Bound-State Model of Weak and Strong Interactions*

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(Received June 13, 1961)

The pion-nucleon coupling constant is calculated from first principles by use of the N/D matrix method. Three models are introduced which contain pions, nucleons, and weakly interacting intermediate bosons of the scalar, pseudoscalar, and vector variety. The basic interactions are taken to be parity and isotopic spin conserving. Certain physical assumptions in the nature of boundary conditions and the known fact that the weak coupling is very weak, together with use of the Born approximation for $\mathbf{N}$, enable us to obtain an eigenvalue equation which expresses the pion-nucleon coupling constant in terms of the three masses in the problem. The correct value for g_π^2 can be obtained for an intermediate vector meson of mass comparable to the nucleon mass with essentially no cutoff employed; on the other hand, the experimental value is also obtained with a spin-zero boson and a relatively small cutoff energy.

I. INTRODUCTION

ONE of the most intriguing problems in physics is that of understanding the relation (which most people believe exists) between coupling constants and masses. In an effort to understand this fundamental question, there have been many attempts to produce field-theoretical models in which not all the parameters are independent. As an example of such theories, van Hove[1] has considered a modified Lee model in which weak-interaction effects appear in the presence of only basically strong-coupling terms in the interaction Lagrangian. Fermi and Yang[2] have pictured the pion as a bound state of a nucleon-antinucleon pair, the binding effect by nonlinear terms in the interaction. This idea has been extended to the strange particles by Sakata and Goldhaber.[3] Nambu and Jona-Lasinio[4] have constructed a model nonlinear theory of elementary particles based upon an analogy with the theory of superconductivity.

In spite of the attractiveness of these various models, there has not been too much progress because of the difficulty of making convincing calculations. This is not a particularly rare phenomenon in field theory. The two standard approaches to the calculation of bound states are the Tamm-Dancoff method[5] and the Bethe-Salpeter equation.[6] Both of these schemes have well-known shortcomings. Thirring[7] has recently devised an extremely clever method of solving the Bethe-Salpeter equation for two fermions in the limit of large binding. This method was applied to a model of a pion assumed to be a bound state of a nucleon-antinucleon pair. The latter are taken to interact via the exchange of a vector meson of very large mass weakly coupled to the nucleon field. It is necessary to regularize the interaction because of the singular nature of the spinor Bethe-Salpeter equation. When the weak-coupling constant and cutoff are chosen to fit the pion mass, the effective pion-nucleon coupling constant (which is easily computed from the pion wave function) turns out to be

$$\frac{g^2}{4\pi}=\frac{2\pi}{\ln(m_V/M)^2}\approx 1.$$

The experimental value is about fourteen[8]; nevertheless this is not bad agreement from such a simple calculation. There is however one very weak point: Once the pion is formed as a bound state it will contribute by far the most important part of the potential. It has a longer range and stronger coupling to the nucleon than the weakly interacting vector meson. What this means is that crossing symmetry is being badly violated in the Bethe-Salpeter ladder approximation used by Thirring.

We shall present a method of calculation which at least avoids this criticism. It is a generalization of the procedure of Blankenbecler and Cook in their treatment of the deuteron binding energy.[9] Our model is basically the same as Thirring's in that we have a weakly coupled boson (hereafter called a V meson), nucleons, and pions. In our dispersion theoretic approach we need not commit ourselves as to whether the pion or the V meson are bound states. Except for the fact that we shall ultimately make certain physical assumptions in the nature of boundary conditions, we treat the V and the pion quite symmetrically. We derive a set of coupled integral equations for the vertex functions describing pion-nucleon and V-nucleon interactions. This set of equations is solved assuming that the V-nucleon vertex function vanishes for large energies and also assuming

* Supported in part by the U. S. Atomic Energy Commission and the Air Force Office of Scientific Research, Air Research and Development Command.

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[9] R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960).

that the V-nucleon interaction is very weak. We obtain an eigenvalue condition which expresses the pion-nucleon coupling constant in terms of the three masses. We try various models for the V meson, scalar, pseudoscalar, and vector, where in all cases it is taken to have isotopic spin unity and to interact in such a way as to conserve parity.

Naturally we will have to make approximations, many of which are difficult to assess. It is our feeling that our approach is a sensible and perphaps amusing one. We share what will probably be the great skepticism of the reader about some of the approximations and the final numerical results.

II. N/D MATRIX METHOD

We present in this section a sketch of the essential steps of our procedure which enables us to calculate the pion-nucleon coupling constant from first principles by use of the N/D matrix method. The arguments for making various approximations are discussed in some length, but many of the specific details are deferred to the following sections in order to clarify the procedure.

We first introduce the vertex functions

$$G=\langle 0|f|NV\rangle(2E_1k_0/M)^{\frac{1}{2}}, \tag{2.1}$$

and

$$F=\langle 0|f|N\pi\rangle(2E_2q_0/M)^{\frac{1}{2}}, \tag{2.2}$$

where one nucleon has been taken off the mass shell at each vertex. The symbols $M(p)E$, $m(k)k_0$, and $\mu(q)q_0$ label the mass (momentum) energy of the nucleon, weakly interacting V meson, and pion. The operator f represents the Heisenberg current for the nucleon. The nucleon-V meson and nucleon-pion scattering states are always to be regarded as "in" states and are referred to as channel 1 and channel 2, respectively. In the time-honored fashion we ignore the fact that the mesons are unstable; and we shall postpone for the moment answering the pertinent question as to which particle or particles, if any, are to be regarded as composite.

The two vertex functions can be expressed in terms of several invariant form factors in the usual fashion, with the explicit construction depending upon the transformation properties of the mesons. The construction is spelled out in detail in the next three sections for V particles of the pseudoscalar, scalar, and vector varieties. If we assume isotopic spin conservation, it is already clear from the above equations that only total angular momentum $J=\frac{1}{2}$ and isotopic spin $I=\frac{1}{2}$ states are of interest in discussions of the scattering states.

Next we consider the $N+V$ and $N+\pi$ scattering amplitudes,

$$T_{11}=\bar{u}(p')\langle V'|f|NV\rangle(4k_0'k_0E_1/M)^{\frac{1}{2}}, \tag{2.3}$$

$$T_{12}=\bar{u}(p')\langle V'|f|N\pi\rangle(4k_0'q_0E_2/M)^{\frac{1}{2}}, \tag{2.4}$$

and

$$T_{22}=\bar{u}(p')\langle \pi'|f|N\pi\rangle(4q_0'q_0E_2/M)^{\frac{1}{2}}. \tag{2.5}$$

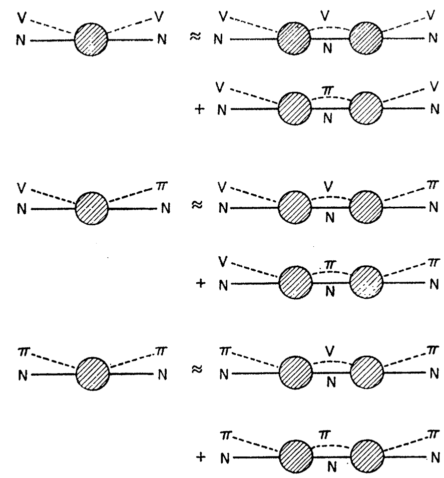

FIG. 1. Dispersion diagrams for the scattering amplitudes T_{11}, T_{12}, and T_{22} in the "one-meson" approximation.

These Feynman matrix elements will later be written in terms of invariant scattering amplitudes which can be more conveniently expanded in partial waves. Consider the unitarity conditions for these matrix elements in the "one-meson" approximation where only the nucleon-V meson and nucleon-pion intermediate states are retained. These relations can be illustrated graphically by the dispersion diagrams in Fig. 1. Let $\mathbf{M}$ represent a two-by-two matrix describing the two channel scattering amplitudes in a particular partial wave. The row index refers to the final channel, and the column index refers to the initial channel. The unitarity conditions in matrix notation for the given partial wave then become simply

$$\mathrm{Im}\mathbf{M}=\mathbf{M}^*\boldsymbol{\varrho}\mathbf{M} \tag{2.6}$$

along the physical branch cuts in the W^2 plane. The variable W is the center-of-mass energy of the NV or $N\pi$ state, and $\boldsymbol{\varrho}$ is a diagonal matrix whose elements contain phase space factors which vanish below the relevant thresholds.

The unitarity conditions for the vertex functions G and F in our "one-meson" approximation are pictured by the dispersion diagrams in Fig. 2. Let $\mathbf{H}$ represent the two-channel row matrix whose two elements consist

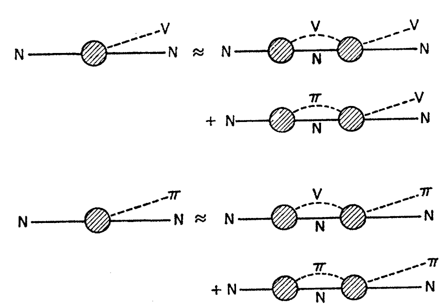

FIG. 2. Dispersion diagrams for the vertex functions G and F in the "one-meson" approximation.

of the partial-wave form factors analogous to the same partial wave elements of $\mathbf{M}$. The unitarity conditions for these form factors in matrix notation are then given by

$$\mathrm{Im}\mathbf{H}=\mathbf{H}^*\rho\mathbf{M}. \quad (2.7)$$

At this point, we introduce the N/D technique of Chew and Mandelstam[10] as generalized to multichannel processes independently by Bjorken and Nauenberg.[11] Again in matrix notation, we define

$$\mathbf{M}=\mathbf{N}\mathbf{D}^{-1}, \quad (2.8)$$

and

$$\mathbf{H}=\boldsymbol{\lambda}\mathbf{D}^{-1}, \quad (2.9)$$

where $\mathbf{N}$ and $\mathbf{D}$ are two-by-two matrices and $\boldsymbol{\lambda}$ is a row matrix similar to $\mathbf{H}$. The D matrix is analytic everywhere except for the physical branch cuts along the positive W^2 axis, while $\mathbf{N}$ contains the dynamic singularities on the left. By means of the Mandelstam representation, Frautschi and Walecka[12] and Frazer and Fulco[13] have located the dynamic singularities for the partial wave pion-nucleon scattering amplitudes. They find that the singularities lie not only along the negative W^2 axis but also in the complex W^2 plane. Since the only singularities of the partial-wave form factors lie along the positive W^2 branch cuts of $\mathbf{D}$, the two λ matrix elements are merely constants, (λ_1,λ_2). We shall set $\lambda_1=0$ and in the following try to justify this choice.

