In this section, we derive the effective interaction Hamiltonian in Eq. (2) of the main text from a microscopic model of the system–environment interaction. The Hamiltonian of the closed system was described in the main text and is given by

\[ H(s) = E_0 (1 - s) (I - |\psi_0\rangle\langle\psi_0|) + E_0 s (I - |m\rangle\langle m|), \]

where \( |m\rangle \) is the marked state and \( |\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x\rangle \). We assume that this Hamiltonian is implemented using \( L \) qubits, such that the states \( |x\rangle \) in the search space are represented by the \( N = 2^L \) eigenstates of the Pauli operators \( \{\sigma_j^x\} \), with \( j = 1, \ldots, L \), acting on the individual qubits. We consider the situation where each qubit is coupled to an independent, bosonic bath, described by the generic interaction Hamiltonian

\[ V = \sum_{j=1}^{L} \sum_{\mu=x,y,z} \sigma^\mu_j \otimes \sum_k g^{\mu j}_k (b_k^{\mu j} + b_k^{\mu j \dagger}). \]

(S2)

Here \( b^{\mu j}_k \) are independent bosonic annihilation operators and \( g^{\mu j}_k \) is the coupling strength to a particular mode. We further assume that the baths are identical such that the bath Hamiltonian is given by

\[ H_B = \sum_k \omega_k \sum_{j=1}^{L} \sum_{\mu=x,y,z} b_k^{\mu j \dagger} b_k^{\mu j}. \]

(S3)

We note that many of the present assumptions can be relaxed without affecting our results qualitatively. For instance, our calculation readily carries over to the situation where all qubits couple to the same environment provided the interaction remains local.

In a closed system, the excited states at energy \( E_0 \) are completely decoupled from the non-trivial subspace \( S = \text{span} \{|m\rangle, |m_\perp\rangle\} \). Although this is not the case in an open system, we can describe the dynamics of the subspace \( S \) near the avoided crossing by an effective Hamiltonian provided the steady-state population in the excited levels is negligible. In thermal equilibrium with a bath at temperature \( T \), this gives rise to the condition \( T \ll E_0/L \) near the avoided level crossing. Since \( E_0 \), the overall energy scale of the system, is an extensive quantity, this condition can be satisfied by a small but intensive temperature. In this limit, the effective Hamiltonian for the system and environment can be derived using the general formalism in reference [S1], yielding

\[ H_{\text{eff}} = PH(s)P + PV P + \frac{1}{2} \sum_{a,b,c} \left( \frac{1}{E_a - E_c} + \frac{1}{E_b - E_c} \right) |a\rangle\langle a| |e\rangle\langle e| V |b\rangle\langle b|, \]

where \( P \) is the projection operator onto \( S \). In the sum, the indices \( a, b \) run over the eigenstates of \( H(s) \) in \( S \) (low-energy states), while \( e \) refers to the states in the orthogonal subspace (excited states). There are two contributions to the interaction of the environment with the low-energy states: a direct interaction and one that is mediated by the excited states through virtual processes. We have neglected higher order terms, which include couplings between different excited states. Such processes cannot be described in terms of the parameters of the avoided crossing alone and are therefore beyond the scope of our discussion. Furthermore, these processes do not affect our results qualitatively as discussed in more detail in section 6.

The quantum speedup in the closed system is enabled by tunneling near the avoided level crossing. We will therefore restrict ourselves to that region, which allows us to replace the energy differences in the denominator of the last term in Eq. (S4) by \(-E_0/2\), neglecting terms of order \( \sqrt{\varepsilon^2 + \Delta^2}/E_0 \). This drastically simplifies the expression to

\[ H_{\text{eff}} = H_S + H_B + PV P - \frac{1}{2E_0} PV (I - P) VP, \]

(S5)

where

\[ H_S(s) = PH(s)P = \frac{E_0}{2} - \frac{1}{2} [\varepsilon(s) \tau^x + \Delta(s) \tau^y]. \]

(S6)
Here $\tau^\mu$ are the Pauli matrices acting on $\mathcal{S}$. In the $\{ |m\rangle, |m_\perp\rangle \}$ basis, the projections of the Pauli operators acting on the physical qubits are given by (see the appendix of reference S2 for details)

$$P \sigma^\tau_j P = \frac{1}{N-1} \left( \begin{array}{cc} 0 & \sqrt{N-1} \\ \sqrt{N-1} & N-2 \end{array} \right), \quad P \sigma^\tau_j P = \frac{s_j}{\sqrt{N-1}} \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad P \sigma^\tau_j P = \frac{s_j}{N-1} \left( \begin{array}{cc} N-1 & 0 \\ 0 & -1 \end{array} \right), \quad (S7)$$

where $s_j = \langle m | \sigma^\tau_j | m \rangle = \pm 1$. We observe that the off-diagonal terms in these matrices are of order $O(N^{-1/2})$ so that coupling between $|m\rangle$ and $|m_\perp\rangle$ is suppressed in the limit of large $N$. This fact has the simple physical interpretation that $|m\rangle$ and $|m_\perp\rangle$ are macroscopically distinct, while the environment acts only locally. In the following, we neglect all terms of order $O(N^{-1/2})$ and below.

