/N_y$ , the grid spacings for the reservoir PDE,  $\zeta_m$  of the derivative terms can be chosen as  $\zeta_1 = \zeta_2 = \zeta_3 = 1$ .

## 5. Computational examples

In order to test the performance of the algorithm thoroughly, we will introduce a hypothetical reservoir for which the true properties are assumed to be known. The assumed fluid and reservoir properties are shown in table 1. The assumed true absolute permeability distribution is given by

$$k(x, y) = 0.3 - 0.1 \sin \left( \frac{2\pi x}{x_L} \right) \sin \left( \frac{\pi y}{y_L} \right) \quad (33)$$

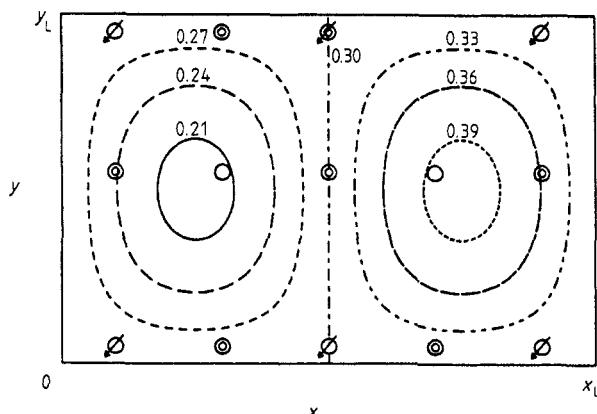
in units of darcies ( $1 \text{ Darcy} = 0.987 \times 10^{-12} \text{ m}^2$ ) for  $(x, y) \in \Omega$ . The location of wells and the true absolute permeability contour map are shown in figure 1. The governing PDES (1)–(9) are solved on a  $15 \times 10$  mesh with the time-step size of 23.1 days. The absolute permeability  $k$  is spline approximated on a  $15 \times 10$  mesh. The observation data are taken from nine observation wells that include two production wells with observation time interval 23.1 days and perturbed by uniformly distributed random numbers (generated by IMSL subroutine GGNML on VAX 11/780) with zero mean and standard deviations

**Table 1.** Specification of reservoir model: (a) properties of water and oil, (b) properties of reservoir.

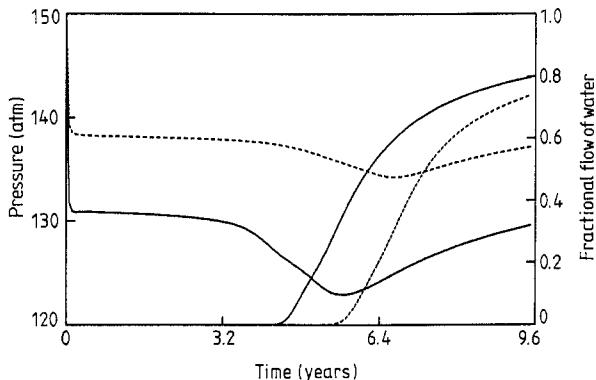
(a)	
$a_w = 0.9$	$a_o = 1.0$
$b_w = 2.5$	$b_o = 2.0$
$S_{iw} = 0.1$	$S_{ro} = 0.2$
$\mu_w = 10^{-3}$ Pa s	$\mu_o = 3 \times 10^{-3}$ Pa s
$c_w = 1.94 \times 10^{-9}$ Pa $^{-1}$	$c_o = 0.97 \times 10^{-9}$ Pa $^{-1}$
Production wells	
$q_w = 0.003 f_w \text{ m}^3 \text{ s}^{-1}$	$q_o = 0.003 (1 - f_w) \text{ m}^3 \text{ s}^{-1}$
Injection wells	
$q_w = 0.001 \text{ m}^3 \text{ s}^{-1}$	$q_o = 0$
(b)	
$c_f = 2.91 \times 10^{-9}$ Pa $^{-1}$	
$\varphi = 0.2 - 0.05 \sin(2\pi x/x_L) \sin(\pi y/y_L)$	
$x_L \times y_L \times h = 1500 \times 1000 \times 10 \text{ m}^3$	
$p(x, y, 0) = 1.52 \times 10^7$ Pa	
$S_w(x, y, 0) = 0.1$	

0.34 atm and 0.0085 for  $p$  and  $f_w$ , respectively. These noisy data are then used to attempt to recover  $(k, b_o, b_w)$ .