The form factors associated with the vertex functions of (2.1) and (2.2) are related to the renormalized vertex function $\Gamma_i(W^2)$ of field theory by[14]

$$H_i(W^2)=S_F^{-1}S_{Fr}'\Gamma_i(W^2), \quad i=1,\,2, \quad (2.10)$$

where S_F^{-1} is the inverse propagator of the nucleon in the Born approximation, and S_{Fr}' is the corresponding fully renormalized propagator. In the asymptotic limit $W^2\to\infty$, $S_F^{-1}S_{Fr}'\to Z_2^{-1}$ (where Z_2 is the nucleon field renormalization constant) and the above reduces to

$$\lim_{W^2\to\infty} H_i(W^2)=Z_2^{-1}\lim_{W^2\to\infty}\Gamma_i(W^2). \quad (2.11)$$

Under the rather general assumption that no ghost states exist in relativistic field theory, Lehmann *et al.*[15] have shown that the vertex function $\Gamma_i(W^2)$ must vanish in the asymptotic limit if the theory is to possess exact solutions. Drell and Zachariasen[14] have further pointed out in an analogous situation that if the renormalization constant Z_2^{-1} is finite, then it follows necessarily that $H_i(W^2)\to 0$ as $W^2\to\infty$. For our purposes, only the ratio of the partial wave form factor in channel 1 compared to that in channel 2 is of interest, so that we need not make any definitive remarks about the finite vs infinite nature of Z_2^{-1}. We simply write

$$\lim_{W^2\to\infty}\frac{H_1(W^2)}{H_2(W^2)}=\lim_{W^2\to\infty}\frac{\Gamma_1(W^2)}{\Gamma_2(W^2)}. \quad (2.12)$$

We now argue that the above limit is in fact zero. To do this, we make the following two observations:

(1) Evaluated on the mass shell of the nucleon, $\Gamma_1(M^2)$ and $\Gamma_2(M^2)$ are proportional to g_V and g_π, respectively, by definition of the renormalized coupling constants g_V and g_π. Moreover, we know experimentally from studies of beta decay and collision processes that if the weak intermediate V meson actually exists, then surely $g_V\ll g_\pi$.

(2) Imagine a model in which the V meson and pion are regarded as bound states of the nucleon-pair system. Lee and Yang[16] have pointed out that the absence of any known K meson decays involving a V meson as one of the decay products demands $m_V>\mu_K$. Therefore, on the basis of a bound state model, the V particle must be less strongly bound than the pion.

These two observation suggest that $\Gamma_1(W^2)$ is initially several orders of magnitude smaller than $\Gamma_2(W^2)$ and has in addition fewer high-frequency Fourier components; it thus seems plausible that $\Gamma_1(W^2)$ asymptotically approaches zero more rapidly than $\Gamma_2(W^2)$ as $W^2\to\infty$. Equation (2.12) then becomes

$$\lim_{W^2\to\infty}\frac{H_1(W^2)}{H_2(W^2)}=0. \quad (2.13)$$

We offer this bound-state model simply as a physical interpretation of the choice of boundary conditions imposed by the above procedure. We are well aware that this is not the only plausible interpretation and the reader may feel free to substitute some other model as he sees fit.

From Eq. (2.9) we form the ratio

$$H_1/H_2=(\lambda_1 d_{22}-\lambda_2 d_{21})/(-\lambda_1 d_{12}+\lambda_2 d_{11}). \quad (2.14)$$

The normalization of the D matrix is arbitrary, and we choose to fix it by setting

$$\lim_{W^2\to\infty}\mathbf{D}(W^2)=\mathbf{1}, \quad (2.15)$$

where the right-hand side represents the unit matrix. It then follows from Eq. (2.14) that the limit of this ratio is given by

$$\lim_{W^2\to\infty}\frac{H_1(W^2)}{H_2(W^2)}=\frac{\lambda_1}{\lambda_2}; \quad (2.16)$$

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[10] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[11] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960); M. Nauenberg, thesis, Cornell University, 1960 (unpublished).
[12] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).
[13] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).
[14] See for example S. D. Drell and F. Zachariasen, Phys. Rev. **119**, 463 (1960). This paper treats the electron-photon vertex function in quantum electrodynamics with the photon taken off the mass shell, but the two expressions are comparable. Note, however, the misprint in Eq. (14) and the following sentence in which Z_3 should be replaced by Z_3^{-1}.
[15] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **2**, 425 (1955).
[16] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960).

and in view of Eq. (2.13), we are led to set $\lambda_1=0$, which was to be shown. For arbitrary W^2, Eq. (2.14) now reduces to

$$H_1(W^2)/H_2(W^2)=-d_{21}(W^2)/d_{11}(W^2); \quad (2.17)$$

and in particular, on the nucleon mass shell, corresponding to $W^2=M^2$, the ratio becomes

$$g_V/g_\pi=-d_{21}(M^2)/d_{11}(M^2). \quad (2.18)$$

Now, the unitarity conditions (2.6) and (2.7) are satisfied if we set

$$\mathrm{Im}\mathbf{D}=-\rho\mathbf{N} \quad (2.19)$$

along the physical W^2 branch cuts, and

$$\mathrm{Im}\mathbf{N}=[\mathrm{Im}\mathbf{M}]\mathbf{D} \quad (2.20)$$

along all the dynamic singularities. An application of Cauchy's theorem together with unitarity then requires that

$$\mathbf{D}(W^2)=\mathbf{1}-\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} dW'^2\frac{\rho(W'^2)\mathbf{N}(W'^2)}{W'^2-W^2-i\epsilon}, \quad (2.21)$$

$$\mathbf{N}(W^2)=\frac{1}{\pi}\int_{-\infty} dW'^2\frac{[\mathrm{Im}\mathbf{M}(W'^2)]\mathbf{D}(W'^2)}{W'^2-W^2-i\epsilon}. \quad (2.22)$$

In writing the above, we have made a subtraction at infinity in $\mathbf{D}$ consistent with the limit (2.15), but we have explicitly assumed that no other subtractions are necessary. Moreover, we have ignored the possibility that the partial-wave scattering amplitudes might vanish in the physical range which automatically precludes the existence of the familiar ambiguities found in the solution of the Low equation by Castillejo *et al.*[17] Both of these assumptions are absolutely vital to the success of our prescription for computing the pion-nucleon coupling constant; if either is invalidated, arbitrary constants must be introduced into the above equations which prove self-defeating. Needless to say, we proceed under the above assumptions. It is perhaps instructive for the reader to consider the number of assumptions made in treating bound states by more standard methods, the Bethe-Salpeter equation, for example.[6]

If Eq. (2.22) is then substituted into Eq. (2.21), one obtains a system of coupled integral equations for the matrix elements of $\mathbf{D}(W^2)$. Even in our "one-meson" approximation, an exact solution of these equations would be possible only if $\mathrm{Im}\mathbf{M}(W^2)$ were known along the dynamic singularities, which are all contained in $\mathbf{N}(W^2)$. The same situation obtains for the pion-pion and pion-nucleon problems where the integral equations are uncoupled. In such cases, approximate solutions have been found by the use of iteration procedures[18] or by the replacement of the dynamical singularities by a set of poles.[12] Our program is considerably less ambitious, and we simply replace $\mathbf{N}(W^2)$ in the integrand of (2.21) by the first Born approximation $\mathbf{B}(W^2)$ for the partial-wave scattering amplitude $\mathbf{M}(W^2)$. In this approximation, it is more desirable to work in the cut W plane rather than the cut W^2 plane. Therefore, we redefine all our amplitudes accordingly and rewrite (2.21)

$$\mathbf{D}(W)=\mathbf{1}-\frac{1}{\pi}\int_{M+\mu}^{\infty} dW' \times\left[\frac{\rho(W')\mathbf{B}(W')}{W'-W-i\epsilon}-\frac{\rho(-W')\mathbf{B}(-W')}{W'+W+i\epsilon}\right], \quad (2.23)$$

or

$$d_{ij}(W)=\delta_{ij}-(g_Vg_\pi/8\pi^2)I_{ij}(W),$$

where as indicated, the physical branch cuts in the W^2 plane along the positive axis divide into two sets of branch cuts lying along both the positive and negative real axes in the W plane. The second expression above simply serves to define the integral $I_{ij}(W)$ from which the two coupling constants have been extracted.