The direct interaction of the environment with the low-energy subspace is hence given by

$$V_1 = PV P \approx \frac{1}{2} \sum_{j=1}^L \sum_k \left[ g_{jk}^x (I - \tau^x) \otimes (b_{jk}^x + b_{jk}^\dagger) + g_{jk}^y s_j (I + \tau^y) \otimes (b_{jk}^x + b_{jk}^\dagger) \right], \quad (S8)$$

while the excited states mediate a two-boson interaction of the form

$$V_2 = -\frac{1}{2E_0} PV (I - P) V \approx -\sum_{j=1}^L \sum_{k,l} \left[ \frac{g_{jk}^x g_{jl}^x}{4E_0} (I + \tau^x) \otimes (b_{jk}^x + b_{jk}^\dagger) (b_{jl}^x + b_{jl}^\dagger) + \frac{g_{jk}^y g_{jl}^y}{2E_0} I \otimes (b_{jk}^y + b_{jk}^\dagger) (b_{jl}^y + b_{jl}^\dagger) + g_{jk}^y g_{jl}^y (I - \tau^x) \otimes (b_{jk}^x + b_{jk}^\dagger) (b_{jl}^x + b_{jl}^\dagger) \right]. \quad (S9)$$

This follows from the observation that $P \sigma^\tau_i \sigma^\tau_j P \approx P \sigma^\tau_i P \sigma^\tau_j P$ when $i \neq j$. We note that the field $b_{ik}^\sigma$ only couples to the identity and therefore does not affect the dynamics of the system. In order to further simplify the expressions, let us focus on a single $x$ mode and simplify the notation by only retaining a single subscript for the mode label. The bath plus effective interaction Hamiltonian involving this mode takes the form

$$H_x = I \otimes \left[ \sum_k \omega_k b_k^\dagger b_k + \frac{g_k}{2} (b_k^\dagger + b_k) - \sum_{k,l} \frac{g_k g_l}{4E_0} (b_k + b_k^\dagger) (b_l + b_l^\dagger) \right] - \tau^x \otimes \left[ \sum_k \frac{g_k}{2} (b_k^\dagger + b_k) + \sum_{k,l} \frac{g_k g_l}{4E_0} (b_k + b_k^\dagger) (b_l + b_l^\dagger) \right]. \quad (S10)$$

The terms coupling to the identity can be understood as the backaction of the system on the environment. We account for this effect by diagonalizing these terms, resulting in a set of renormalized bosonic operators. It is straightforward to check that the single-boson term can be accounted for by introducing the shifted operators

$$c_k = b_k + \frac{1}{1 - a} \frac{g_k}{2 \omega_k}, \quad (S11)$$

where

$$a = \frac{1}{E_0} \sum_k \frac{g_k^2}{\omega_k} = \frac{1}{E_0} \int_0^\infty d\omega \frac{J(\omega)}{\omega}. \quad (S12)$$

The existence of the integral requires that $J(\omega)$ decays sufficiently fast as $\omega \to \infty$. Furthermore, the power-law dependence $J(\omega) \propto \omega^\eta$ must satisfy $\eta > 0$. The Hamiltonian $H_x$ can hence be written as

$$H_x = I \otimes \left[ \sum_k \omega_k c_k^\dagger c_k - \sum_{k,l} \frac{g_k g_l}{4E_0} (c_k + c_k^\dagger) (c_l + c_l^\dagger) \right] - \tau^x \otimes \left[ \frac{g}{2} \frac{1 - 2a}{1 - a} \sum_k \frac{g_k}{2} (c_k^\dagger + c_k) + \sum_{k,l} \frac{g_k g_l}{4E_0} (c_k + c_k^\dagger) (c_l + c_l^\dagger) \right]. \quad (S13)$$
where we introduced the environment-induced bias

\[ \tilde{\varepsilon} = \frac{E_0 a(3a - 2)}{4(1-a)^2}. \]  

(S14)

This induced bias results in a shift of the avoided level crossing. Since the location of the avoided level crossing bears no significance, as long as it is known, we will drop the environment-induced bias in what follows.

The quadratic term coupling to the identity can be diagonalized perturbatively using the techniques outlined in section \( \cdot \) The result is that

\[ H_k = I \otimes \sum_k \tilde{\omega}_k d_k^\dagger d_k - \tau^2 \otimes \left[ \sum_k \tilde{g}_k \left( d_k + d_k^\dagger \right) + \sum_{k,l} \frac{\tilde{g}_k \tilde{g}_l}{E} \left( d_k + d_k^\dagger \right) \left( d_l + d_l^\dagger \right) \right], \]  

(S15)

where

\[ \tilde{\omega}_k = \omega_k - \frac{g_k^2}{2E_0}, \]  