Water-breakthrough time has an important significance in the identifiabilities of the parameters. Watson *et al* (1984) have shown, in the two-phase one-dimensional reservoir where water is injected at one end and oil is produced at the other end, that absolute permeability can be estimated from the data up to the water-breakthrough time and that the prebreakthrough production data carry little information about the relative permeabilities. Figure 2 shows the transient pressure and fractional flow of water at the production wells located at (450 m, 550 m) and (1050 m, 550 m) calculated on this basis of the true  $(k, b_o, b_w)$ . We note that for the conditions of this example the water-breakthrough time of this reservoir model occurs at about 6.4 years after the inception of water injection. In the following examples, two different time periods of the observed data will be chosen, one of 9.5 and the other of 6.4 years.



**Figure 1.** Contours of the assumed true absolute permeability profile and location of wells.  $\circ$ , injection wells;  $\square$ , production wells;  $\odot$ , observation wells.



**Figure 2.** Transient pressure and fractional flow of water at the two production wells. —, well at (450 m, 550 m); ---, well at (1050 m, 550 m).

The convergence criteria of the minimisation are

$$\left| \frac{\partial J_{LS}}{\partial \bar{k}} \right| < 10 \quad \text{and} \quad \| \mathbf{g}_b \|_\infty < 10 \quad (34)$$

for step 1 and

$$\| \mathbf{g}_w \|_\infty < 2 \quad \text{and} \quad \| \mathbf{g}_b \|_\infty < 5 \quad (35)$$

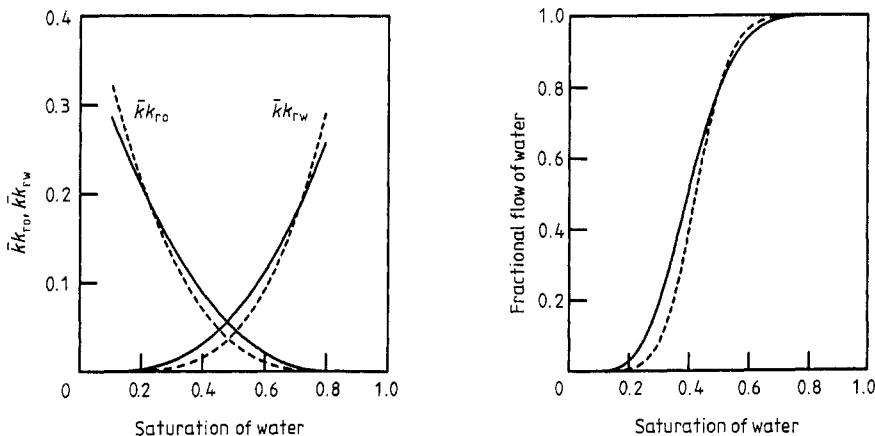
for steps 2 and 3. Although the same conditions are used for steps 2 and 3, they are in effect more strict for step 3 due to the additional term  $\beta J_{ST}$ . In case of step 3', only the first criterion in equation (35) is used to terminate the iteration.