In the above approximation, Eq. (2.18) for the ratio of the form factors then becomes

$$\frac{g_V}{g_\pi}=\frac{(g_Vg_\pi/8\pi^2)I_{21}(M)}{1-(g_V^2/8\pi^2)I_{11}(M)}. \quad (2.24)$$

We have thus succeeded in obtaining an eigenvalue condition between the coupling constants g_V, g_π, and the masses of the nucleon, V boson, and pion. If we now make use of the experimental fact that g_V^2 is small, we finally find

$$g_\pi^2/4\pi=2\pi/I_{21}(M), \quad (2.25)$$

which can be regarded as an expression for the pion-nucleon coupling constant as a function of the weak intermediate boson mass, m. This is a rather amusing result. It is very similar to the eigenvalue condition on the binding energy of the deuteron in terms of the pion-nucleon coupling constant and masses that was found by Blankenbecler and Cook in considering the deuteron problem by dispersion techniques.[9] Their calculation was simplified by the presence of an anomalous threshold, which of course is not present in our problem for the mass values of interest.

It is clear that this matrix method can, in principle, be applied to any choice of transformation properties for the V boson. One need merely determine the partial wave amplitudes which are relevant for the construction of the form-factor and scattering matrices $\mathbf{H}$ and $\mathbf{M}$. We consider first the cases of pseudoscalar and scalar V mesons where the N/D method derived above can be applied directly. The case of a vector meson is treated last, because a suitable modification of the matrix method is required.

[17] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[18] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960).

III. PSEUDOSCALAR CASE

A pseudoscalar V meson represents the simplest of the three cases considered, for here its transformation properties are identical to those of the pion. The vertex functions defined in Eqs. (2.1) and (2.2) must transform like pseudoscalars in spinor space. We write for F in the center-of-mass system of the pion-nucleon state

$$F=-i\{F_1+F_2[i\gamma\cdot(p+q)+M]\}\gamma_5 u(p)$$
$$=-i\tfrac{1}{2}[\mathfrak{F}_+(1+\gamma_4)+\mathfrak{F}_-(1-\gamma_4)]\gamma_5 u(p), \quad (3.1)$$

or, in two-component notation,

$$F=-i[2M(E_2+M)]^{-\frac{1}{2}}\begin{bmatrix}\mathfrak{F}_+\boldsymbol{\sigma}\cdot\mathbf{q}\\ -\mathfrak{F}_-(E_2+M)\end{bmatrix}\chi_2, \quad (3.2)$$

where

$$\mathfrak{F}_+=F_1+(M-W)F_2, \quad (3.3)$$

$$\mathfrak{F}_-=F_1+(M+W)F_2. \quad (3.4)$$

All γ matrices are Hermitian, and the spinors are normalized to $\bar{u}u=1$, with χ the usual two-component spinor. The invariant form factors F_1 and F_2 are functions of W^2, and $\mathfrak{F}_+$ is the ${}^2P_{\frac{1}{2}}$ or $(1-)$ partial wave form factor with $\mathfrak{F}_-$ the corresponding ${}^2S_{\frac{1}{2}}$ or $(0+)$ partial wave amplitude. All form factors refer to the $J=\frac{1}{2}$, $I=\frac{1}{2}$ amplitudes, and hence these labels will always be suppressed. Note that on the mass shell of the nucleon, the coefficient of F_2 vanishes and $F_1=\sqrt{3}g_\pi$ by definition of the coupling constant.[19] An analogous construction exists for the V-nucleon vertex function G in terms of the form factors G_1, G_2, $\mathcal{G}_+$, and $\mathcal{G}_-$. The substitution of symbols is so trivial that we do not spell it out here.

The kinematics of the pion-nucleon scattering problem have been summarized in detail by Frautschi and Walecka.[12] We include here only the most relevant formulas. We relate the scattering amplitude T_{22} defined in Eq. (2.5) to the S matrix by

$$S_{22}=1+(2\pi)^4 i(M^2/4E_2'E_2q_0'q_0)^{\frac{1}{2}}$$
$$\times T_{22}\delta(p'+q'-p-q). \quad (3.5)$$

Since T_{22} transforms like a scalar, in the center-of-mass system we set

$$T_{22}=\bar{u}(p')[A_2-\tfrac{1}{2}i\gamma\cdot(q+q')B_2]u(p)$$
$$=(4\pi W/M)\chi_2'^*[f_1+f_2\boldsymbol{\sigma}\cdot\hat{q}'\boldsymbol{\sigma}\cdot\hat{q}]\chi_2, \quad (3.6)$$

where f_1 and f_2 are related to A_2 and B_2 by

$$f_1=\frac{(W+M)^2-\mu^2}{16\pi W^2}[A_2+(W-M)B_2], \quad (3.7)$$

$$f_2=\frac{(W-M)^2-\mu^2}{16\pi W^2}[-A_2+(W+M)B_2]. \quad (3.8)$$

The symbol $\hat{q}$ signifies a unit vector in the direction of the momentum $\mathbf{q}$. In the language peculiar to the Mandelstam representation, the amplitudes A_2 and B_2 are functions of the invariants s, t, and u, two of which are independent; for example, one can choose as the independent variables W^2 and $\cos\theta$. If f_1 and f_2 are expanded in l waves according to

$$f_i=\textstyle\sum_l(2l+1)f_i{}^lP_l(\hat{q}'\cdot\hat{q}), \quad (3.9)$$

the $1-$ and $0+$ partial waves are given by

$$f_{1-}=f_1{}^1+f_2{}^0, \quad (3.10)$$

and

$$f_{0+}=f_1{}^0+f_2{}^1. \quad (3.11)$$

Since the partial waves $A_2{}^l$ and $B_2{}^l$ are functions only of W^2, the f_{1-} and f_{0+} amplitudes are not independent but are related by the following reflection principle first pointed out by MacDowell[20]

$$f_{0+}(W)=-f_{1-}(-W). \quad (3.12)$$

In what follows, we can thus restrict our attention to $f_{1-}(W)$.

The $N+V$ scattering and reaction amplitudes are completely analogous to the $N+\pi$ scattering amplitudes discussed above. Corresponding to Eq. (3.6), we have

$$T_{11}=\bar{u}(p')[A_1-\tfrac{1}{2}i\gamma\cdot(k+k')B_1]u(p)$$
$$=(4\pi W/M)\chi_1'^*[g_1+g_2\boldsymbol{\sigma}\cdot\hat{k}'\boldsymbol{\sigma}\cdot\hat{k}]\chi_1, \quad (3.13)$$

for the $N+V$ scattering amplitude and

$$T_{12}=\bar{u}(p')[A_{12}-\tfrac{1}{2}i\gamma\cdot(q+k')B_{12}]u(p)$$
$$=(4\pi W/M)\chi_1'^*[t_1+t_2\boldsymbol{\sigma}\cdot\hat{k}'\boldsymbol{\sigma}\cdot\hat{q}]\chi_2, \quad (3.14)$$

for the $N+\pi\rightarrow N+V$ reaction amplitude. The amplitudes g_{1-}, g_{0+}, t_{1-}, and t_{0+} are written in a fashion analogous to (3.10) and (3.11) and obey similar reflection principles. In particular,

$$g_{1-}=g_1{}^1+g_2{}^0, \quad (3.15)$$

$$t_{1-}=t_1{}^1+t_2{}^0. \quad (3.16)$$

After these preliminaries, we are now ready to make use of the reduction formulas in order to determine the unitary conditions in the physical region. Consider the $N+\pi$ scattering matrix element T_{22} defined in Eq. (2.5). If the nucleon is contracted according to the reduction techniques of Lehmann *et al.*[21] the corresponding absorptive part along the physical cuts can be written

$$\mathrm{Im}T_{22}=\tfrac{1}{2}(2\pi)^4(4q_0'q_0)^{\frac{1}{2}}\bar{u}(p')$$
$$\times\{\sum_{p'',q''}\langle\pi'|f|N''\pi''\rangle\langle N''\pi''|\bar{f}|\pi\rangle$$
$$\times\delta(p''+q''-p-q)+\sum_{p'',k''}\langle\pi'|f|N''V''\rangle$$
$$\times\langle N''V''|\bar{f}|\pi\rangle\delta(p''+k''-p-q)\}u(p), \quad (3.17)$$

when only the $N\pi$ and NV intermediate states are re-

[19] The factor $\sqrt{3}$ enters from isotopic spin considerations.