(S16)

\[ \tilde{g}_k = \frac{1 - 2a}{1-a} \left( 1 - \frac{g_k}{2E_0} \sum_{l \neq k} \frac{g_l}{\omega_k - \omega_l} + \frac{g_k}{2E_0} \sum_l \frac{g_l}{\omega_k + \omega_l} \right) \frac{g_k}{2}, \]  

(S17)

\[ \tilde{E} = \left( \frac{1-a}{1-2a} \right)^2 E_0, \]  

(S18)

and \( d_k \) are new bosonic operators. They can be related to \( c_k \) by

\[ d_k \approx c_k + \frac{g_k}{2E_0} \sum_{l \neq k} \frac{g_l}{\omega_k - \omega_l} c_l - \frac{g_k}{2E_0} \sum_l \frac{g_l}{\omega_k + \omega_l} c_l^\dagger. \]  

(S19)

Before proceeding, it is worth verifying the validity of perturbation theory employed here. The correction of the energy eigenvalues is certainly small since \( g_k^2 \) is inversely proportional to the volume of the bath. It vanishes entirely in the thermodynamic limit and we will therefore neglect it below. In addition, we require that the correction to the bosonic operators be small, i.e.,

\[ \frac{g_k^2}{4E_0^2} \left[ \sum_{l \neq k} \frac{g_l^2}{(\omega_k - \omega_l)^2} + \sum_l \frac{g_l^2}{(\omega_k + \omega_l)^2} \right] \ll 1. \]  

(S20)

Let us consider the first sum by re-writing it in terms of the noise spectral density \( J(\omega) \). We introduce the mode spacing \( \Delta \omega \) at frequency \( \omega_k \) such that

\[ \frac{g_k^2}{4E_0^2} \sum_{l \neq k} \frac{g_l^2}{(\omega_k - \omega_l)^2} = \frac{1}{4E_0^2} \int_{\omega_k - \Delta \omega}^{\omega_k + \Delta \omega} d\omega J(\omega) \left[ \int_{\omega_k - \Delta \omega}^{\omega_k + \Delta \omega} d\omega' J(\omega') \left( \frac{1}{(\omega_k - \omega')^2} + \frac{1}{(\omega_k + \omega')^2} \right) \right]. \]  

(S21)

We are interested in the continuum limit \( \Delta \omega \to 0 \), which yields

\[ \frac{g_k^2}{4E_0^2} \sum_{l \neq k} \frac{g_l^2}{(\omega_k - \omega_l)^2} = \frac{J(\omega_k)}{2E_0^2} \lim_{\Delta \omega \to 0} \Delta \omega \int_{0}^{\omega_k - \Delta \omega} d\omega' \frac{J(\omega')}{(\omega_k - \omega')^2} + \int_{\omega_k + \Delta \omega}^{\infty} d\omega' \frac{J(\omega')}{(\omega_k - \omega')^2} - \frac{J(\omega_k)^2}{E_0^2}, \]  

(S22)

where we employed L’Hôpital’s rule to evaluate the limit. This shows that the first sum in Eq. (S20) is small as long as the weak coupling limit \( J(\omega) \ll E_0 \) is satisfied. The second sum can be bounded from above by

\[ \frac{g_k^4}{16E_0^4} \sum_l \frac{g_l^2}{(\omega_k + \omega_l)^2} \leq \frac{g_k^4}{16E_0^4} \sum_{l \neq k} \frac{g_l^2}{(\omega_k - \omega_l)^2} + \frac{g_k^4}{16E_0^4} \omega_k \]  

(S23)

The last term is again small in the weak coupling regime since we can always take \( \omega_k \geq \Delta \omega \) without modifying the spectrum of the bath significantly. Hence, perturbation theory is valid in the weak coupling regime.

The Hamiltonian in Eq. (S15) indeed has the form of the effective Hamiltonian introduced in the main text. There will be a similar contribution for each qubit and polarization of the bath modes. Since these contributions all commute, we expect the dynamics to be well described by a single contribution, with the only modification being that \( J(\omega) \) should multiplied by the number of channels that couple to the environment.
ADIABATIC RENORMALIZATION

We argued in the main text that it is possible to treat the fast oscillators of the environment by introducing a renormalized tunneling rate between the two well states. More specifically, we consider the system and bath Hamiltonian

\[
H = -\frac{1}{2}(\varepsilon \tau^z + \Delta \tau^z) + \sum_k \omega_k b_k^\dagger b_k + \tau^z \otimes \left[ \sum_k g_k \left(b_k + b_k^\dagger\right) + \sum_{k,l} \frac{g_k g_l}{E} \left(b_k + b_k^\dagger\right) \left(b_l + b_l^\dagger\right) \right]
\]  

(S24)

in the absence of tunneling, \(\Delta = 0\). The eigenstates in this case can be written as

\[
|\tau, n\rangle = e^{-i\tau^z S} |\tau\rangle \otimes \prod_k \frac{1}{\sqrt{n_k!}} \left(b_k^\dagger\right)^{n_k} |0\rangle = e^{-i\tau^z S} |\tau\rangle \otimes |n\rangle
\]

