Abstract—This paper presents a new method for synthesizing stochastic control Lyapunov functions for a class of nonlinear stochastic control systems. The technique relies on a transformation of the classical nonlinear Hamilton-Jacobi-Bellman partial differential equation to a linear partial differential equation for a class of problems with a particular constraint on the stochastic forcing. This linear partial differential equation can then be relaxed to a linear differential inclusion, allowing for relaxed solutions to be generated using sums of squares programming. The resulting relaxed solutions are in fact viscosity super/subsolutions, and by the maximum principle are pointwise upper and lower bounds to the underlying value function, even for coarse polynomial approximations. Furthermore, the pointwise upper bound is a stochastic control Lyapunov function, yielding a method for generating nonlinear controllers with pointwise bounded distance from the optimal cost when using the optimal controller. These approximate solutions may be computed with non-increasing error via a hierarchy of semidefinite optimization problems. Finally, this paper develops a-priori bounds on trajectory suboptimality when using these approximate value functions, as well as demonstrates that these methods, and bounds, can be applied to a more general class of nonlinear systems not obeying the constraint on stochastic forcing. Simulated examples illustrate the methodology.

I. INTRODUCTION

The study of system stability is a central theme of control engineering. A primary tool for such studies is Lyapunov theory, wherein an energy-like function is used to show that some measure of distance from a stability point decays over time. The construction of Lyapunov functions that certify system stability advanced considerably with the introduction of Sums of Squares (SOS) programming, which has allowed for Lyapunov functions to be synthesized for both polynomial systems [1] and more general vector fields [2].

To address the more challenging problem of stabilization, rather than the analysis of an existing closed loop system, it is possible to generalize Lyapunov functions to incorporate control inputs. The existence of a Control Lyapunov Function (CLF) (see [3–5]) is sufficient for the construction of a stabilizing controller. However, the synthesis of a CLF for general systems remains an open question. Unfortunately, the SOS-based methods cannot be naively extended to the generation of CLF solutions, due to the bilinearity between the Lyapunov function and control input.

However, for several large and important classes of systems, CLFs are in fact known and may be used for stabilization, with a review of the theory available in [5]. The drawback is that these CLFs are hand-constructed and may be shown to be arbitrarily suboptimal. One way to circumvent this issue is through the use of Receding Horizon Control (RHC), which allows for the incorporation of optimality criteria. Euler-Lagrange equations are used to construct a locally optimum trajectory [6], and stabilization is guaranteed by constraining the terminal cost in the RHC problem to be a CLF. Suboptimal CLFs have found extensive use, with applications in legged locomotion [7] and distributed control [8]. Adding stochasticity to the governing dynamics compounds these difficulties [9], [10].

A complementary area in control engineering is the study of the Hamilton-Jacobi-Bellman (HJB) equation that governs the optimal control of a system. Methods to calculate the solution to the HJB equation via semidefinite programming have been proposed previously by Lasserre et al. [11]. In their work, the solution and the optimality conditions are integrated against monomial test functions, producing an infinite set of moment constraints. By truncating to any finite list of monomials, the optimal control problem is reduced to a semidefinite optimization problem. The method is quite general, applicable to any system with polynomial nonlinearities.

In this work, we propose an alternative line of study based on the linear structure of a particular form of the HJB equation. Since the late 1970s, Fleming [12], Holland [13] and other researchers thereafter [14], [15] have made connections between stochastic optimal control and reaction-diffusion equations based on the linear structure of a particular form of the HJB equation. The linearity of this class of problems has given rise to a growing body of research, with an overview available in [18]. Kappen’s work focused on calculating solutions via path integral techniques. Todorov began with the analysis in [18]. Kappen’s work focused on calculating solutions via path integral techniques. Todorov began with the analysis of particular Markov decision processes, and showed the connection between the two paradigms. This work was built upon by Theodorou et al. [19] into the Path Integral framework in use with Dynamic Motion Primitives. These results have been developed in many compelling directions [18], [20–22].