[20] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

[21] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955).

tained. Use of the definitions (2.4) and (2.5) together with the equations given in this section, then leads one to write

$$\mathrm{Im} f_{1-} = q|f_{1-}|^2 + k|t_{1-}|^2, \tag{3.18}$$

for the "one-meson" approximation. The nucleons can be contracted in the remaining scattering matrix elements T_{11} and T_{12}, and one finds, in a similar manner, the unitarity conditions

$$\mathrm{Im} g_{1-} = q|t_{1-}|^2 + k|g_{1-}|^2, \tag{3.19}$$

$$\begin{aligned}\mathrm{Im} t_{-1} &= q t_{1-}^* f_{1-} + k g_{1-}^* t_{1-} \\ &= q f_{1-}^* t_{1-} + k t_{1-}^* g_{1-}.\end{aligned} \tag{3.20}$$

The symbols q and k refer to the magnitudes of the three-momenta whose squares are given by

$$q^2 = [(W+M)^2 - \mu^2][(W-M)^2 - \mu^2]/4W^2, \tag{3.21}$$

and

$$k^2 = [(W+M)^2 - m^2][(W-M)^2 - m^2]/4W^2. \tag{3.22}$$

Now consider the pion-nucleon vertex function F defined by (2.2). If the nucleon is contracted, in the "one-meson" approximation the absorptive part is given by

$$\begin{aligned}\mathrm{Im}F = \tfrac{1}{2}(2\pi)^4(2q_0)^{\frac{1}{2}} &\\ \times\{ \sum_{p'',q''} \langle 0|f|N''\pi''\rangle\langle N''\pi''|\bar{f}|\pi\rangle &\\ \times\delta(p''+q''-p-q) + \sum_{p'',k''} \langle 0|f|N''V''\rangle &\\ \times\langle N''V''|\bar{f}|\pi\rangle\delta(p''+k''-p-q)\}u(p), &\end{aligned} \tag{3.23}$$

and the unitarity condition for $\mathfrak{F}_+$ becomes

$$\mathrm{Im}\mathfrak{F}_+ = q\mathfrak{F}_+^* f_{1-} + k\left(\frac{E_1-M}{E_2-M}\right)^{\frac{1}{2}} \mathcal{G}_+^* t_{1-}. \tag{3.24}$$

Likewise, for the V-nucleon vertex function, one is led to write

$$\mathrm{Im}\mathcal{G}_+ = q\left(\frac{E_2-M}{E_1-M}\right)^{\frac{1}{2}} \mathfrak{F}_+^* t_{1-} + k\mathcal{G}_+^* g_{1-}. \tag{3.25}$$

The five unitarity conditions given above now suggest that we define the two-channel partial wave scattering matrix $\mathbf{M}$ and form factor $\mathbf{H}$ by setting

$$\mathbf{M} = \frac{4\pi W}{M}\begin{bmatrix} g_{1-} & \left(\dfrac{E_1-M}{E_2-M}\right)^{\frac{1}{2}} t_{1-} \\ \left(\dfrac{E_2-M}{E_1-M}\right)^{\frac{1}{2}} t_{1-} & f_{1-}\end{bmatrix}, \tag{3.26}$$

$$\mathbf{H} = (\mathcal{G}_+ \mathfrak{F}_+). \tag{3.27}$$

The unitarity conditions (2.6) and (2.7) are then con-

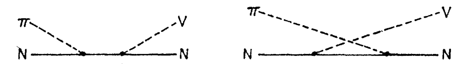

FIG. 3. Feynman diagrams contributing to the Born amplitude B_{21}.

sistent with this choice if we take the ρ matrix equal to

$$\varrho = \frac{M}{4\pi W}\begin{pmatrix} k & 0 \\ 0 & q\end{pmatrix}. \tag{3.28}$$

We now observe from Eq. (3.3) and the remarks following it, that on the nucleon mass shell

$$H_1/H_2 = \mathcal{G}_+/\mathfrak{F}_+ = g_V/g_\pi. \tag{3.29}$$

This ratio can now be set equal to that given in Eq. (2.18), and the prescription of Sec. II can then be carried out directly to determine the D matrix elements.

In particular, we need to evaluate $I_{21}(M)$ which involves $B_{21}(W)$, the $1-$ partial wave Born amplitude for the reaction $N+V \to N+\pi$. The two Feynman diagrams contributing to the scattering matrix element T_{21} in lowest order are pictured in Fig. 3. In order to ensure the adoption of a consistent sign convention, we may write the interaction Lagrangian in the form

$$\mathcal{L}_I = -ig_V\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}_V\psi - ig_\pi\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}_\pi\psi. \tag{3.30}$$

This agrees with the explicit construction of the vertex function F, Eq. (3.1), on the nucleon mass shell in lowest order perturbation theory. Taking into account a relationship analogous to Eq. (3.5) between the T and S matrix elements, one then finds that the Born amplitude for T_{21} in the isotopic spin $I=\frac{1}{2}$ state is given by

$$\begin{aligned}T_{21} = -g_V g_\pi \bar{u}(p')\bigg[3\frac{i\gamma\cdot(p+k)+M}{M^2-W^2} &\\ -\frac{i\gamma\cdot(p-q')+M}{(p-q')^2+M^2}\bigg]u(p). &\end{aligned} \tag{3.31}$$

From this we identify

$$t_1 = [(E_1+M)(E_2+M)]^{\frac{1}{2}}\frac{W-M}{8\pi W}(\alpha+\beta), \tag{3.32}$$

$$t_2 = [(E_1-M)(E_2-M)]^{\frac{1}{2}}\frac{W+M}{8\pi W}(\alpha+\beta), \tag{3.33}$$

with

$$\alpha = g_V g_\pi(3/M^2 - W^2), \tag{3.34}$$

$$\beta = g_V g_\pi[(p'-k)^2 + M^2]. \tag{3.35}$$

Next, the t_{1-} amplitude is projected out and $B_{21} = M_{21}$ is calculated from the definition in (3.26).

We now observe from Eqs. (3.28) and (3.21) that $\rho_{22}(W)$ is an even function of W. The integral $I_{21}(M)$ appearing in the expression (2.23) for $d_{21}(M)$ then becomes

$$I_{21}(M)=\frac{2M}{g_V g_\pi}\int_{M+\mu}^{\infty}\frac{dW}{W}q(W)\left(\frac{B_{21}(W)}{W-M}-\frac{B_{21}(-W)}{W+M}\right)$$
$$=M\int_{M+\mu}^{\infty}\frac{dW}{W}q(W)J(W), \tag{3.36}$$

which also serves to define $J(W)$. For a V meson of the pseudoscalar variety, we find

$$J_{PS}(W)=6\frac{\mu^2}{(W^2-M^2)^2}+\frac{1}{E_1^2-M^2}$$
$$+\left(\frac{(W+E_1)\mu^2m^2}{W(E_1^2-M^2)(W^2-M^2)}-\frac{E_1(W^2-M^2)}{W(E_1^2-M^2)}\right)$$
$$\times\frac{1}{4qk}\ln\frac{2E_2k_0-m^2+2qk}{2E_2k_0-m^2-2qk}. \tag{3.37}$$

We shall complete the calculation of $I_{21}(M)$ and $g_\pi^2/4\pi$ for the pseudoscalar case in Sec. VI, but first we turn to an analysis of the scalar and vector V meson cases.

IV. SCALAR CASE

It is of some interest to see what effect a particular choice of parity for the V meson has upon the final results obtained in Sec. VI for the pion-nucleon coupling constant. This is most easily demonstrated in the spin-zero case where the procedure for a scalar V meson can be contrasted with that for a pseudoscalar V meson as treated in the previous section. The main difference to be expected between these two cases is that the $1-$ and $0+$ partial wave channels are mixed for the reaction amplitude $N+\pi\rightarrow N+V$.

Since the transformation properties of the pion-nucleon vertex function and the elastic pion-nucleon scattering amplitude T_{22} remain unchanged, Eqs. (3.1) through (3.12) are still valid. In addition, the structure of the elastic V-nucleon scattering amplitude as given by Eq. (3.13) remains unaltered. We must thus reconsider only the V-nucleon vertex function and the reaction amplitude T_{12}.

The vertex function G defined in (2.1) now transforms like a scalar in spinor space. In the center-of-mass system, we write for G

$$G=\{G_1+G_2[i\gamma\cdot(p+k)+M]\}u(p)$$
$$=\tfrac{1}{2}[\mathcal{G}_+(1+\gamma_4)+\mathcal{G}_-(1-\gamma_4)]u(p), \tag{4.1}$$

or in two-component notation

$$G=[2M(E_1+M)]^{-\frac{1}{2}}\begin{bmatrix}\mathcal{G}_+(E_1+M)\\ -\mathcal{G}_-\boldsymbol{\sigma}\cdot\mathbf{k}\end{bmatrix}\chi_1, \tag{4.2}$$

where as before

$$\mathcal{G}_+=G_1+(M-W)G_2, \tag{4.3}$$

$$\mathcal{G}_-=G_1+(M+W)G_2. \tag{4.4}$$

Here, however, $\mathcal{G}_+$ is the ${}^2S_{\frac{1}{2}}$ or $0+$ partial wave form factor and $\mathcal{G}_-$ the ${}^2P_{\frac{1}{2}}$ or $1-$ partial wave amplitude which is just the reverse of the situation found in Sec. III.

The structure of the scattering matrix T_{12} in the present case is given by

$$T_{12}=-i\bar{u}(p')[A_{12}-\tfrac{1}{2}i\gamma\cdot(k'+q)B_{12}]\gamma_5u(p)$$
$$=-(4\pi W/M)i\chi_1'^*[t_1\boldsymbol{\sigma}\cdot\hat{k}'+t_2\boldsymbol{\sigma}\cdot\hat{q}]\chi_2. \tag{4.5}$$

Here again the amplitudes t_1 and t_2 can be related to the invariant amplitudes A_{12} and B_{12} by

$$t_1=\frac{[(W-M)^2-m^2]^{\frac{1}{2}}[(W+M)^2-\mu^2]^{\frac{1}{2}}}{16\pi W^2}\times(-A_{12}+WB_{12}), \tag{4.6}$$

$$t_2=\frac{[(W+M)^2-m^2]^{\frac{1}{2}}[(W-M)^2-\mu^2]^{\frac{1}{2}}}{16\pi W^2}\times(A_{12}+WB_{12}). \tag{4.7}$$

Time-reversal invariance then enables one to write the construction for the scattering amplitude T_{21} in terms of t_1 and t_2 as follows:

$$T_{21}=(4\pi W/M)i\chi_2'^*[t_1\boldsymbol{\sigma}\cdot\hat{k}+t_2\boldsymbol{\sigma}\cdot\hat{q}']\chi_1. \tag{4.8}$$

An l wave expansion of t_1 and t_2 can be carried out similar to that performed in Eq. (3.9) for the f_i. For the only total angular momentum state of interest, $J=\frac{1}{2}$, the two partial wave amplitudes are given by

$$\Delta_+=t_1^1+t_2^0 \tag{4.9}$$

for the reaction $N+\pi\rightarrow N+V$ from an initial ${}^2P_{\frac{1}{2}}$ $(1-)$ $N\pi$ state to a final ${}^2S_{\frac{1}{2}}$ $(0+)$ NV state, and conversely; and

$$\Delta_-=t_1^0+t_2^1 \tag{4.10}$$

for scattering of pions from nucleons in an initial ${}^2S_{\frac{1}{2}}$ $(0+)$ state into V mesons and nucleons in a ${}^2P_{\frac{1}{2}}$ $(1-)$ final state, and conversely. In order to avoid the labeling confusion which can arise due to the fact that the initial and final partial wave states are mixed, we have designated the above partial wave amplitudes by Δ_+ and Δ_- rather than by t_{1-} and t_{0+} as was previously done in the pseudoscalar case. A reflection principle analogous to (3.12) connects these two partial wave amplitudes.