(S25)

where \(\tau = m, m_\perp\) labels the two eigenstates of \(\tau^z\). The unitary \(S\) diagonalizes the system bath interaction,

\[
e^{i\tau^z S} \left\{ \sum_k \omega_k b_k^\dagger b_k + \tau^z \otimes \left[ \sum_k g_k \left(b_k + b_k^\dagger\right) + \sum_{k,l} \frac{g_k g_l}{E} \left(b_k + b_k^\dagger\right) \left(b_l + b_l^\dagger\right) \right] \right\} e^{-i\tau^z S} = \sum_k \omega_k b_k^\dagger b_k,
\]

(S26)

where we dropped all terms acting only on the system on the right-hand side of the equation. We have also neglected corrections to the spectrum of the environment by the system–environment interaction since they are inversely proportional to the volume of the bath, as already observed in the derivation of the effective Hamiltonian in section 4. The Hamiltonian to be diagonalized here is in fact very similar to that in section 4 and the same methods can be applied. We obtain

\[
e^{-i\tau^z S} = e^{-i\tau^z S_1} e^{-i\tau^z S_2}
\]

(S27)

where

\[
S_1 = i \sum_k \delta_k \left( b_k - b_k^\dagger \right), \quad \delta_k = \frac{1}{1 - \alpha^2 \omega_k}, \quad a = \frac{4}{E} \int d\omega \frac{J(\omega)}{\omega}
\]

(S28)

affects a displacement to remove the single-boson terms, while

\[
S_2 = \sum_{k,l} \left[ A_{kl} b_k^\dagger b_l + \frac{i}{2} B_{kl} \left(b_k b_l - b_k^\dagger b_l^\dagger\right) \right], \quad A_{kl} = -\frac{2i}{\omega_k - \omega_l} \frac{g_k g_l}{E}(1 - \delta_{kl}), \quad B_{kl} = -\frac{2}{\omega_k + \omega_l} \frac{g_k g_l}{E}
\]

(S29)

diagonalizes the two-boson terms in the weak coupling limit, \(J(\omega) \ll E\). We point out that we omitted a term proportional to \(\tau^z\) in the expression for the displacement \(\delta_k\). Such a term merely gives rise to a state-independent displacement in Eq. (S27), which does not affect the renormalized tunneling rate as will be apparent shortly. In addition, we will drop the pre-factor involving \(a\) since we are only interested in the asymptotic scaling with the size of the search space.

For a given set of occupation number \(n\), the renormalized tunneling rate can now be expressed as

\[
\tilde{\Delta}_n = \Delta \langle m_\perp, n|\tau^x|m, n\rangle' = \Delta \langle n|e^{-is_2}e^{-2is_1}e^{-is_3}|n\rangle'.
\]

(S30)

Here the prime reminds us that we should only consider processes that are fast compared to the dynamics of the system. At zero bias, the only time scale of the system is set by the renormalized tunneling rate. Hence, the renormalized tunneling rate may be determined self-consistently by evaluating the expectation value in Eq. (S30) with a low-frequency cutoff \(\Omega = p\Delta_n\), where \(p\) is an unimportant numerical factor as long as \(p \gg 1\). We point out this argument readily generalizes to the case of finite bias, where the cutoff should be taken to be \(\Omega = p\sqrt{\varepsilon^2 + \Delta_0^2}\). We do not discuss this more complicated case here since the transition between the coherent and incoherent regime first occurs at the smallest gap of the system, that is, at zero bias. As argued in the main text, any potential quantum speedup is lost when the system is rendered incoherent during any part of the evolution. Thus, the coherence properties at zero bias fully determine the performance of the algorithm.

The renormalized tunneling rate in Eq. (S30) clearly depends on the occupation numbers \(n\) and it is therefore not unique at finite temperature. Nevertheless, we can obtain a typical value \(\tilde{\Delta}\) by taking a thermal expectation value

\[
\tilde{\Delta} = \Delta \text{Tr} \left\{ \rho e^{-is_2}e^{-2is_1}e^{-is_3} \right\}',
\]

(S31)

where \(\rho = e^{-\sum \omega_k b_k^\dagger b_k / T} / Z\) is the thermal state at temperature \(T\).
Single-boson processes

Before we consider Eq. (S31) fully, it is instructive to compute the renormalized tunneling rate in the absence of two-boson processes, i.e., setting $S_2 = 0$. The trace in Eq. (S31) is most readily evaluated by observing that

$$\tilde{\Delta} \approx \Delta \exp \left[ -2 \sum_k \delta_k (1 + 2N(\omega_k)) \right] = \Delta \exp \left[ -2 \int_{p\tilde{\Delta}}^{\infty} d\omega \frac{J(\omega)}{\omega^2} \coth \frac{\omega}{2T} \right],$$

where we cut off the integral at $p\tilde{\Delta}$ in accordance with the prescription of adiabatic renormalization. At zero temperature, the integral is convergent for $\eta > 1$ and we can safely extend the lower limit to 0:

$$\tilde{\Delta} = \Delta \exp \left[ -2 \int_0^{\infty} d\omega J(\omega) \coth \frac{\omega}{2T} \right] \quad \text{if } \eta > 1.$$  