Over a period of 9.5 year, 150 pressure and 150 production data are taken at each of the nine observation wells and  $(W, b_o, b_w)$  is estimated using the suggested three-step algorithm. The results of the estimation are summarised in table 2. The first step is to estimate the set  $(\bar{k}, b_o, b_w)$  that minimises  $J_{LS}$  where  $\bar{k}$  denotes a spatially uniform  $k$ . Although the resultant  $\bar{k}$  is not an acceptable estimate of a spatially varying  $k$  in most cases, it is a reasonable average of the spatially varying  $k$ . Two different sets of  $(\bar{k}, b_o, b_w)$ , (0.2 Darcy, 1.5, 1.5) and (0.4 Darcy, 3.0, 3.0) were chosen as the starting point of this step. The convergence results, (0.289 Darcy, 2.09, 2.51) and (0.286 Darcy, 2.06, 2.48), show good agreement indicating the robustness of this step. In figure 3,  $\bar{k} k_{rw}(S_w)$ ,  $\bar{k} k_{ro}(S_w)$  and  $f_w(S_w)$  calculated from these values are depicted by the full curves. This step makes the remainder of the algorithm insensitive to the choice of initial guess of  $(\bar{k}, b_o, b_w)$ . The next step is the pure least-squares estimation of  $(k, b_o, b_w)$  with  $\beta = 0$ , where  $k$  is represented by the set of spline coefficients  $W$ . In this step,  $\sigma_p$  and  $\sigma_f$  decrease substantially and approach those calculated from the true  $(k, b_o, b_w)$ . The estimated  $k$  is shown in figure 4 and  $(b_o, b_w) = (1.98, 2.50)$ . From the resultant  $W_p \sigma_p^2$  and  $J_{ST}$ ,  $\beta = 2.63 \text{ Darcy}^{-2}$ . Step 3 is the final regularised estimation of  $(k, b_o, b_w)$  with  $\beta$  determined from step 2. The resultant  $k$  is shown in figure 4 and  $(b_o, b_w) = (1.98, 2.50)$ . Table 2 shows that, in step 3,  $J_{ST}$  is reduced significantly due to its inclusion,  $J_{LS}$  is reduced slightly due to continued minimisation and  $J_{SM}$  is increased compared with the values in step 2. Comparison of the  $k$  contours in figure 4 shows the smoothing effect of regularisation on the 'hump' near the lower right corner of the reservoir. As an alternative of step 3, step 3' is the regularised estimation of  $W$  while  $b_o$  and  $b_w$  are fixed to the values determined by step 2 and the same  $\beta$  is used as

**Table 2.** Performance of estimation of  $(k, k_o, b_w)$  from  $9 \times 150$  data.

	$\bar{k}$ (Darcy)	$b_o$	$b_w$	$\beta$ (Darcy $^{-2}$ )	$\sigma_p$ (atm)	$\sigma_r$	$J_{LS}$ (Darcy $^2$ )	$J_{SR}$ (Darcy $^2$ )	$J_{SM}$ (Darcy $^2$ )	CPU time <sup>a</sup> (s)	Number of iterations <sup>b</sup>
Initial guess (A)	0.2	1.5	1.5	2.98	0.0896	181					
Step 1 (from A)	0.289	2.09	2.51	2.20	0.0378	59.3					
Initial guess (B)	0.4	3.0	3.0	2.51	0.0621	104					
Step 1 (from B)	0.286	2.06	2.48	2.20	0.0380	59.6					
Step 2		1.98	2.50	0.0	0.37	0.0095	2.34	0.430	2.34	116	30
Step 3		1.98	2.50	2.63	0.36	0.0089	2.14	0.274	2.87	127	31
Step 3'		1.98	2.50	2.63	0.36	0.0086	2.04	0.239	2.67	54	13
True value		2.0	2.5	0.34	0.0087	2	0.144			65	16

<sup>a</sup> On Cray X MP/48<sup>b</sup> Number of solutions of state and adjoint PDES.



**Figure 3.**  $\bar{k}k_{rw}$ ,  $\bar{k}k_{ro}$  and  $f_w$  plotted against  $S_w$  calculated from the resultant  $(\bar{k}, b_o, b_w)$  of Step 1. —, 9 × 150 data; ---, 9 × 100 data.

step 3. The contours of the resultant  $k$  are shown in figure 4, which shows more smoothing effect compared with that of step 3. Both the discrepancy and the stabilising functional terms are smaller than those of step 3 while step 3' required more computing time as shown in table 2. Throughout the estimation of process,  $(b_o, b_w)$  is estimated accurately even in step 1. The entire algorithms, steps 1, 2 and 3, required 63 and 74 iterations (solutions of state and adjoint PDEs), corresponding to 252 and 297 s of computing time and steps 1, 2 and 3', 66 and 77 iterations corresponding to 263 and 308 s (4.0 s per iteration) on a Cray X-MP/48 for the given initial guesses (0.2 Darcy, 1.5, 1.5) and (0.4 Darcy, 3.0, 3.0), respectively.