This paper combines these previously disparate fields of linearly solvable optimal control and Lyapunov theory, and provides a systematic way to construct stabilizing controllers...
with guaranteed performance. The result is a hierarchy of SOS programs that generates stochastic CLFs (SCLF) for arbitrary linearly solvable systems. Such an approach has many benefits. First and foremost, this approach generates stabilizing controllers for an important class of nonlinear, stochastic systems even when the optimal controller is not found. We prove that the approximate solutions generated by the SOS programs are pointwise upper and lower bounds to the true solutions. In fact, the upper bound solutions are SCLFs which can be used to construct stabilizing controllers, and they bound the performance of the system when they are used to construct suboptimal controllers. Existing methods for the generation of SCLFs do not have such performance guarantees. Additionally, we demonstrate that, although the technique is based on linearly solvability, it may be readily extended to more general systems, including deterministic systems, while inheriting the same performance guarantees.

A preliminary version of this work appeared in [23] and [24], where the use of sum of squares programming for solving the HJB were first considered. This paper builds on this recent body of research, studying the stabilization and optimality properties of the resulting solutions. These previous works focused on path planning, rather than stabilization, and did not include the stability analysis or suboptimality guarantees presented in this paper. A short version of this work appeared in [25] which included less details and did not include the extensions in Section VII.

The rest of this paper is organized as follows. Section II reviews linearly solvable HJB equations, SCLFs, and SOS programming. Section III introduces a relaxed formulation of the HJB solutions which is efficiently computable using the SOS methodology. Section IV analyzes the properties of the relaxed solutions, such as approximation errors relative to the exact solutions. This section shows that the relaxed solutions are SCLFs, and that the resulting controller is stabilizing. The upper bound solution is also shown to bound the performance when using the suboptimal controller. Section V summarizes some extensions of the method to handle issues such as approximation of optimal control problems which are not linearly solvable, robust controller synthesis, and non-polynomial systems. Two examples are presented in Section VI to illustrate the optimization technique and its performance. Section VII summarizes the findings of this work and discusses future research directions.

## II. BACKGROUNDS

This section briefly describes the paper’s notation and reviews necessary background on the linear HJB equation, SCLFs, and SOS programming.

### A. Notation

Table I summarizes the notation of different sets appearing in the paper.

A compact domain in $\mathbb{R}^n$ is denoted as $\Omega$ where $\Omega \subset \mathbb{R}^n$, and its boundary is denoted as $\partial \Omega$. A domain $\Omega$ is a basic closed semialgebraic set if there exists $g_i(x) \in \mathbb{R}[x]$ for $i = 1, 2, \ldots, m$ such that $\Omega = \{x \mid g_i(x) \geq 0 \ \forall i = 1, 2, \ldots, m\}$.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_+$</td>
<td>All positive integers</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>All real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}_+$</td>
<td>All nonnegative real numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>All $n$-dimensional real vectors</td>
</tr>
<tr>
<td>$\mathbb{R}[x]$</td>
<td>All real polynomial functions in $x$</td>
</tr>
<tr>
<td>$\mathbb{R}^{n \times m}$</td>
<td>All $n \times m$ matrices</td>
</tr>
<tr>
<td>$\mathbb{R}^{n \times m}[x]$</td>
<td>All $M \in \mathbb{R}^{n \times m}$ such that $M_{i,j} \in \mathbb{R}[x] \ \forall i,j$</td>
</tr>
<tr>
<td>$\mathcal{K}$</td>
<td>All continuous nondecreasing functions $\mu : \mathbb{R}<em>+ \to \mathbb{R}</em>+$ such that $\mu(0) = 0$, $\mu(r) &gt; 0$ if $r &gt; 0$, and $\mu(r) \geq \mu(r')$ if $r &gt; r'$</td>
</tr>
<tr>
<td>$C^{k,k'}$</td>
<td>All functions $f$ such that $f$ is $k$-differentiable with respect to the first argument and $k'$-differentiable with respect to the second argument</td>
</tr>
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</table>

A point on a trajectory, $x(t) \in \mathbb{R}^n$, at time $t$ is denoted $x_t$, while the segment of this trajectory over the interval $[t, T]$ is denoted by $x_{t:T}$.