When $\eta < 1$, the integral diverges with small $\tilde{\Delta}$ as $\tilde{\Delta}^{\eta-1}$. Thus,

$$\tilde{\Delta} \approx \Delta \exp \left[ -\frac{2\alpha}{1 - \eta} \left( \frac{1}{p\tilde{\Delta}} \right)^{\eta-1} + \log c \right],$$

where the constant $c$ depends on the high frequency behavior of the noise spectrum $J(\omega)$. We re-arrange the expression to

$$z \left( \frac{\tilde{\Delta}}{c\Delta} \right)^{1-\eta} \log \left( \frac{\tilde{\Delta}}{c\Delta} \right) = -\frac{2\alpha}{1 - \eta} \left( \frac{1}{p\tilde{\Delta}} \right)^{1-\eta}.$$

The expression on the left has a global minimum of $-1/e(1 - \eta)$ such that a non-zero solution for $\tilde{\Delta}$ only exists if

$$2\alpha \left( \frac{1}{p\tilde{\Delta}} \right)^{1-\eta} < \frac{1}{e},$$

or

$$\alpha < \frac{1}{2e} \left( \frac{1}{p\tilde{\Delta}} \right)^{1-\eta}.$$  

This shows that there exists a critical coupling strength

$$\alpha^* \propto \Delta^{1-\eta} = O(N^{(\eta-1)/2}) \quad \text{at } \eta = 1,$$

above which $\tilde{\Delta} = 0$. Our simple argument does not predict the precise value of $\alpha^*$ due to the dependence on $p$. Nevertheless, more detailed studies have confirmed that the form of Eq. (S39) is qualitatively correct [S3, S4]. This result implies that for a fixed $\alpha$, the dynamics are incoherent even at zero temperature for sub-ohmic environments in the limit of large $N$.

At $\eta = 1$, the exponent in Eq. (S33) diverges logarithmically with $\tilde{\Delta}$ such that

$$\tilde{\Delta} \propto \Delta^{1/(1-2\alpha)} = O(N^{-1/2(1-2\alpha)}).$$

This expression is only valid if $\tilde{\Delta} < \Delta$, which implies that $\alpha < 1/2$. If $\alpha > 1/2$, the renormalized tunneling rate vanishes. The critical coupling strength is therefore independent of $N$ at $\eta = 1$, as expected from Eq. (S39).

The above arguments can be readily generalized to the case of finite temperature. Assuming that $T \gg \Delta$, as will be naturally the case for large systems, we can approximate $\coth \omega/2T \approx 2T/\omega$ near the lower limit of the integral. The integral is convergent for $\eta > 2$ and we obtain to a good approximation

$$\tilde{\Delta} = \Delta \exp \left[ -2 \int_0^{\infty} d\omega \frac{J(\omega)}{\omega^2} \coth \frac{\omega}{2T} \right] \quad \text{if } \eta > 2.$$  

For $\eta > 2$, adiabatic renormalization predicts that $\tilde{\Delta} = 0$ unless

$$\alpha < \frac{1}{4eT} (pc\Delta)^{2-\eta}. \quad (S42)$$

For fixed temperature, there exists a critical coupling strength

$$\alpha^* \propto \frac{\Delta^{2-\eta}}{T} = O(N^{(\eta-2)/2}) \quad (S43)$$

above which $\tilde{\Delta} = 0$. For fixed $\alpha$, we can alternatively identify a critical coupling temperature with the same scaling as the critical coupling strength,

$$T^* \propto \frac{\Delta^{2-\eta}}{\alpha} = O(N^{(\eta-2)/2}). \quad (S44)$$

The renormalized tunneling rate vanishes for any $T > T^*$. At $\eta = 2$, the renormalized tunneling rate vanishes unless $\alpha T < 1/4$, showing that the critical coupling strength and temperature are again independent of the search space size.

We point out that Eq. (S44) is only valid if $\eta > 1$ since otherwise the assumption that $T \gg \tilde{\Delta}$ cannot be satisfied in the limit of large $N$. For $\eta < 1$, we found above that the incoherent tunneling rate vanishes even at zero temperature in the limit of large $N$ and fixed $\alpha$. This can be summarized as

$$T^* = \begin{cases} 0, & \text{if } \eta < 1 \\ O(N^{(\eta-2)/2}), & \text{if } \eta > 1. \end{cases} \quad (S45)$$

At $\eta = 1$, a non-zero critical temperature, which scales as $T^* = O(N^{-1/2})$ only exists if $\alpha < 1/2$.

