ON THE ORDER OF DIRICHLET $L$-FUNCTIONS

G. Kolesnik

1. Introduction. Let $L(s, \chi)$ be a Dirichlet $L$-function, where $\chi$ is a nonprincipal character $(\text{mod } q)$ and $s = \sigma + it$. A standard estimate for $L(s, \chi)$ based on bounds for $\zeta(s, w)$, is

$$|L(s, \chi)| \leq C_1 \tau \eta(1-\sigma)^{1/2} q^{1-\sigma} \log^{2/3} \tau, \quad \frac{1}{2} \leq \sigma \leq 1,$$

where $\tau = |t| + 2$, $c = 1/6$ (see, for example, Prachar [5, (4.12)]), and in fact, $c$ can be replaced by a constant $< 1/6$. An immediate application of Richert’s work [6] gives

$$|L(s, \chi)| \leq C_2 \tau^{100(1-\sigma)/3} q^{1-\sigma} \log^{2/3} \tau, \quad \frac{1}{2} \leq \sigma \leq 1,$$

which is better than (1) if $\sigma$ is near 1.

Another estimate can easily be obtained from $|L(1 + it, \chi)| \leq C_3 \log \tau q$ and the functional equation of $L(s, \chi)$ as follows. First,

$$|L(it, \chi)| = 2 \cdot |(2\pi)^{it-1} q^{1/2-it} \times \cos \frac{1}{2} \pi \left(1 - it + \frac{1}{2} - \frac{1}{2} \chi(-1)\right) \Gamma(1 - it)L(1 - it, \chi)|$$

$$\leq C_4 \sqrt{\tau q} \log \tau q.$$

Now the convexity principle yields for

$$|L(s, \chi)| \leq (C_3 \sqrt{\tau q} \log \tau q)^{1-\sigma} \cdot (C_5 \log \tau q)^{\sigma} \leq C_6 (\tau q)^{1/2(1-\sigma)}$$

$$\times \log \tau q, \quad 0 \leq \sigma \leq 1.$$

Neglecting dependence on $\tau$, Davenport [2], improved (3):

$$|L(s, \chi)| \leq C_6 (\tau)^{1/2(1-\sigma)}, \quad 0 \leq \sigma \leq 1.$$

Also, Burgess [1] improved (4) by establishing

$$|L(s, \chi)| \leq C_7 (\varepsilon, \tau) q^{1/2(1-\sigma)} \log \tau, \quad \frac{1}{2} \leq \sigma \leq 1.$$

By examining Burgess’ proof, it can be seen that the constant $C(\varepsilon, \tau)$ can be taken to be $C_8 (\varepsilon) \pi^{2(1-\sigma)}$ and his result can be further sharpened to yield

$$|L(s, \chi)| \leq C_8 \tau^{11(1-\sigma)} q^{3/8(1-\sigma)} C^\omega \log \tau, \quad \frac{1}{2} \leq \sigma \leq 1,$$

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where $\omega = \log q / \log \log q$. The estimates (3), (4), and (5) are better than (1) if $q$ is large compared to $\tau$.

For $\sigma = 1/2$, the previous estimates were improved by Fujii, Gallagher and Montgomery, [3], who showed that if $P$ is a fixed set of primes and $q$ is composed only of primes in $P$, then

$$ L\left(\frac{1}{2} + it, \chi\right) \leq C(e, P)(\tau q)^{1/8 + \epsilon}. \quad (6) $$

In this paper we prove two more estimates which imply (1), (4), and (5) and which are better than (2), (3), and (6) in some range of $\sigma$, $\tau$, and $q$. We prove:

**Theorem 1.** Let $\chi$ be a nonprincipal character $\pmod q$. Let $1/2 \leq \sigma \leq 1$, $\tau = |t| + 2$ and $\omega = \log q / \log \log q$. Then

$$ |L(s, \chi)| \ll \tau^{-\sigma} q^{\frac{1}{2} \log^2 \tau} C^\omega \log \tau, \quad (7) $$

where $C$ is some absolute constant.