Given a polynomial $p(x)$, $p(x)$ is positive on domain $\Omega$ if $p(x) > 0 \ \forall x \in \Omega$. $p(x)$ is nonnegative on domain $\Omega$ if $p(x) \geq 0 \ \forall x \in \Omega$, and $p(x)$ is positive definite on domain $\Omega$ where $0 \in \Omega$, if $p(0) = 0$ and $p(x) > 0$ for all $x \in \Omega \setminus \{0\}$.

If it exists, the infinity norm of a function is defined as $\|f\|_{\infty} = \sup_{x \in \Omega} |f(x)|$. To improve readability, a function, $f(x_1, \ldots, x_n)$, is abbreviated as $f$ when the arguments of the function are clear from the context.

### B. Linear Hamilton-Jacobi-Bellman (HJB) Equation

Consider the following affine nonlinear dynamical system,

$$dx_t = (f(x_t) + G(x_t)u_t) \ dt + B(x_t) \ d\omega_t$$

(1)

where $x_t \in \Omega$ is the state at time $t$ in a compact domain $\Omega \subset \mathbb{R}^n$, and $u_t \in \mathbb{R}^m$ is the control input, $f(x) \in \mathbb{R}^n[x]$, $G(x) \in \mathbb{R}^{n \times m}[x]$, and $B(x) \in \mathbb{R}^{n \times m}$ are real polynomial functions of the state variables $x$, and $\omega_t \in \mathbb{R}^d$ is a vector consisting of Brownian motions with covariance $\Sigma_\omega$, i.e., $\omega_t^i$ has independent increments with $\omega_t^i - \omega_s^i \sim \mathcal{N}(0, \Sigma_\omega(t-s))$, for $\mathcal{N}(\mu, \sigma^2)$, a normal distribution. The domain $\Omega$ is assumed to be a basic closed semialgebraic set defined as $\Omega = \{x \mid g_i(x) \in \mathbb{R}[x], g_i(x) \geq 0 \ \forall i = 1, 2, \ldots, m\}$. Extensions to non-polynomial functions is discussed in Section V-D. Without loss of generality, let $0 \in \Omega$ and $x = 0$ be the equilibrium point, whereby $f(0) = 0$, $G(0) = 0$ and $B(0) = 0$.

The goal is to minimize the following functional,

$$J(x,u) = \mathbb{E}_{\omega_t} \left[ \phi(x_T) + \int_0^T q(x_t) + \frac{1}{2} u_t^T R u_t \ dt \right]$$

(2)

subject to (1), where $\phi \in \mathbb{R}[x]$, $\phi : \Omega \to \mathbb{R}_+$ represents a state-dependent terminal cost, $q \in \mathbb{R}[x]$, $q : \Omega \to \mathbb{R}_+$ is state dependent cost, and $R \in \mathbb{R}^{m \times m}$ is a positive definite matrix. The final time, $T$, unknown a priori, is the time at which the system reaches the domain boundary or the origin. This problem is generally called the first exit problem. The expectation $\mathbb{E}_{\omega_t}$ is taken over all realizations of the noise $\omega_t$.