**Two-boson processes**

We repeat the above analysis for two-boson processes, ignoring single boson processes for the moment. The expectation value in Eq. (S31) is harder to compute in this case since $S_2$ is a quadratic operator involving terms of the form $b^\dagger_k b_l, b_k b^\dagger_l$, and $b^\dagger_k b_l b^\dagger_l b^\dagger$. We will only evaluate it perturbatively by expanding the exponential to the lowest non-trivial order. By ignoring terms that scale inversely with the volume of the bath, we hence obtain

$$\tilde{\Delta} \approx \Delta \exp \left[-2 \sum_{k,l} A_{kl} A_{lk} N(\omega_k) (1 + N(\omega_l)) - \sum_{k,l} B_{kl} B_{lk} [(1 + N(\omega_k))(1 + N(\omega_l)) - N(\omega_k)N(\omega_l)] \right] \quad (S46)$$

$$\approx \Delta \exp \left[-\phi - \chi \right], \quad (S47)$$

where

$$\phi = 2 \int' d\omega d\omega' \frac{J(\omega)J(\omega')}{(\omega - \omega')^2} N(\omega) (1 + N(\omega')) , \quad (S48)$$

$$\chi = \int' d\omega d\omega' \frac{J(\omega)J(\omega')}{(\omega + \omega')^2} [(1 + N(\omega))(1 + N(\omega')) - N(\omega)N(\omega')]. \quad (S49)$$

In both cases, the integral is to be taken over processes that are fast compared to the low-frequency cutoff $p\tilde{\Delta}$. For $\phi$, which describes two-boson scattering processes, this corresponds to $|\omega - \omega'| > p\tilde{\Delta}$, stating that the beating frequency of the two modes is fast. The two-boson absorption and emission processes are captured by $\chi$, for which we therefore impose that $\omega + \omega' > p\tilde{\Delta}$. We point out that the above expansion is only justified if $A_{kl}$ and $B_{kl}$ as well as $\sqrt{N(\omega_k)N(\omega_l)} A_{kl}$ and $\sqrt{N(\omega_k)N(\omega_l)} B_{kl}$ are small matrices in the sense that each column forms a vector with magnitude much less than one. This is indeed the case for $A_{kl}$ and $B_{kl}$ in the weak coupling limit $J(\omega) \ll E$, as shown explicitly in section . A similar treatment can be applied to the other two matrices, giving rise to the condition

$$J(\omega)N(\omega) \ll E. \quad (S50)$$
At high frequencies $\omega \gg T$, this is trivially satisfied in the weak coupling limit. At low frequencies $\omega \ll T$, however, this leads to the additional constraint

$$\frac{J(\omega)}{\omega} \ll \frac{E}{T}. \tag{S51}$$

It is important to note that this inequality can only be satisfied as $\omega \to 0$ for ohmic and super-ohmic environments. Restricting ourselves to this particular parameter regime is not a significant limitation since single-boson processes alone will render the dynamics incoherent even at zero temperature for sub-ohmic environments. Therefore, two-boson processes are expected to modify the dynamics qualitatively only for ohmic and super-ohmic environments.

We now investigate the low-frequency divergences of $\phi$ and $\chi$ to identify the critical coupling strength and temperature as in the case of single-boson processes. At zero temperature, $\phi$ vanishes while $\chi$ is always finite. Hence, two-boson processes only weakly modify the tunneling rate at zero temperature. At non-zero temperatures, $\chi$ remains finite whereas $\phi$ exhibits an infrared divergence for any $\eta$. The functional form of the divergence with the low-frequency cutoff $p\Delta$ can be found to be given by

$$\phi \propto \frac{\alpha^2 T^{2\eta+1}}{E^2 \Delta^2}. \tag{S52}$$

Using similar arguments to the ones for the single-boson processes, this allows us to identify a critical coupling strength

$$\alpha^* \propto ET^{-(\eta+1/2)} \Delta^{1/2} = O(N^{-1/4}) \tag{S53}$$

and a critical temperature

$$T^* \propto \alpha^{-2/(2\eta+1)} E^{2/(2\eta+1)} \Delta^{1/(2\eta+1)} = O(N^{-1/(4\eta+2)}). \tag{S54}$$

The divergence of $\phi$ as $\Delta \to 0$ originates from the denominator in Eq. (S48), the form of which is dictated by conservation of energy during the scattering process. It is for this reason that higher-order terms in the effective Hamiltonian Eq. (S44) are not expected to modify the scaling of the critical temperature or the critical coupling strength.

**Combined effects**

We will now show that the single-boson and two-boson processes approximately decouple in the regime of interest. We start by expressing Eq. (S31) in a coherent state basis

$$\hat{\Delta} = \Delta \int D\alpha D\beta \langle \alpha | e^{-iS_2} p e^{-iS_2} | \beta \rangle \langle \beta | e^{-2iS_1} | \alpha \rangle, \tag{S55}$$

where we introduced the short-hand notation $D\alpha = \prod_k d^2 \alpha_k / \pi$ and $| \alpha \rangle = \prod_k | \alpha_k \rangle$. The state $| \alpha_k \rangle$ is a coherent state of the mode $b_k$ and the integral runs over the entire complex plane for each mode. We note that the matrix elements may be written as

$$\langle \alpha | e^{-iS_2} p e^{-iS_2} | \beta \rangle = e^{-f(\alpha^*,\beta)} \langle \alpha | \beta \rangle, \tag{S56}$$