**Theorem 2.** Let $\chi$ be a character $\pmod q$. Let $1/2 \leq \sigma \leq 1$ and $\tau = |t| + 2$. Then

$$ |L(s, \chi)| \ll \tau^{35/108(1-\sigma)} q^{-\sigma} \log^3 \tau q. \quad (8) $$

In particular, (7) and (8) imply

$$ \left| L\left(\frac{1}{2} + it, \chi\right) \right| \ll \sqrt{\tau} q^{3/16} C^\omega \log \tau $$

and

$$ \left| L\left(\frac{1}{2} + it, \chi\right) \right| \ll \tau^{35/216} q \log^3 \tau q. $$

The estimates of $L(s, \chi)$ for $\sigma \in [0, 1/2]$ can be obtained by using (7) or (8) and the functional equation of $L(s, \chi)$.

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2. Notation.

$$ e(f(x)) = \exp(2\pi i f(x)). $$

$$ \omega = \log q / \log \log q. $$

$$ s = \sigma + it, \quad \frac{1}{2} \leq \sigma \leq 1. $$

$$ \tau = |t| + 2. $$
C denotes some appropriate absolute constant, not always the same.

3. Application of the estimate of Burgess. In this section we will show that

$$|L(s, \chi)| \ll \pi^{1-\epsilon} \tau^{3/8(1-\epsilon)} C^w \log^k \tau.$$  

We need the following result of E. Bombieri:

**Lemma.** Let $N$ and $m$ be nonnegative integers. Let $\alpha_j, \beta_j$ be numbers such that $|\alpha_j - \beta_j| \leq (2\pi m N^2)^{-1}$ for $1 \leq j \leq m$, and let $f(x) = \alpha_1 x + \cdots + \alpha_m x^m, g(x) = \beta_1 x + \cdots + \beta_m x^m$. Let $c_1, c_2, \cdots$ be complex, and let

$$S(\alpha, N) = \max_{1 \leq n_1 \leq N} \left| \sum_{1 \leq n \leq N} c_n e(f(n)) \right|,$$

where $\alpha = (\alpha_1, \cdots, \alpha_m)$. Then $S(\beta, N) \leq 6S(\alpha, N)$.

**Proof.** For every $N_i \in [1, N]$ we have:

$$\sum_{1 \leq n \leq N_1} c_n e(g(n)) = \sum_{1 \leq n \leq N_1} c_n e(f(n)) \prod_{j=1}^m e((\beta_j - \alpha_j)n^j)$$

$$= \sum_{k_1, \ldots, k_m = 0}^\infty \left( \prod_{j=1}^m \frac{2\pi i (\beta_j - \alpha_j)^{k_j}}{k_j!} \right) \sum_{1 \leq n \leq N_1} c_n n^{k_1 + \cdots + k_m} e(f(n)).$$

Using Abel's summation formula, we obtain:

$$S(\beta, N) \leq \sum_{k_1, \ldots, k_m = 0}^\infty \prod_{j=1}^m \frac{|2\pi (\beta_j - \alpha_j)|^{k_j}}{k_j!} N^{k_1 + \cdots + k_m} 2S(\alpha, N)$$

$$\leq 2S(\alpha, N) \cdot \sum_{k_1, \ldots, k_m = 0}^\infty \prod_{j=1}^m \frac{|2\pi (\beta_j - \alpha_j) N^j|^ {k_j}}{k_j!}$$

$$\leq 2S(\alpha, N) \left( \sum_{k=0}^\infty m^{-k} |k!| \right)^m \leq 6S(\alpha, N).$$

**Lemma 2.** Let $q \geq 2$ and let $M, N$ be integers. Let $\chi$ be a primitive character (mod $q$). Then

$$|\sum_{1 \leq n \leq N} \chi(n + M)| \leq \sqrt{N} q^{3/16} C^w.$$

This lemma can be proven similarly to Theorem 2, [1].