In terms of the matrix element definitions and our new partial wave amplitudes, the three partial wave unitarity conditions become

$$\mathrm{Im}f_{1-}=q|f_{1-}|^2+k|\Delta_+|^2, \tag{4.11}$$

$$\mathrm{Im}\Delta_+=q\Delta_+^*f_{1-}+kg_{0+}^*\Delta_+$$
$$=qf_{1-}^*\Delta_++k\Delta_+^*g_{0+}, \tag{4.12}$$

$$\mathrm{Im}g_{0+}=q|\Delta_+|^2+k|g_{0+}|^2. \tag{4.13}$$

These relations take the place of the unitarity conditions (3.18)–(3.20). Likewise, the unitarity conditions for the form factors now become

$$\mathrm{Im}\mathcal{F}_+ = q\mathcal{F}_+{}^* f_{1-} + k\left(\frac{E_1+M}{E_2-M}\right)^{\frac{1}{2}} \mathcal{G}_+{}^*\Delta_+, \quad (4.14)$$

$$\mathrm{Im}\mathcal{G}_+ = q\left(\frac{E_2-M}{E_1+M}\right)^{\frac{1}{2}} \mathcal{F}_+{}^*\Delta_+ + k\mathcal{G}_+{}^* g_{0+}. \quad (4.15)$$

These are to be compared with Eqs. (3.24) and (3.25) for the pseudoscalar case.

The N/D matrix method is now introduced just as before. Taking note of Eqs. (2.6) and (2.7), we find the above unitarity relations are satisfied identically if we set

$$\mathbf{M} = \frac{4\pi W}{M}\begin{bmatrix} g_{0+} & \left(\frac{E_1+M}{E_2-M}\right)^{\frac{1}{2}}\Delta_+ \\ \left(\frac{E_2-M}{E_1+M}\right)^{\frac{1}{2}}\Delta_+ & f_{1-} \end{bmatrix}, \quad (4.16)$$

$$\mathbf{H} = (\mathcal{G}_+ \mathcal{F}_+), \quad (4.17)$$

and

$$\boldsymbol{\varrho} = \frac{M}{4\pi W}\begin{pmatrix} k & 0 \\ 0 & q \end{pmatrix}. \quad (4.18)$$

With this choice of matrices, the pion-nucleon coupling constant can be calculated directly following the procedure developed in Sec. II and applied to the pseudoscalar case in Sec. III.

For this purpose we need the Born approximation for the reaction amplitude T_{21}. We maintain a consistent sign convention for the coupling constants by writing for the interaction Lagrangian

$$\mathcal{L}_I = g_V\bar{\psi}\boldsymbol{\tau}\cdot\boldsymbol{\phi}_V\psi - ig_\pi\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}_\pi\psi, \quad (4.19)$$

where the V meson is assumed to be of the isotopic vector variety. The two Feynman diagrams contributing in lowest order are illustrated in Fig. 3. Taking into account the relationship between the S and T matrix elements as given in Eq. (3.5), we find for the isotopic spin $I=\frac{1}{2}$ Born amplitude

$$T_{21} = -ig_Vg_\pi\bar{u}(p')\left[3\frac{i\gamma\cdot(p+k)+M}{M^2-W^2} + \frac{i\gamma\cdot(p-q')-M}{(p-q')^2+M^2}\right]\gamma_5 u(p). \quad (4.20)$$

If this amplitude is then expressed in two-component spinor notation, comparison with Eq. (4.8) enables us to identify

$$t_1 = [(E_1-M)(E_2+M)]^{\frac{1}{2}}(W-M/8\pi W)(\alpha-\beta), \quad (4.21)$$

$$t_2 = [(E_1+M)(E_2-M)]^{\frac{1}{2}}(W+M/8\pi W)(\alpha-\beta), \quad (4.22)$$

where α and β have been defined in Eqs. (3.34) and (3.35). If the Δ_+ amplitude is then projected out according to (4.9), the M_{21} matrix element of (4.16) in Born approximation, i.e., B_{21} as it is referred to in Sec. II, can be calculated.

Finally the integral I_{21} of (2.23) must be evaluated in order to find the pion-nucleon coupling constant according to Eq. (2.25). In terms of the function $J(W)$ defined in Eq. (3.36), we find for the scalar case

$$J_S(W) = J_{PS}(W) + \frac{4\mu^2}{W^2-M^2}\frac{1}{4kq} \times \ln\frac{2E_2k_0-m^2+2kq}{2E_2k_0-m^2-2kq}, \quad (4.23)$$

which is analogous to (3.37) for the pseudoscalar V meson case.

V. VECTOR CASE

We now turn our attention to the more interesting case of a vector meson. The V meson must transform like a vector (or axial vector) particle, if it is to be identified with the "physical" intermediate boson of beta decay as formulated by Lee and Yang.[16] In principle, our plan of attack for this case is identical to that of the pseudoscalar and scalar cases, but the N/D matrix method must be modified in order to accommodate the complications introduced by the spin of the V particle. As noted in Sec. IV for the scalar case, Eqs. (3.1) through (3.12) for the pion-nucleon vertex functions and pion-nucleon elastic scattering amplitudes remain valid, while the corresponding expressions involving the V meson must be altered.

We represent by ξ_μ the polarization four-vector of the V particle. In the rest frame of this particle, we demand that the time component ξ_0 vanish so that

$$\xi\cdot k = 0. \quad (5.1)$$

Since the left-hand side of this expression is an invariant, the above equation holds in any frame of reference and corresponds to the Lorentz condition of electrodynamics. Unlike the case of electrodynamics, however, we are not free to choose a particular gauge in which, say, $\xi_0 \approx 0$. At most, ξ_0 can be related to $\boldsymbol{\xi}$ through Eq. (5.1).

With these considerations in mind, we construct the V meson vertex function (2.1) by writing

$$G = -i\{G_1+G_2[i\gamma\cdot(p+k)+M]\}\gamma\cdot\xi u(p) + \{G_3+G_4[i\gamma\cdot(p+k)+M]\}\xi\cdot p u(p); \quad (5.2a)$$

here again the G_i are functions of W^2. In two-component notation, this becomes in light of (5.1)

$$G = [2M(E_1+M)]^{-\frac{1}{2}}\begin{pmatrix} (\mathcal{H}_+ + \mathcal{G}_+)\boldsymbol{\xi}\cdot\mathbf{k} + \mathcal{G}_+ i\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\times\mathbf{k} \\ \{[\mathcal{H}_- + (E_1+M)\mathcal{G}_-]\boldsymbol{\xi}\cdot\mathbf{k} + (E_1+M)\mathcal{G}_- i\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\times\mathbf{k}\}\boldsymbol{\sigma}\cdot\mathbf{k} \end{pmatrix}\chi_1, \quad (5.2b)$$

where

$$\mathcal{G}_{\pm}=G_1+(M\mp W)G_2, \tag{5.3}$$

$$\mathcal{G}'_{\pm}=G_3+(M\mp W)G_4, \tag{5.4}$$

$$\mathcal{H}_+=[\mathcal{G}_+-W\mathcal{G}'_+](E_1+M)/(W-E_1), \tag{5.5}$$

and

$$\mathcal{H}_-=[\mathcal{G}_-+W\mathcal{G}'_-](E_1^2-M^2)/(W-E_1). \tag{5.6}$$

Now there are four invariant form factors present; however, on the nucleon mass shell, two of them contribute to the structure of G, as seen from

$$\bar{u}(p+k)G=\bar{u}(p+k)[-iG_1\gamma\cdot\xi+G_3\xi\cdot p]u(p). \tag{5.7}$$

This situation may be contrasted with that existing in the pseudoscalar and scalar cases, where only one form factor remains when the vertex function is evaluated on the mass shell of the nucleon. This new complication will demand some further discussion of our technique for calculating the pion-nucleon coupling constant, and this point will be returned to later.