For stability of the resultant controller to the origin, $q(\cdot)$ and $\phi(\cdot)$ are also required to be positive definite functions.
The solution to this minimization problem is known as the value function, \( V : \Omega \rightarrow \mathbb{R}_+ \), where beginning from an initial point \( x_t \) at time \( t \)

\[
V(x_t) = \min_{u_{t+T}} J(x_{t:T}, u_{t:T}).
\]  

(3)

Based on dynamic programming arguments [26, Ch. III.7], the HJB equation associated with this problem is

\[
0 = \min_u \left( q + \frac{1}{2} u^T R u + (\nabla_x V)^T (f + G u) + \frac{1}{2} T r \left( (\nabla_{xx} V) B \Sigma_e B^T \right) \right)
\]  

(4)

whereby the optimal control effort, \( u^* \), can be found analytically, and takes the form

\[
u^* = -R^{-1} G^T \nabla_x V.
\]  

(5)

Substituting the optimal control, \( u^* \), into (4) yields the following nonlinear, second order partial differential equation (PDE):

\[
0 = q (\nabla_x V)^T f - \frac{1}{2} (\nabla_x V)^T G R^{-1} G^T (\nabla_x V) + \frac{1}{2} T r \left( (\nabla_{xx} V) B \Sigma_e B^T \right)
\]  

(6)

with boundary condition \( V(x) = \phi(x) \). For the stabilization problem on a compact domain, it is appropriate to set the boundary condition to be \( \phi(x) = 0 \) for \( x = 0 \), indicating zero cost accrued for achieving the origin, and \( \phi(x) > 0 \) for \( x \in \partial \Omega \setminus \{0\} \). In practice, \( \phi(x) \) at the exterior boundary is usually chosen to be a large number that depends on application to impose large penalty for exiting the predefined domain.

In general, (6) is difficult to solve due to its nonlinearity. However, with the assumption that there exists a \( \lambda > 0 \), a control penalty cost \( R \) satisfying the equation

\[
\lambda G(x_t) R^{-1} G(x_t)^T = B(x_t) \Sigma_e B(x_t)^T \triangleq \Sigma_t
\]  

(7)

and using the logarithmic transformation

\[
V = -\lambda \log \Psi,
\]  

(8)

it is possible [17, 27, 28] after substitution and simplification, to obtain the following linear PDE from (6)

\[
0 = -\frac{1}{\lambda} \Psi + f^T(\nabla_x \Psi) + \frac{1}{2} T r \left( (\nabla_{xx} \Psi) \Sigma_t \right) \quad x \in \Omega
\]  

(9)

This transformation of the value function has been deemed the desirability function [17, Table 1]. For brevity, define the following expression

\[
L(\Psi) \triangleq f^T(\nabla_x \Psi) + \frac{1}{2} T r \left( (\nabla_{xx} \Psi) \Sigma_t \right)
\]

and the function \( \psi(x) \) at the boundary as

\[
\psi(x) \triangleq e^{-\frac{\phi(x)}{\lambda}} \quad x \in \partial \Omega.
\]

Condition (7) is trivially met for systems of the form \( dx_t = f(x_t) \ dt + G(x_t) (u_t \ dt + dw_t) \), a pervasive assumption in the adaptive control literature [29]. This constraint restricts the design of the control penalty \( R \), such that control effort is highly penalized in subspaces with little noise, and lightly penalized in those with high noise. Additional discussion is given in [17, SI Sec. 2.2].

C. Stochastic Control Lyapunov Functions (SCLF)

Before the stochastic control Lyapunov function (SCLF) is introduced, the definitions for two forms of stability are provided, following the definitions in [30, Ch. 5].

Definition 1. Given \( \{1\} \), the equilibrium point at \( x = 0 \) is stable in probability for \( t \geq 0 \) if for any \( s \geq 0 \) and \( \epsilon > 0 \),

\[
\lim_{x \to 0} P \left\{ \sup_{t \geq s} |X^{x,s}(t)| > \epsilon \right\} = 0
\]

where \( X^{x,s} \) is the trajectory of \( \{1\} \) starting from \( x \) at time \( s \).

Intuitively, Definition 1 is similar to the notion of stability for deterministic systems. The following is a stronger stability definition that is similar to the notion of asymptotic stability for deterministic systems.