The identifiability condition of relative permeabilities given by Watson *et al* (1984) is not directly applicable to the two-dimensional reservoir with multiple injection and multiple production wells considered in this study. To investigate the effect of observation time period we consider the case in which 100 pressure and 100 production data are taken over a period of 6.4 years. Both of the production wells begin to produce water as well as oil but flow data after water breakthrough are not available from the well located at (450 m, 550 m) by the time period of 6.4 years for the given reservoir. As is shown in table 3, this example shows the same tendency as the previous example in terms of the insensitivity of the result of step 1 to the choice of initial guess and the slight improvement of data match in steps 3 and 3' as compared to step 2. Step 1 is started with (0.2 Darcy, 1.5, 1.5) and (0.4 Darcy, 3.0, 3.0) and coverages to (0.329 Darcy, 2.75, 3.49) and (0.328 Darcy, 2.74, 3.40), respectively. The resultant  $\bar{k}$  (0.328–0.329 Darcy) is a reasonable average of spatially varying  $k$  given in equation (33) although it is 0.4 Darcy higher than that estimated in the previous case. In contrast to the previous case, however, the values of  $(b_o, b_w)$  are far from the true ones. Nevertheless, as shown in figure 3 by the broken curve,  $\bar{k}k_{rw}(S_w)$ ,  $\bar{k}k_{ro}(S_w)$  and  $f_w(S_w)$  do not disagree substantially with those values calculated in the previous case. Step 2 is started with (0.329 Darcy, 2.75, 3.49) and  $\beta=0$ . The resultant  $k$  surface is shown in figure 5 and  $(b_o, b_w)$  is (1.99, 2.53) where  $(b_o, b_w)$  shows good agreement with true one. Comparison of  $(b_o, b_w)$  in steps 1 and 2 shows that the flow data up to the water breakthrough can be fitted by the wide range of different values of  $(b_o, b_w)$  and in this case the estimation of  $(b_o, b_w)$  should be carried out based on the  $k$  that matches