We now turn our attention to the structure of the reaction amplitude T_{12} given in Eq. (2.4). This matrix element is very similar to that for photoproduction of mesons as treated by Bincer, for example.[22] Six invariant amplitudes characterize the problem. Since it is easier to work only with the two-component expressions, for ease of writing we refer the reader to Eqs. (30) and (31) of reference 22 for the covariant four-component structure of T_{12}. In two-component notation we have

$$\begin{aligned}T_{12}=&(4\pi W/M)i\chi_1'^*[t_1\boldsymbol{\sigma}\cdot\xi^*+t_2(i\boldsymbol{\sigma}\cdot\hat{k}\times\xi^*)\boldsymbol{\sigma}\cdot\hat{q}\\&+t_3\boldsymbol{\sigma}\cdot\hat{k}\xi^*\cdot\hat{q}+t_4\boldsymbol{\sigma}\cdot\hat{q}\xi^*\cdot\hat{q}+t_5\boldsymbol{\sigma}\cdot\hat{k}\xi^*\cdot\hat{k}\\&+t_6\boldsymbol{\sigma}\cdot\hat{q}\xi^*\cdot\hat{k}]\chi_2\\=&(4\pi W/M)i\chi_1'^*T(\hat{k},\hat{q})\chi_2.\end{aligned} \tag{5.8}$$

We simply note at this point that the last two terms vanish in the meson photoproduction process if one chooses to work in the Coulomb gauge. As remarked earlier, such considerations do not apply in the presence of the weak vector boson.

It is customary to make a multipole expansion of the t_i. But since only the $J=\frac{1}{2}$ amplitudes are of interest, we find it more convenient to expand the t_i directly in partial waves up through $l=2$ according to

$$t_i=t_i^0+3t_i^1P_1(\hat{k}\cdot\hat{q})+5t_i^2P_2(\hat{k}\cdot\hat{q}),\quad(i=1,\cdots,6). \tag{5.9}$$

We can then determine the partial wave transition amplitudes as follows. We first apply the proper projection operators to the right of the matrix $T(\hat{k},\hat{q})$ which select the total angular momentum $J=\frac{1}{2}$ and appropriate l wave in the initial pion-nucleon state. From the left we then project out the final V-nucleon orbital angular momentum of interest. In particular, we find

$$\begin{aligned}t(P_{\frac{1}{2}}\leftarrow P_{\frac{1}{2}})&=\int\frac{d\Omega''}{4\pi}\frac{d\Omega'}{4\pi}3P_1(\hat{k}\cdot\hat{k}'')T(\hat{k}'',\hat{q}')\boldsymbol{\sigma}\cdot\hat{q}'\boldsymbol{\sigma}\cdot\hat{q}\\&=(a+b)\xi^*\cdot\hat{k}\boldsymbol{\sigma}\cdot\hat{q}+a(i\boldsymbol{\sigma}\cdot\xi^*\times\hat{k})\boldsymbol{\sigma}\cdot\hat{q},\end{aligned} \tag{5.10}$$

$$\begin{aligned}t(S_{\frac{1}{2}}\leftarrow S_{\frac{1}{2}})&=\int\frac{d\Omega''}{4\pi}\frac{d\Omega'}{4\pi}T(\hat{k}'',\hat{q}')\\&=(c+\tfrac{1}{3}d)\boldsymbol{\sigma}\cdot\xi^*,\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}t(D_{\frac{1}{2}}\leftarrow S_{\frac{1}{2}})&=\int\frac{d\Omega''}{4\pi}\frac{d\Omega'}{4\pi}5P_2(\hat{k}\cdot\hat{k}'')T(\hat{k}'',\hat{q}')\\&=d(-\tfrac{1}{3}\boldsymbol{\sigma}\cdot\xi^*+\boldsymbol{\sigma}\cdot\hat{k}\xi^*\cdot\hat{k}),\end{aligned} \tag{5.12}$$

where

$$a=t_1^1-t_2^0-\tfrac{1}{3}t_3^0+\tfrac{1}{3}t_3^2, \tag{5.13}$$

$$b=t_2^0+\tfrac{2}{3}t_3^0+\tfrac{1}{3}t_3^2+t_4^1+t_5^1+t_6^0, \tag{5.14}$$

$$c=t_1^0-t_2^1+\tfrac{1}{3}t_4^0-\tfrac{1}{3}t_4^2, \tag{5.15}$$

$$d=t_2^1+t_3^1+t_4^2+t_5^0+t_6^1. \tag{5.16}$$

In a similar fashion, we would find for $f(P_{\frac{1}{2}}\leftarrow P_{\frac{1}{2}})$,

$$f(P_{\frac{1}{2}}\leftarrow P_{\frac{1}{2}})=f_{1-}\boldsymbol{\sigma}\cdot\hat{q}'\boldsymbol{\sigma}\cdot\hat{q}. \tag{5.17}$$

We emphasize here that the letter appearing in the spectroscopic notation refers only to the *orbital* angular momentum. Since the pion transforms like a pseudoscalar and the polarization vector of the V meson changes sign under space inversion, the parity of the initial and final l waves must be equal as indicated in (5.10–12).

Finally we remark that the scattering matrix element T_{21} is given by

$$\begin{aligned}T_{21}=&\bar{u}(p')\langle\pi'|f|NV\rangle\left(\frac{4E_1q_0k_0}{M}\right)^{\frac{1}{2}}\\=&-\frac{4\pi W}{M}i\chi_2'^*[t_1\boldsymbol{\sigma}\cdot\xi-t_2\boldsymbol{\sigma}\cdot\hat{q}(i\boldsymbol{\sigma}\cdot\hat{k}\times\xi)\\&+t_3\boldsymbol{\sigma}\cdot\hat{k}\xi\cdot\hat{q}+t_4\boldsymbol{\sigma}\cdot\hat{q}\xi\cdot\hat{q}+t_5\boldsymbol{\sigma}\cdot\hat{k}\xi\cdot\hat{k}\\&+t_6\boldsymbol{\sigma}\cdot\hat{q}\xi\cdot\hat{k}]\chi_1\end{aligned} \tag{5.18}$$

as demanded by time-reversal invariance. Moreover, the partial wave transition amplitudes now assume the form

$$t(P_{\frac{1}{2}}\leftarrow P_{\frac{1}{2}})=(a+b)\xi\cdot\hat{k}\boldsymbol{\sigma}\cdot\hat{q}-a\boldsymbol{\sigma}\cdot\hat{q}(i\boldsymbol{\sigma}\cdot\xi\times\hat{k}), \tag{5.19}$$

$$t(S_{\frac{1}{2}}\leftarrow S_{\frac{1}{2}})=(c+\tfrac{1}{3}d)\boldsymbol{\sigma}\cdot\xi, \tag{5.20}$$

$$t(S_{\frac{1}{2}}\leftarrow D_{\frac{1}{2}})=d(-\tfrac{1}{3}\boldsymbol{\sigma}\cdot\xi+\boldsymbol{\sigma}\cdot\hat{k}\xi\cdot\hat{k}), \tag{5.21}$$

where a, b, c, and d are again given by Eqs. (5.13)–(5.16).

To complete the kinematical preliminaries for the vector case, we should now construct the elastic nucleon-V meson T_{11} matrix element defined in (2.3). This construction is completely analogous to the elastic neutron-deuteron scattering problem. Here one finds that twelve invariant amplitudes are necessary to describe the problem. Fortunately, we can forego the specific details and simply refer the interested reader to

[22] A. M. Bincer, Phys. Rev. 118, 855 (1960).

the paper of Blankenbecler *et al.*[23] The reason for this is that the weak coupling constant is so small that in our approximation procedure the specific structure of the partial wave amplitudes for the T_{11} elastic channel never enters (cf. the previous sections). Moreover, we shall be able to construct the partial wave scattering matrix **M** and form factor matrix **H** without detailed knowledge of the elastic nucleon-V meson matrix elements.

We now form the unitarity conditions for the partial wave scattering amplitudes and form factors in the "one-meson" approximation illustrated in Figs. 1 and 2. It is sufficient for us to consider only the p-wave amplitudes. Following the reduction formalism of LSZ[21] to calculate the absorptive parts of the various scattering amplitudes, we find from equations similar to (3.17)

$$\operatorname{Im} f(P_{\frac{1}{2}} \leftarrow P_{\frac{1}{2}}) = q \sum_{\Omega''} |f(P_{\frac{1}{2}} \leftarrow P_{\frac{1}{2}})|^2 + k \sum_{\Omega'',\xi} |t(P_{\frac{1}{2}} \leftarrow P_{\frac{1}{2}})|^2, \quad (5.22\text{a})$$

and

$$\operatorname{Im} t(P_{\frac{1}{2}} \leftarrow P_{\frac{1}{2}}) = q \sum_{\Omega''} t^*(P_{\frac{1}{2}} \leftarrow P_{\frac{1}{2}}) f(P_{\frac{1}{2}} \leftarrow P_{\frac{1}{2}}) + \cdots, \quad (5.23\text{a})$$

for the unitarity conditions along the positive branch cuts. The dots indicate the term neglected according to the prescription given in the preceding paragraph. If the integrations over solid angle and summation over polarization direction are carried out, use of Eqs. (5.10) and (5.17) enables one to identify

$$\operatorname{Im} f_{1-} = q|f_{1-}|^2 + k[3|a|^2 + a^*b + b^*a + |b|^2]. \quad (5.22\text{b})$$

$$\operatorname{Im}(a+b) = q(a^*+b^*)f_{1-} + \cdots, \quad (5.23\text{b})$$

$$\operatorname{Im} a = q a^* f_{1-} + \cdots. \quad (5.23\text{c})$$

In an entirely analogous fashion, we find for the partial-wave form factors

$$\operatorname{Im}\mathfrak{F}_+ = q\mathfrak{F}_+^* f_{1-} + k\left(\frac{E_1-M}{E_2-M}\right)^{\frac{1}{2}} \times[-(\mathfrak{K}_+^* + \mathfrak{G}_+^*)(a+b) + 2\mathfrak{G}_+^* a], \quad (5.24)$$

$$\operatorname{Im}(\mathfrak{K}_+ + \mathfrak{G}_+) = -q\left(\frac{E_2-M}{E_1-M}\right)^{\frac{1}{2}} \mathfrak{F}_+^*(a+b) + \cdots, \quad (5.25\text{a})$$

and

$$\operatorname{Im}\mathfrak{G}_+ = q\left(\frac{E_2-M}{E_1-M}\right)^{\frac{1}{2}} \mathfrak{F}_+^* a + \cdots. \quad (5.25\text{b})$$