Definition 2. Given \( \{1\} \), the equilibrium point at \( x = 0 \) is asymptotically stable in probability if it is stable in probability and

\[
\lim_{x \to 0} P \left\{ \lim_{t \to \infty} |X^{x,s}(t)| = 0 \right\} = 1
\]

where \( X^{x,s} \) is the trajectory of \( \{1\} \) starting from \( x \) at time \( s \).

These notions of stability can be ensured through the construction of SCLFs, as follows.

Definition 3. A stochastic control Lyapunov function (SCLF) for system \( \{1\} \) is a positive definite function \( \mathcal{V} \in C^2(1) \) on a compact domain \( \mathcal{O} = \Omega \cup \{0\} \times \{t > 0\} \) such that

\[
\mathcal{V}(0, t) = 0, \quad \mathcal{V}(x, t) \geq \mu(|x|) \quad \forall \ t > 0,
\]

\[
\quad \exists \ u(x, t) \text{ s.t. } L(\mathcal{V}(x, t)) \leq 0 \quad \forall \ (x, t) \in \mathcal{O} \setminus \{(0, t)\}
\]

where \( \mu \in \mathcal{K} \), and

\[
L(\mathcal{V}) = \partial_x \mathcal{V} + \nabla_x \mathcal{V}^T (f + G u) + \frac{1}{2} T r ((\nabla_{xx} \mathcal{V}) \Sigma_e B^T).
\]

(10)

Theorem 4. [30, Thm. 5.3] For system \( \{1\} \), assume that there exists a SCLF and a \( u \) satisfying Definition 3. Then, the equilibrium point \( x = 0 \) is stable in probability, and \( u \) is a stabilizing controller.

To achieve the stronger condition of asymptotic stability in probability, we have the following result.

Theorem 5. [30, Thm. 5.5 and Cor. 5.1] For system \( \{1\} \) suppose that in addition to the existence of a SCLF and a \( u \) satisfying Definition 3, \( u \) is time-invariant,

\[
\forall \ t > 0, \quad L(\mathcal{V}(x, t)) < 0 \quad \forall \ (x, t) \in \mathcal{O} \setminus \{(0, t)\}
\]

where \( \mu' \in \mathcal{K} \). Then, the equilibrium point \( x = 0 \) is asymptotically stable in probability, and \( u \) is an asymptotically stabilizing controller.

D. Sum of Squares (SOS) Programming

Sum of Squares (SOS) programming is the primary tool by which approximate solutions to the HJB equation are generated in this paper. In particular, we will show how
the PDE that governs the HJB may be relaxed to a set of nonnegativity constraints. SOS methods will then allow for the construction of an optimization problem where these nonnegativity constraints may be enforced. A complete introduction to SOS programming is available in [11]. Here, we review the basic definition of SOS that is used throughout the paper.

**Definition 6.** A multivariate polynomial $f(x)$ is a SOS polynomial if there exist polynomials $f_0(x), \ldots, f_m(x)$ such that

$$f(x) = \sum_{i=0}^m f_i^2(x).$$

The set of SOS polynomials in $x$ is denoted as $\mathbb{S}[x]$. Accordingly, a sufficient condition for nonnegativity of a polynomial $f(x)$ is that $f(x) \in \mathbb{S}[x]$. Membership in the set $\mathbb{S}[x]$ may be tested as a convex program [11].

**Theorem 7.** (1 Thm. 3.3) The existence of a SOS decomposition of a polynomial in $n$ variables of degree $2d$ can be decided by solving a semidefinite programming (SDP) feasibility problem. If the polynomial is dense (no sparsity), the dimension of the matrix inequality in the SDP is equal to $(n + d) \times (n + d)$. Hence, by adding SOS constraints to the set of all positive polynomials, testing nonnegativity of a polynomial becomes a tractable SDP. The converse question, is a nonnegative polynomial necessarily a SOS, is unfortunately false, indicating that this test is conservative [11]. Nonetheless, SOS feasibility is sufficiently powerful for our purposes.