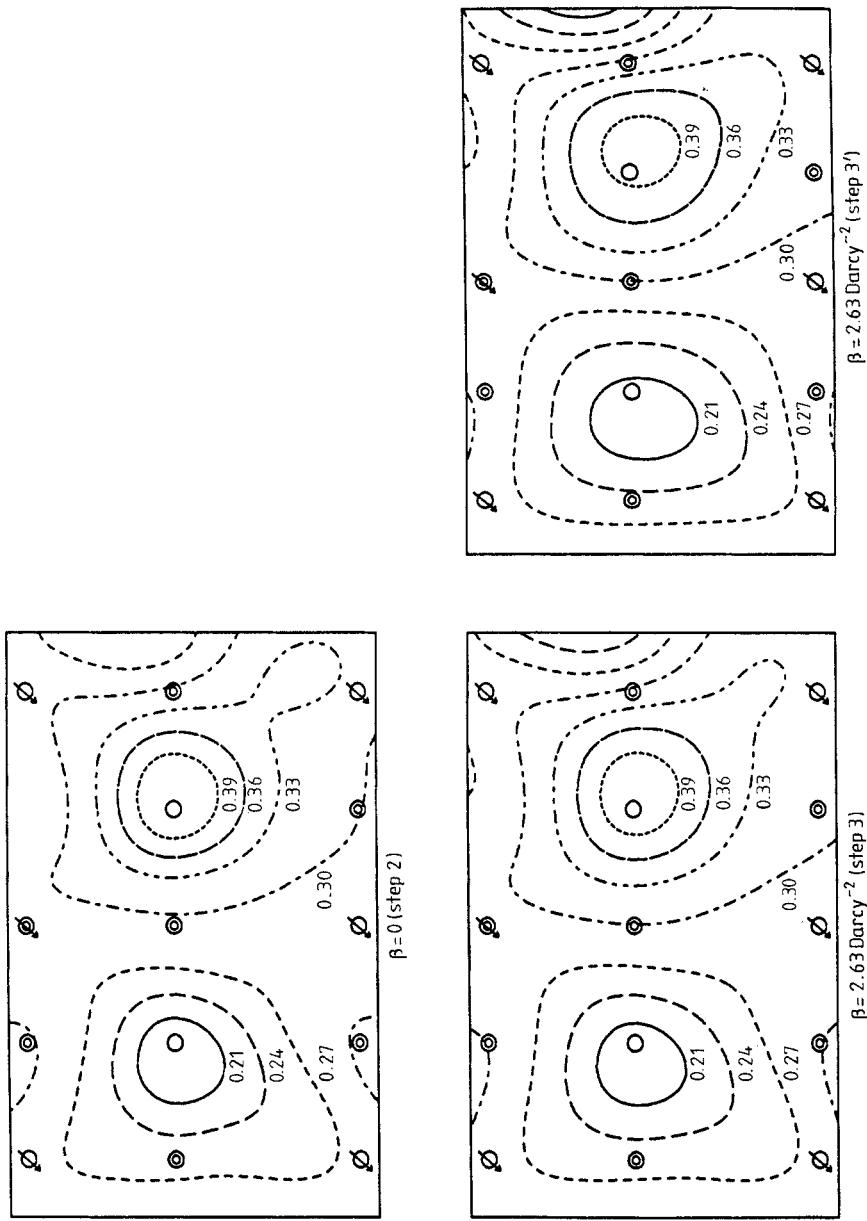


Figure 4. Estimated  $k$  surfaces from 150 data points at each well.  $\varnothing$ , injection wells;  $\circ$ , production wells;  $\circ$ , observation wells.

**Table 3.** Performance of estimation of  $(k, b_o, b_w)$  from  $9 \times 100$  data.

$\bar{k}$ (Darcy)	$b_o$	$b_w$	$\beta$ (Darcy $^{-2}$ )	$\sigma_p$ (atm)	$\sigma_r$ (atm)	$J_{ls}$ (Darcy $^2$ )	$J_{st}$ (Darcy $^2$ )	$J_{sm}$ (Darcy $^2$ )	CPU time <sup>a</sup> (s)	Number of iterations <sup>b</sup>
Initial guess (A)	0.2	1.5	1.5	3.20	0.0963	209				
Step 1 (from A)	0.329	2.75	3.49	2.15	0.0502	72.2				50
Initial guess (B)	0.4	3.0	3.0	2.47	0.0555	91.9				
Step 1 (from B)	0.328	2.74	3.40	2.12	0.0481	68.4				61
Step 2	1.99	2.53	0.0	0.43	0.0091	2.64	0.464	2.64	114	22
Step 3	1.99	2.51	3.30	0.36	0.0087	2.11	0.222	2.84	115	40
Step 3'	1.98	2.53	3.30	0.38	0.0088	2.22	0.253	3.06	31	42
True value	2.0	2.5	0.34	0.0087	2	0.144				11

<sup>a</sup> On Cray X-MP/48.<sup>b</sup> Number of solutions of state and adjoint PDES.