It is immediately obvious from the above unitarity conditions that we can *not* construct a two-by-two partial-wave scattering matrix **M** and corresponding form factor matrix **H** from which the above conditions can be read off according to Eqs. (2.6) and (2.7). For example, the presence of the factor 3 in (5.22b) rules out this possibility. Clearly the difficulty arises because of the presence of two different p waves, ${}^2P_{\frac{1}{2}}$ and ${}^4P_{\frac{1}{2}}$, in the V meson-nucleon states as opposed to only the single p wave, ${}^2P_{\frac{1}{2}}$, in the pion-nucleon states. This suggests that we replace the single matrix element M_{11} for elastic channel 1 scattering by a two-by-two matrix whose elements are related to the partial wave amplitudes in the V-nucleon elastic scattering amplitude

$$g(P_{\frac{1}{2}} \leftarrow P_{\frac{1}{2}}) = g_{11}\xi^*\cdot\hat{k}'\xi\cdot\hat{k} + g_{12}\xi^*\cdot\hat{k}'(-i\boldsymbol{\sigma}\cdot\xi\times\hat{k}) + g_{21}(i\boldsymbol{\sigma}\cdot\xi^*\times\hat{k}')\xi\cdot\hat{k} + g_{22}(i\boldsymbol{\sigma}\cdot\xi^*\times\hat{k}')(-i\boldsymbol{\sigma}\cdot\xi\times\hat{k}). \quad (5.26)$$

A detailed study of T_{11} would bring out this dependence. It suffices for us to recognize that the N/D matrix method of Sec. II can be employed if the partial-wave scattering matrix **M** now represents a three-by-three matrix whose first two rows and columns refer to the nucleon-V meson channel and whose third row and third column refer to the nucleon-pion channel. Likewise the form-factor matrix **H** is now understood to represent a row matrix with three elements. A brief study of the above unitarity conditions then reveals that the choice

$$\mathbf{M} = \frac{4\pi W}{M}\begin{bmatrix} \cdots & \cdots & -\left(\frac{E_1-M}{E_2-M}\right)^{\frac{1}{2}}(a+b) \\ \cdots & \cdots & \left(\frac{E_1-M}{E_2-M}\right)^{\frac{1}{2}} a \\ -\left(\frac{E_2-M}{E_1-M}\right)^{\frac{1}{2}}(a+b) & \left(\frac{E_2-M}{E_1-M}\right)^{\frac{1}{2}} a & f_{1-} \end{bmatrix}, \quad (5.27)$$

$$\mathfrak{K} = ((\mathfrak{K}_+ + \mathfrak{G}_+) \quad \mathfrak{G}_+ \quad \mathfrak{F}_+), \quad (5.28)$$

and

$$\varrho = \frac{M}{4\pi W}\begin{pmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & q \end{pmatrix}, \quad (5.29)$$

[23] R. Blankenbecler, M. L. Goldberger, and F. R. Halpern, Nuclear Phys. 12, 629 (1959).

leads to a consistent set of unitarity relations when Eqs. (2.6) and (2.7) are applied.

We now introduce the N/D matrix method in which the definitions $\mathbf{M}=\mathbf{N}\mathbf{D}^{-1}$ and $\mathbf{H}=\boldsymbol{\lambda}\mathbf{D}^{-1}$ refer to three-by-three matrices for N and D and a row matrix similar to $\mathbf{H}$ with constant coefficients for $\boldsymbol{\lambda}$. The matrices $\mathbf{N}$ and $\mathbf{D}$ satisfy the same system of coupled integral equations given in (2.21) and (2.22). By an argument similar to that presented in Sec. II, we are led to set $\lambda_1=\lambda_2=0$; consequently, we find from (2.9)

$$(\det\mathbf{D})(\mathcal{H}_+ + \mathcal{G}_+)=\lambda_3(d_{21}d_{32}-d_{22}d_{31}), \quad (5.30a)$$

$$(\det\mathbf{D})\mathcal{G}_+=\lambda_3(d_{31}d_{12}-d_{32}d_{11}), \quad (5.30b)$$

and

$$(\det\mathbf{D})\mathcal{F}_+=\lambda_3(d_{11}d_{22}-d_{12}d_{21}). \quad (5.30c)$$

If we now form the ratio of two of the above equations and make use of the fact that the diagonal elements d_{11} and d_{22} are essentially unity while the off-diagonal elements are small, we obtain

$$\mathcal{G}_+/\mathcal{F}_+\approx -d_{32}/d_{22}\approx -d_{32}. \quad (5.31)$$

On the nucleon mass shell, $\mathcal{G}_+(M)=G_1(M^2)$ and the nucleon-V meson vertex function assumes the structure indicated in Eq. (5.7). In the absence of G_3, we would identify $G_1(M^2)$ (up to a factor $\sqrt{3}$)[19] with the renormalized V-nucleon coupling constant corresponding to the unrenormalized coupling constant appearing in the structure of the interaction Lagrangian

$$\mathcal{L}_I=-g_V\bar{\psi}i\gamma_\mu\boldsymbol{\tau}\cdot\boldsymbol{\phi}_{V\mu}\psi-g_\pi i\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}_\pi\psi. \quad (5.32)$$

In analogy with quantum electrodynamics, we have assumed the validity of a principle of "minimal intermediate boson interaction" in writing down the vector coupling appearing in the above Lagrangian. The presence of G_3 in Eq. (5.7) demands some reinterpretation. One could introduce into the interaction Lagrangian a derivative coupling term with suitable coupling constant which serves to generate the G_3 term in (5.7). We find this rather distasteful. An alternative is to regard the G_3 term as being induced by the weak boson coupling of (5.32). This interpretation is rather attractive but without a detailed dynamics, we are unable to express the dependence of G_1 and G_3 on W^2 and, in particular, to relate these form factors to the renormalized coupling constant g_V exactly. To resolve this dilemma, we shall assume that $G_3\approx 0$ and set $G_1=\sqrt{3}g_V$ on the nucleon mass shell. Equation (5.31) then reduces to

$$g_V/g_\pi=-d_{32}(M)=(g_Vg_\pi/8\pi^2)I_{32}(M) \quad (5.33)$$

or

$$g_\pi^2/4\pi=2\pi/I_{32}(M),$$

where

$$I_{32}(M)=\frac{8\pi}{g_Vg_\pi}\int_{M+\mu}^{\infty}dW\rho_3(W)\left(\frac{B_{32}(W)}{W-M}-\frac{B_{32}(-W)}{W+M}\right)$$

$$\equiv M\int_{M+\mu}^{\infty}dW\frac{q(W)}{W}J(W). \quad (5.34)$$

Recall that $B_{32}(W)$ refers in our approximation to the Born amplitude for the partial wave matrix element M_{32}. A comparison of the above equations with their counterparts, Eqs. (2.25) and (3.36), in the pseudoscalar V meson case reveals that both sets formally have the same structure. Thus, according to our approximation procedure, the detailed differences among the three types of V mesons considered are all buried in the factor $J(W)$.

Turning now to the calculation of this factor, we find for the Feynman amplitude T_{21} in the isotopic spin $\frac{1}{2}$ state

$$T_{21}=g_Vg_\pi\bar{u}(p')\left[3\frac{i\gamma\cdot(p'+q')+M}{M^2-W^2}\gamma\cdot\xi\right.$$

$$\left.+\gamma\cdot\xi\frac{-i\gamma\cdot(p'-q)+M}{(p'-q)^2+M^2}\right]\gamma_5 u(p), \quad (5.35)$$

with use of the lowest-order diagrams in Fig. 3 and the interaction Lagrangian (5.32). If this matrix element is then spelled out in two-component notation and compared with the structure given in Eq. (5.18), one can immediately identify the amplitudes

$$t_1=[(E_1+M)(E_2+M)]^{\frac{1}{2}}(W-M/8\pi W)(\alpha-\beta), \quad (5.36)$$

$$t_2=-[(E_1-M)(E_2-M)]^{\frac{1}{2}}(W+M/8\pi W)(\alpha-\beta), \quad (5.37)$$

$$t_3=-[(E_1-M)(E_2-M)]^{\frac{1}{2}}[2(E_2+M)/8\pi W]\beta, \quad (5.38)$$

$$t_4=[(E_1+M)(E_2+M)]^{\frac{1}{2}}[2(E_2-M)/8\pi W]\beta, \quad (5.39)$$

$$t_5=[(E_1+M)(E_2+M)]^{\frac{1}{2}}(E_1-M/W-E_1)(1/8\pi W)$$
$$\times[(W-M)\alpha-(2E_2-W-M)\beta], \quad (5.40)$$

and

$$t_6=[(E_1-M)(E_2-M)]^{\frac{1}{2}}(1/W-E_1)(1/8\pi W)$$
$$\times\{-(W+M)^2\alpha+[2E_2(E_1+M)+m^2]\beta\}, \quad (5.41)$$

where α and β have been defined in Eqs. (3.34) and

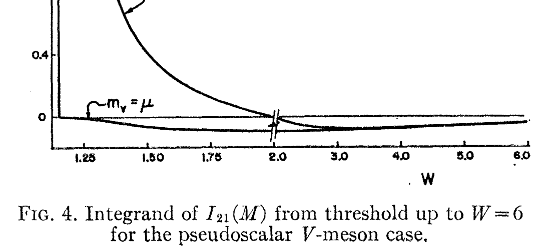

FIG. 4. Integrand of $I_{21}(M)$ from threshold up to $W=6$ for the pseudoscalar V-meson case.