$$\langle \beta | e^{-2iS_1} | \alpha \rangle = e^{g(\beta^*,\alpha)} \langle \beta | \alpha \rangle, \tag{S57}$$

where $g(\beta^*,\alpha)$ is a linear function, while $f(\alpha^*,\beta)$ contains only quadratic terms. Computing the function $f$ is rather cumbersome due to the presence of the squeezing terms $b_k b_1$ and $b_k^\dagger b_1^\dagger$ in $S_2$. For our purposes, it suffices to exploit the general structure of a Gaussian integral over a real vector $v$,

$$\int \left( \prod_n \frac{dv_n}{\sqrt{2\pi}} \right) \exp \left[ -\frac{1}{2} v^T M v + w^T v \right] = \frac{1}{\det M} \exp \left[ w^T M^{-1} w \right]. \tag{S58}$$

Applied to Eq. (S55), we can see that the matrix $M$ is determined by $S_2$, while the vector $w$ follows from $S_1$. We may thus write

$$\hat{\Delta} = \Delta \langle e^{-2iS_2} \rangle \langle e^{-2iS_1^*} \rangle, \tag{S59}$$
where the operator $S'_i$ accounts for both single-boson processes and the coupling between single-boson and two-boson processes. Under the same conditions that we were able to expand $e^{-2iS_2}$ in the previous section, we can also expand $S'_i$ in powers of $A_{kl}$ and $B_{kl}$. To leading order, we clearly must have $S'_i \approx S_1$, which shows that the single-boson and two-boson processes decouple under the assumption that $J(\omega) \ll E$ and $J(\omega) N(\omega) \ll E$. The nature of the dynamics of the system may thus be deduced by considering the two processes separately, as done in the main text.

**THERMALIZATION RATES**

**Incoherent regime**

We argued in the main text that it is necessary to determine the scaling of the thermalization rate in order to exclude the possibility of a quantum speedup in the incoherent regime. In particular, a speedup over the classical algorithm is possible if the thermalization rate decays slower with the size of the search space than $N^{-1}$. In the incoherent regime, adiabatic renormalization predicts that the system is localized. However, adiabatic renormalization does not take into account incoherent tunneling. To estimate the rate of incoherent tunneling, we perform perturbation theory in the bare tunneling rate $\Delta$. It is convenient to switch to the basis that diagonalizes the Hamiltonian in the absence of tunneling,

$$e^{iS}He^{-iS} = -\frac{1}{2}(\varepsilon \tau^z + \Delta e^{iS} \tau^x e^{-iS}) + \sum_k \omega_k b_k^\dagger b_k,$$

(S60)

where $S$ is given by Eq. (S27). We move to the interaction picture, where the time-evolution is fully governed by

$$V(t) = -\frac{\Delta}{2}e^{-i\varepsilon t}e^{iS(t)} \tau^+ e^{-iS(t)} + \text{h.c.},$$

(S61)

with $\tau^+ = (\tau^x + i\tau^y)/2$ and

$$S(t) = e^{it} \sum_k \omega_k b_k^\dagger b_k S e^{-it} \sum_k \omega_k b_k^\dagger b_k.$$

(S62)

Starting with the initial state $|\psi(0)\rangle = |m_1, n\rangle$, the probability that the system ends up in $|m\rangle$ after time $t$ is given to lowest order in perturbation theory by

$$p(t) = \sum_\perp \langle m, n | \int_0^t dt' V(t') | m_1, n \rangle |^2 = \int_0^t dt' \int_0^t dt'' \langle m_\perp, n | V(t') | m \rangle \langle m | V(t'') | m_\perp, n \rangle.$$

(S63)

By expressing $e^{iS}$ in terms of $S_1$ and $S_2$ and taking a thermal average over the initial state, we obtain

$$p(t) = \left(\frac{\Delta}{2}\right)^2 \int_0^t dt' \int_0^t dt'' e^{i\varepsilon(t' - t'')} \left\langle e^{iS_2(t')} e^{2iS_1(t')} e^{iS_2(t')} e^{-iS_2(t'')} e^{-2iS_1(t'')} e^{-iS_2(t'')} \right\rangle.$$

(S64)

We note that the expectation value in the integrand is a function of $t' - t''$ only, which allows us to write

$$p(t) = \int_0^t dt' \Gamma(t'),$$

(S65)

where

$$\Gamma(t) = \left(\frac{\Delta}{2}\right)^2 \int_{-t}^t dt' e^{i\varepsilon t'} \left\langle e^{iS_2(t')} e^{2iS_1(t')} e^{iS_2(t')} e^{-iS_2(t')} e^{-2iS_1(t')} e^{-iS_2(t')} \right\rangle.$$

(S66)

is the instantaneous decay rate at time $t$. Typically, we can extend the limits of this integral to infinity to obtain a single decay rate

$$\Gamma = \left(\frac{\Delta}{2}\right)^2 \int_{-\infty}^\infty dt' e^{i\varepsilon t'} \left\langle e^{iS_2(t')} e^{2iS_1(t')} e^{iS_2(t')} e^{-iS_2(t')} e^{-2iS_1(t')} e^{-iS_2(t')} \right\rangle.$$