**Lemma 3.** Let $q$ and $N$ be integers such that $q \geq 2$ and $N \leq \tau q$. Let $\chi$ be a primitive character (mod $q$). Then

$$|S = \max_{N \geq N_1 \leq 2N} \left| \sum_{N + 1 \leq n \leq N} \chi(n)^{n^{-it}} \right| \ll \sqrt{N} \log \tau \cdot q^{3/16} C^w.$$
Proof. We can obviously suppose that $\tau \log \tau q \leq N$ since otherwise the estimate is trivial. Taking $H = \lfloor N(\tau \log \tau q)^{-1} \rfloor$ and $m = \lfloor \log \tau q \rfloor$, and dividing the sum in $S$ into $\leq 2NH^{-1}$ subsums, we obtain:

$$|S| \leq 2NH^{-1} \max_{N \leq m \leq 2N} \max_{1 \leq H \leq H} \sum_{n \leq M} |\chi(n)n^{-i\tau}|.$$

For every $M$ and $H_i$ in the above range, we get

$$\sum_{M+1 \leq n \leq M+H_i} \chi(n)n^{-i\tau} = \left| \sum_{1 \leq n \leq H_i} X(n + M) \left( \frac{n + M}{M} \right)^{-i\tau} \right| \leq \left| \sum_{1 \leq n \leq H_i} \chi(n + M) e\left( -\frac{2\pi}{t} \left( \frac{n}{M} - \frac{n^2}{2M^2} + \cdots + \frac{(-1)^m \cdot n^m}{mM^m} \right) \right) \right| + \frac{|t|H^{m+1}}{M^{m+1}}.$$  

Let $\beta_j = 0$ and $\alpha_j = (-1)^j t/2\pi jm^j$. Then for $1 \leq j \leq m |\alpha_j - \beta_j| = |t| \cdot (2\pi jm^j)^{-1} \leq (2\pi m H^j)^{-1}$. Applying Lemmas 1 and 2, we obtain:

$$|S| \leq 2NH^{-1} \max_{N \leq m \leq 2N} \max_{1 \leq H \leq H} \sum_{n \leq M} \chi(n + M) + 2 \frac{\tau H^{m+1}}{N^m} \ll NH^{-1} \sqrt{\log \tau q}^{3/16} C^\omega + N\tau(\tau \log \tau q)^{-10\log q} \ll \sqrt{N \tau \log \tau q} q^{3/16} C^\omega.$$  

From this, the result is easily obtained.

Now we can prove Theorem 1. First, we suppose that $\chi$ is primitive. Let $N = \lfloor \tau q \rfloor$, $M = \lfloor \tau q^{10} \rfloor$, $L = \log (N/M)/\log 2$, $N_i = M2^l (l = 0, \ldots, L)$. Using Abel’s formula, the Polya-Vinogradov estimate for character sums and Lemma 3, we get:

$$|L(s, \chi)| \ll \sum_{n < M} n^{-s} + \sum_{N \leq n \leq N} \chi(n)n^{-i\tau} + \sum_{N \leq n \leq N} |\chi(n)|$$  

$$\ll M^{1-s} \log M + \sum_{l=0}^{L} \max_{N_j \leq N_1 \leq N} \sum_{N_1 \leq n \leq N} \chi(n)n^{-i\tau}$$  

$$\ll M_{1-s} \log M + \sum_{l=0}^{L} N_i^{-s} \max_{N_j \leq N_1 \leq N} \sum_{N_1 \leq n \leq N} \chi(n)n^{-i\tau} \ll M^{1-s} \log M + \sum_{l=0}^{L} N_i^{1-s} \sqrt{\tau q^{3/16}} C^\omega \sqrt{\log \tau} + \tau \sqrt{\tau q^{3/16}} C^\omega \log q$$  

$$\ll M^{1-s} \log M + L M^{1-s} \sqrt{\tau q^{3/16}} C^\omega \sqrt{\log \tau} + \tau \sqrt{\tau q^{3/16}} C^\omega \log q \ll \tau^{-1} q^{3/16} \log$$

If $X$ is not primitive, then there is a $q_1 | q$ and a primitive
character $\chi_i \pmod{q_i}$, associated with $\chi$, such that we can write (see, for example, [5, (6.12))):

$$|L(s, \chi)| = |L(s, \chi)| \prod_{p|q} \left| 1 - \frac{\chi(p)}{p^s} \right| \leq |L(s, \chi)| \cdot \prod_{p|q} 2 \leq |L(s, \chi)| \cdot 2^r,$$

and the theorem follows.