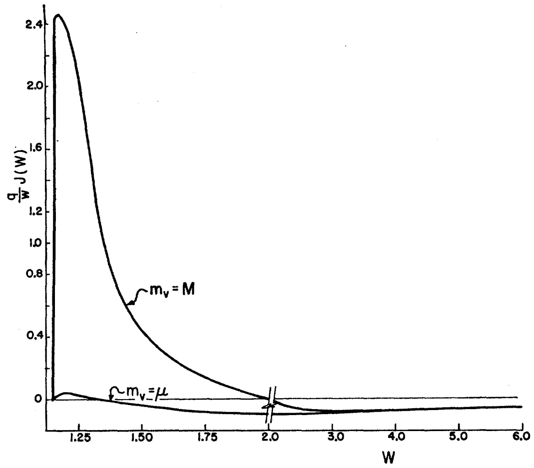

FIG. 5. Integrand of $I_{21}(M)$ from threshold up to $W=6$ for the scalar V-meson case.

(3.35). We must now form

$$B_{32}(W)=\frac{4\pi W}{M}(E_2-M/E_1-M)^{\frac{1}{2}}a, \tag{5.42}$$

by projecting out the appropriate l waves according to Eq. (5.13). Finally we make use of Eq. (5.34) and find for the vector V meson case

$$J_V(W)=-J_{PS}(W)+\frac{12\mu^2}{(W^2-M^2)^2}+\frac{2E_2k_0-m^2}{2k^2(W^2-M^2)}+\frac{2q^2}{W^2-M^2}\left(1-\frac{(2E_2k_0-m^2)^2}{4k^2q^2}\right)\times\frac{1}{4kq}\ln\frac{2E_2k_0-m^2+2kq}{2E_2k_0-m^2-2kq}. \tag{5.43}$$

Let us now turn to a numerical evaluation of the coupling constant for the three types of V mesons considered.

VI. NUMERICAL RESULTS AND CONCLUSIONS

We now complete our calculation of the pion-nucleon coupling constant. In all three cases, we have found that the general structure is of the form

$$g_\pi^2/4\pi=2\pi/I(M), \tag{6.1}$$

where

$$I(M)=\int_{M+\mu}^{\infty}\frac{M}{W}q(W)J(W)dW, \tag{6.2}$$

with the appropriate factor $J(W)$ given by Eqs. (3.37), (4.23), or (5.43).

Two remarks should be made about these expressions for $J(W)$. The logarithmic terms stand as written in the region where the momentum, k, of the V meson is real; however, in the region where k can become imaginary as given by Eq. (3.22), each logarithm must be replaced by an arctangent with the understanding that the principal branch is to be chosen. Secondly, the apparent poles in the logarithm terms of (3.37) and (5.43) at $k=0$ are nonexistent as is easily seen by a simple algebraic regrouping of terms.

As noted earlier, we choose to regard Eq. (6.1) as an equation determining the pion-nucleon coupling constant in terms of the mass parameters involved. Since the mass of the weakly interacting V meson (let alone its very existence) is experimentally unknown, we can at most determine the functional dependence of the pion-nucleon coupling constant upon the V meson mass. This is too difficult to carry out in great detail, however; instead, we have decided to select the pion mass and nucleon mass as rather widely separated masses and typical of the range in which the V meson mass might be expected to lie.

The integrands of $I(M)$ for the three cases under consideration are plotted in Figs. 4–6 from threshold up to $W=6M$ with the above-mentioned choices of mass for the V meson. Since the integral $I(M)$ is dimensionless, units have been chosen such that the nucleon mass is set equal to unity. Above $W=6$, the integrands are well represented by their asymptotic expressions for the pseudoscalar, scalar, and vector cases:

$$\left.\begin{array}{l}\dfrac{q}{W}J_{PS}(W)\\[2ex] \dfrac{q}{W}J_S(W)\end{array}\right\}\xrightarrow[W\to\infty]{}-\frac{2}{W^2}(\ln W-1), \tag{6.3}$$

$$\frac{q}{W}J_V(W)\xrightarrow[W\to\infty]{}\frac{2}{W^2}(\ln W-\tfrac{3}{4}). \tag{6.4}$$

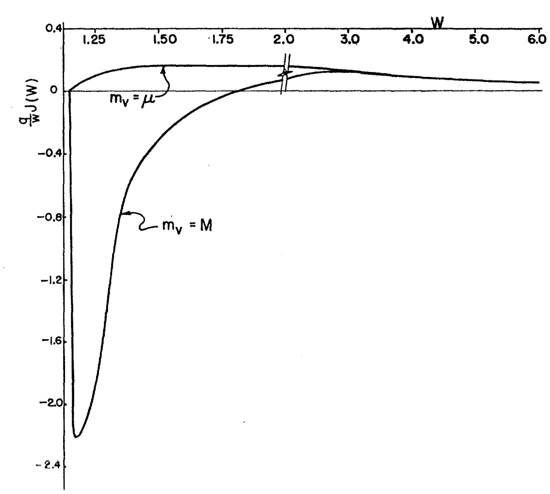

FIG. 6. Integrand of $I_{32}(M)$ from threshold up to $W=6$ for the vector V-meson case.

TABLE I. Calculation of the integral $I(M)$ and $g_\pi^2/4\pi$ for the three types of V mesons with various cutoff energies introduced.

Type V meson	Mass m	$I(M)$ with cutoff at $2M$	$3M$	∞	$g_\pi^2/4\pi$ with cutoff at $2M$	$3M$	∞
Pseudoscalar	μ	-0.06	-0.16	-0.98	-105	-39	-6.4
	M	0.51	0.46	-0.34	12	14	-19
Scalar	μ	-0.03	-0.13	-0.94	-200	-50	-6.7
	M	0.56	0.51	-0.29	11	12	-22
Vector	μ	0.10	0.22	1.23	63	29	5.1
	M	-0.43	-0.32	0.63	-15	-20	10

The results of the numerical integrations are tabulated for $I(M)$ in the accompanying table together with the results for the pion-nucleon coupling constant as determined from Eq. (6.1). Our Born approximation expressions for $J(W)$ are expected to be least accurate at high energies where the (unknown) effects of the more massive intermediate states are presumably quite significant. Therefore, for the sake of comparison, we have also included in the table values of $I(M)$ and g_π^2 when a cutoff is introduced.

A brief study of the three graphs and table reveals that for a V-meson mass equal to the pion mass, the integral maintains the same sign whatever cutoff is used (except at very low cutoff values in the scalar V case); while for a V meson as massive as the nucleon, the integral can have either sign depending upon the cutoff energy. This "anomalous" behavior exhibited in the latter case can be traced directly to the fact that, just above threshold for the pion-nucleon channel, a region exists from which contributions to the integral occur that involves unphysical values of the V-meson momentum. By crossing the V-meson–nucleon threshold, one finds that important terms in the integrand change sign; consequently, the integral $I(M)$ can assume either sign depending upon the size of the cutoff energy. This "anomalous" behavior will always occur for a V-meson mass greater than the pion mass.

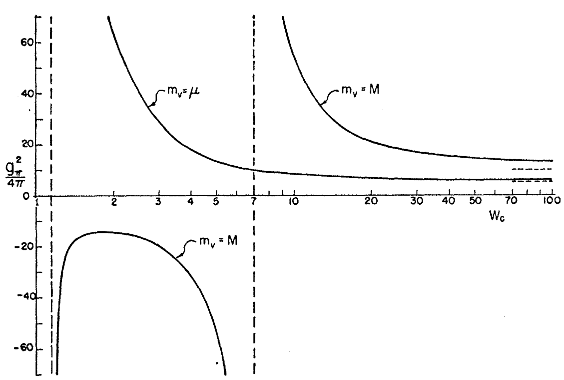

FIG. 7. $g_\pi^2/4\pi$ as a function of the cutoff energy, W_c, for a vector V meson and two choices of the V-meson mass.

Since g_π^2 is inversely proportional to $I(M)$, when the integral passes through zero, the pion-nucleon coupling constant exhibits an infinite discontinuity there. This anomaly is illustrated in Fig. 7 where the pion-nucleon coupling constant is plotted as function of the cutoff energy for the vector V-meson case. The corresponding graphs for the scalar and pseudoscalar cases resemble the mirror image of Fig. 7 obtained by a reflection about the horizontal W_c axis. The negative values of g_π^2 have no physical significance (they presumably herald the appearance of ghosts).

The results for the scalar and pseudoscalar V-meson cases are very similar; on the other hand, the results for a vector V meson differ from those of the spin 0 cases mainly by the appearance of an over-all minus sign. Therefore we find that the correct experimental value of the pion-nucleon coupling constant can be obtained with a vector V meson and a very high cutoff energy, or with a spin 0 meson of mass comparable to the nucleon mass and a low cutoff energy.