(S67)

This is a good approximation for almost all $t$ provided the width over which the integrand contributes significantly is small compared to $1/\Gamma$. Since the width of the integrand is independent of $N$, the decay rate scales as $O(N^{-1})$, and thus the condition for extending the limits of the integral is always fulfilled in the limit of large $N$. We will not evaluate the above expression any further as we are only interested in the scaling with the search space size.
Coherent regime

For completeness we briefly discuss the thermalization rate in the coherent regime, i.e., $\eta > 1$ at zero temperature. The thermalization rate in this regime has no immediate implications for the scalability of the quantum algorithm since a quantum speedup is always available in the coherent regime. However, a thermalization rate that exceeds the classical scaling $O(N^{-1})$ enables a quantum speedup by thermalization alone as discussed in the main text.

In the coherent regime, thermalization occurs via transitions between the eigenstates of the closed system rather than by incoherent tunneling. The thermalization rate is readily obtained by applying Fermi’s Golden rule after adiabatic renormalization. For the single-boson processes, this yields at zero bias

$$\Gamma_1 = 2\pi \sum_k g_k^2 \delta(\omega_k - \tilde{\Delta}) = 2\pi J(\tilde{\Delta}) = O(N^{-\eta/2}).$$

(S68)

Interestingly, the thermalization rate exceeds the classical limit for $1 < \eta < 2$. We further remark that the rate drops below the classical scaling for $\eta > 2$. In this regime, incoherent tunneling and processes coupling to $\tau^x$ and $\tau^y$, which we have neglected, will dominate the thermalization rate.

For the the two-boson processes, the Golden rule rate is given by

$$\Gamma_2 = \frac{2\pi}{\mathcal{F}^2} \int_{0}^{\tilde{\Delta}} d\omega J(\omega) J(\tilde{\Delta} - \omega) = O(N^{-(\eta+1)/2}).$$

(S69)

This vanishes parametrically faster than the single-boson decay rate such that two-boson emission only contributes weakly to the thermalization rate.

**DIAGONALIZATION OF QUADRATIC HAMILTONIANS**

We briefly review the diagonalization of a general quadratic Hamiltonian of the form

$$H = \beta^\dagger M \beta,$$

(S70)

where $M$ is a Hermitian matrix and $\beta = (b_1, b_2, \ldots, b_1^\dagger, b_2^\dagger, \ldots)$ is a vector formed by creation and annihilation operators. We closely follow the notation of reference [S6], where the diagonalization of both fermionic and bosonic Hamiltonians is discussed in detail. For the sake of clarity, we focus on bosons below.

The goal is to introduce new bosonic operators $\gamma = (c_1, c_2, \ldots, c_1^\dagger, c_2^\dagger, \ldots)$ such that the Hamiltonian can be written as

$$H = \sum_i \lambda_i c_i^\dagger c_i$$

up to a constant. The new operators $\gamma$ are related to the original operators by a linear transformation

$$\begin{pmatrix} c \\ c^\dagger \end{pmatrix} = T \begin{pmatrix} b \\ b^\dagger \end{pmatrix}.$$

(S72)

The fact that $b^\dagger$ is the adjoint of $b$ implies that $T$ must take the form

$$T = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}.$$

(S73)

Furthermore, the conservation of canonical commutation relations leads to the additional constraint

$$T^{-1} = \mu T^\dagger \mu, \quad \mu = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

(S74)

Eq. (S70) can hence be written as

$$H = \gamma^\dagger \mu T \mu M T^{-1} \gamma.$$

(S75)
Assuming that \( M \) is positive definite, it is shown in \[S6\] that there exists a transformation \( T \) satisfying Eq. (S73) and Eq. (S74) which diagonalizes \( \mu M \) to give

\[
T\mu M T^{-1} = \frac{1}{2} \begin{pmatrix} \lambda_i \delta_{ij} & 0 \\ 0 & -\lambda_i \delta_{ij} \end{pmatrix}.
\] (S76)

This immediately yields the desired result Eq. (S71) up to a constant.

It is often useful to express the transformation described by the matrix \( T \) as a unitary transformation \( S \) acting on the creation and annihilation operators, i.e.

\[
c_i = S b_i S^\dagger, \quad S^\dagger S = SS^\dagger = I.
\] (S77)

The transformation takes the form \( S = \exp(i\beta^\dagger K \beta/2) \), where \( K \) is a Hermitian matrix. By direct substitution into Eq. (S77) and comparison to Eq. (S72) we obtain

\[
T = e^{-i\mu K}.
\] (S78)

Finally, we note that the vacuum \( |0_c\rangle \), where \( c_i|0_c\rangle = 0 \) for all \( i \), is related to the vacuum \( |0_b\rangle \), for which \( b_i|0_b\rangle = 0 \), by

\[
|0_c\rangle = S |0_b\rangle.
\] (S79)

All other Fock states transform in the same manner.

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