Theorem 7 guarantees a tractable procedure to determine whether a particular polynomial, possibly parameterized, is a SOS polynomial. Our method combines multiple polynomial constraints to an optimization formulation. To do so, we need to define the following polynomial sets.

**Definition 8.** The preordering of polynomials $g_i(x) \in \mathbb{R}[x]$ for $i = 1, 2, \ldots, m$ is the set

$$P(g_1, \ldots, g_m) = \left\{ \sum_{\nu \in \{0,1\}^m} s_\nu(x)g_1(x)^{\nu_1} \cdots g_m(x)^{\nu_m} \mid s_\nu \in \mathbb{S}[x] \right\}. \quad (11)$$

The quadratic module of polynomials $g_i(x) \in \mathbb{R}[x]$ for $i = 1, 2, \ldots, m$ is the set

$$M(g_1, \ldots, g_m) = \left\{ \sum_{i=1}^m s_i(x)g_i(x) \mid s_i \in \mathbb{S}[x] \right\}. \quad (12)$$

The following proposition is trivial, but it is useful to incorporate the domain $\Omega$ in our optimization formulation later.

**Proposition 9.** Given $f(x) \in \mathbb{R}[x]$ and the domain

$$\Omega = \{ x \mid g_i(x) \in \mathbb{R}[x], g_i(x) \geq 0, i \in \{1, 2, \ldots, m\} \},$$

if $f(x) \in P(g_1, \ldots, g_m)$, or $f(x) \in M(g_1, \ldots, g_m)$, then $f(x)$ is nonnegative on $\Omega$. If there exists another polynomial $f'(x)$ such that $f'(x) \geq f(x)$, then $f'(x)$ is also nonnegative on $\Omega$.

**Proof.** Because $g_i(x)$ and $s_i(x)$ are nonnegative, all functions in $M(\cdot)$ and $P(\cdot)$ are nonnegative. The second statement is trivially true if the first statement is true.

To illustrate how this proposition applies, consider a polynomial $f(x)$ defined on the domain $x \in [-1, 1]$. The bounded domain can be equivalently defined by polynomials with $g_1(x) = 1 + x$ and $g_2(x) = 1 - x$. To certify that $f(x) \geq 0$ on the specified domain, construct a function $h(x) = s_1(x)(1 + x) + s_2(x)(1 - x) + s_3(x)(1 + x)(1 - x)$ where $s_i \in \mathbb{S}[x]$ and certify that $f(x) - h(x) \geq 0$. Notice that $h(x) \in P(1 + x, 1 - x)$, so $h(x) \geq 0$. If $f(x) - h(x) \geq 0$, then $f(x) - h(x) \geq 0$. Proposition 9 is applied here. Finding the correct $s_i(x)$ is not trivial in general. Nonetheless, as mentioned earlier, if we further impose that $f(x) - h(x) \in \mathbb{S}[x]$, then checking if there exists $s_i(x)$ such that $f(x) - h(x) \in \mathbb{S}[x]$ becomes a SDP as given by Theorem 7. More concretely, the procedure may begin with a limited polynomial degree for $s_i(x)$, increasing the degree until a certificate is found (if one exists) or the computation resources are exhausted.

To simplify notation in the remainder of this text, given a domain $\Omega = \{ x \mid g_i(x) \in \mathbb{R}[x], g_i(x) \geq 0, i \in \{1, 2, \ldots, m\} \}$, we set the notation $P(\Omega) = P(g_1, \ldots, g_m)$ and $M(\Omega) = M(g_1, \ldots, g_m)$.

**Remark 10.** Choosing either $M(\Omega)$ or $P(\Omega)$ relies on the computational resources available. Although $M(\Omega) \subset P(\Omega)$ and therefore the chances of finding a certificate is larger using $P(\Omega)$, the resulting SDP is also larger. In addition, using other subsets of $P(\Omega)$ apart from $M(\Omega)$ does not change the results. These polynomial sets are often used in the discussions of Schmüdgen’s or Putinar’s Positivstellensatz. Loosely speaking, Schmüdgen’s Positivstellensatz states that if $f(x)$ is positive on a compact domain $\Omega$, then $f(x) \in P(\Omega)$ [11, 17].