4. The proof of Theorem 2. To prove Theorem 2, we need two lemmas.

**Lemma 4.** Let $t \geq 0$, $0 < a < 1$, and let $X$ and $X_1$ be integers such that $0 < X \leq X_1 \leq 2X \leq \tau^{143/108}$. Then

$$S_i = \sum_{X \leq x < X_1} e(t \log (x + a)) \ll \sqrt{X} \frac{35}{216} \log^2 \tau.$$

**Proof.** If $X \leq \sqrt{\tau}$, then the result can be proven similarly to Corollary 2, [4]. The same method yields

$$\sum_{X \leq x < X_1} e(t \log x - ax) \ll \sqrt{X} \frac{35}{216} \log^2 \tau,$$

for $X \leq \sqrt{\tau}$. If $\sqrt{\tau} \leq X \leq \tau^{143/108}$, then, by Lemma 3 of [4]

$$|S_i| \leq \sum_{t \leq t(X + a) \leq X \leq X_1} \frac{\sqrt{t}}{n} \frac{e(t \log n - an)}{n} + O(X^{-1/2}).$$

Here $t/(X + a) \leq \sqrt{\tau}$. With the use of Abel’s inequality, (9) yields the result for $\sqrt{\tau} \leq X \leq \tau^{143/108}$.

**Lemma 5.** Let $1/2 \leq \sigma \leq 1$, $t \geq 1$ and $0 \leq a < 1$. Then

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s} \ll a^{-s} + \tau^{35(1-\sigma)/108} \log^2 \tau.$$

**Proof.** Let $N = \tau^{143/108}$. Using the Euler-Maclaurin formula [see, for example, [5], (1.7), p. 372]), we obtain similarly to [5], (5.8), p. 114:

$$\zeta(s, a) - \sum_{n=0}^{N-1} (n + a)^{-s} = \frac{(N + a)^{-s}}{1-s} - \frac{1}{2} \int_N^{\infty} \frac{x - [x]}{(x + a)^{s+1}} dx$$

$$= \frac{(N + a)^{-s}}{1-s} - \frac{1}{2} \int_N^{\infty} \frac{(x - [x])^2}{(x + a)^{s+1}} dx + \frac{1}{2} \int_N^{\infty} \frac{s(s + 1)}{(x + a)^{s+3}} dx$$

$$\ll 1 + \tau^2 \int_N^{\infty} u^{-s-1} du \leq 1 + \tau^2 \cdot N^{-s-1} \ll \tau^{35(1-\sigma)/108}.$$

If we denote $M = [\tau^{35/108}]$, $L = [\log (N/M)/\log 2] = N_i = M \cdot 2^t$ for $l = 0, \ldots, L$ and $N_{L+1} = N$, then we have
Using Abel's formula and Lemma 4, we obtain:

\[
S \ll a^{-\sigma} + M^{1-\sigma} \log M + \sum_{0 \leq l \leq L} N_l^{-\sigma} \max_{0 \leq l \leq L} \sum_{\lambda=0}^{N_l} \sum_{n \leq N_l} (n + a)^{-\sigma}.
\]

This proves the lemma.

To prove Theorem 2, we can obviously suppose \( t \geq 1 \), otherwise the result follows from (1). Using Lemma 5, we obtain:

\[
|L(s, \chi)| = |q^{-\sigma} \sum_{m=1}^{q} \chi(m) \zeta(s, m/q)|
\]

\[
< q^{-\sigma} \sum_{m=1}^{q} ((q/m)^{\sigma} + \tau^{50(1-\sigma)/108} \log^2 \tau) \ll \tau^{50(1-\sigma)/108} q^{1-\sigma} \log^3 \tau.
\]

Note Added in Proof. We would like to draw attention to a recent paper by D. R. Heath-Brown, "Hybrid bounds for Dirichlet L-function," Inventiones Mathematicae, 44 (1978), 149–170, which contains a better result than our Theorem 7.

REFERENCES

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CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CA 91125

AND

STATE UNIVERSITY OF NEW YORK AT BUFFALO
BUFFALO, NY 14214