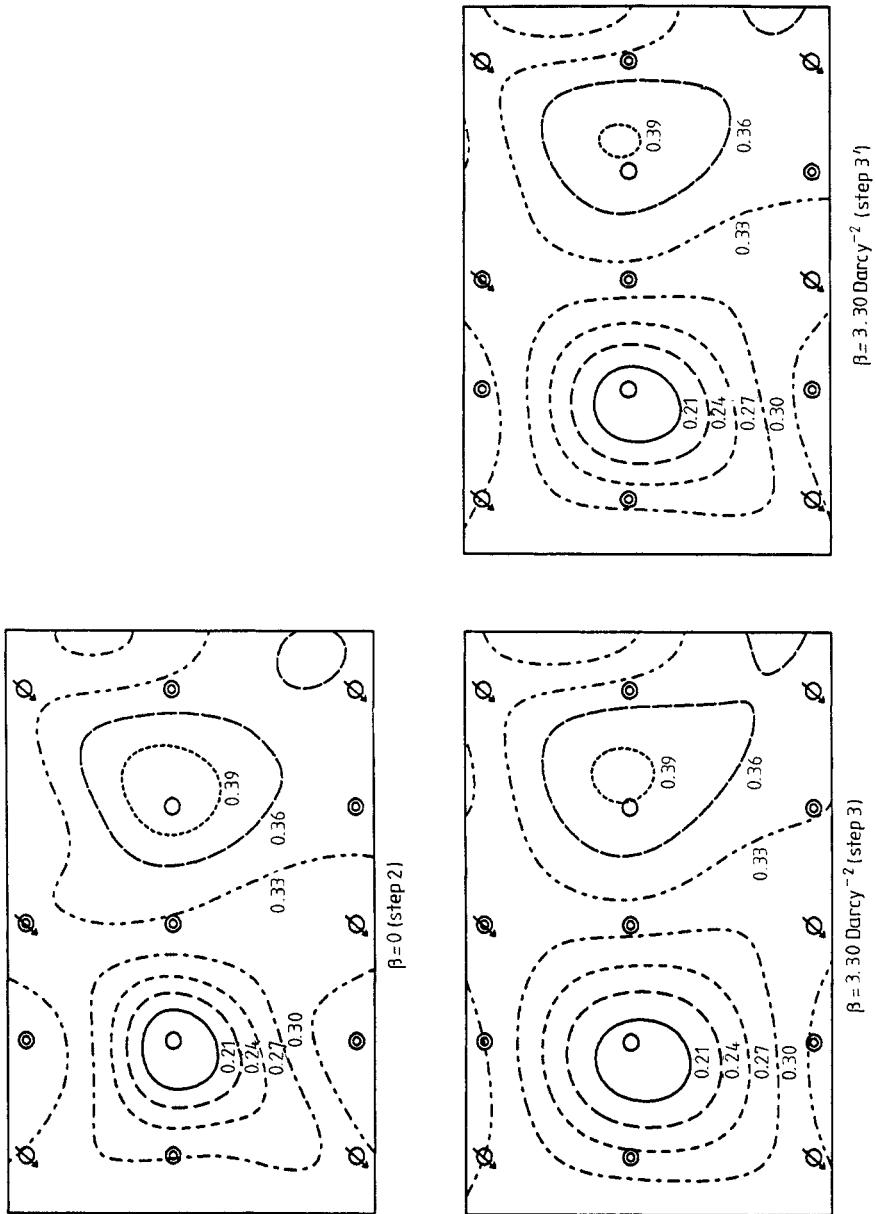


Figure 5. Estimated  $k$  surfaces from 100 data points at each well.  $\varnothing$ , injection wells;  $\circ$ , production wells;  $\odot$ , observation wells.

the observed data accurately. The value of  $\beta$  estimated by the algorithm is 3.30 Darcy<sup>-2</sup>. The final ( $b_o$ ,  $b_w$ ) is (1.99, 2.51) for step 3 and the estimated  $k$  for steps 3 and 3' are shown in figure 5. The discrepancy and stabilising functional term of step 3 are smaller than those of step 3' (see table 3) but figure 5 shows about the same degree of smoothing effect for the two different regularisation steps. The algorithm required 101 and 104 iterations corresponding to 279 and 290 s of computing time (2.8 s per iteration) for step 3 and 70 and 73 iterations corresponding to 195 and 206 s for step 3' for the given initial guesses (0.2 Darcy, 1.5, 1.5) and (0.4 Darcy, 3.0, 3.0), respectively.

## 6. Conclusions

A numerical algorithm is developed to estimate the spatially varying absolute permeability,  $k$ , and the exponents in the relative permeability expressions for two-phase petroleum reservoirs, based on noisy pressure and flow data. The spatially varying absolute permeability is estimated by regularisation with bi-cubic spline approximation. The algorithm developed suggests the choice of the regularisation parameter based on the ratio of the level of the observation error in pressure data to the measure of non-smoothness of parameter. The regularised estimation alleviates the ill-conditioning resulted from the conventional least-squares estimation. We demonstrate conditions under which the absolute and relative permeabilities can be estimated simultaneously.

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