**III. SUM-OF-SQUARES RELAXATION OF THE HJB PDE**

Sum of squares programming has found many uses in combinatorial optimization, control theory, and other applications. This section now adds solving the linear HJB to this list.

We would like to emphasize the following standing assumption, necessary in moment and SOS-based methods [11, 17].

**Assumption 11.** Assume that system (1) evolves on a compact domain $\Omega \subset \mathbb{R}^n$, and $\Omega$ is a basic closed semialgebraic set such that $\Omega = \{ x \mid g_i(x) \in \mathbb{R}[x], g_i(x) \geq 0, i \in \{1, \ldots, k\} \}$ for some $k \geq 1$. Then, the boundary $\partial \Omega$ is polynomial representable. We use the notation $\partial \Omega = \{ x \mid h_i(x) \in \mathbb{R}[x], \prod_{i=1}^m h_i(x) = 0 \}$ for some $m \geq 1$ to describe it.

The following definitions formalize several operators that will prove useful in the sequel.

**Definition 12.** Given a basic closed semialgebraic set $\Omega = \{ x \mid g_i(x) \in \mathbb{R}[x], g_i(x) \geq 0, i \in \{1, \ldots, k\} \}$ and a set of SOS polynomials,

$$S = \{ s_\nu(x) \mid s_\nu(x) \in \mathbb{S}[x], \nu \in \{0, 1\}^k \},$$

define the operator \( D \) as
\[
D(\Omega, S) = \sum_{\nu \in (0,1)^k} s_\nu(x)g_1(x)\nu_1 \cdots g_k(x)\nu_k
\]
where \( D(\Omega, S) \in P(\Omega) \).

**Definition 13.** Given a polynomial inequality, \( p(x) \geq 0 \), the boundary of a compact set \( \partial \Omega = \{ x \mid h_i(x) \in \mathbb{R}[x], \prod_{i=1}^m h_i(x) = 0 \} \) and a set of polynomials, 
\[
T = \{ t_s(x) \mid t_s(x) \in \mathbb{R}[x], s \in \{1, \ldots, m\} \},
\]
define the operator \( B \) as
\[
B(p(x), \partial \Omega, T) = \{ p(x) - t_s(x) h_i(x) \mid s \in \{1, \ldots, m\} \}
\]
where \( B \) returns a set of polynomials that is nonnegative on \( \partial \Omega \).

**A. Relaxation of the HJB equation**

If the linear HJB [9] is not uniformly parabolic [31], a classical solution may not exist. The notion of viscosity solutions is developed to generalize the classical solution. We refer readers to [31] for a general discussion on viscosity solutions and [26] for a discussion on viscosity solutions related to Markov diffusion processes.

**Definition 14. [31, Def. 2.2]** Given \( \Omega \subset \mathbb{R}^N \) and a partial differential equation
\[
F(x, u, \nabla_x u, \nabla_{xx} u) = 0 \tag{13}
\]
where \( F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N) \rightarrow \mathbb{R}, S(N) \) is the set of real symmetric \( N \times N \) matrices, and \( F \) satisfies
\[
F(r, p, X) \leq F(x, s, p, Y) \text{ whenever } r \leq s \text{ and } Y \leq X,
\]
then a viscosity subsolution of \( 13 \) on \( \Omega \) is a function \( u \in USC(\Omega) \) such that
\[
F(x, u, \nabla_x u, \nabla_{xx} u) \leq 0 \quad \forall x \in \Omega, (p, X) \in J_{\Omega}^{2+} u(x)
\]
Similarly, a viscosity supersolution of \( 13 \) on \( \Omega \) is a function \( u \in LSC(\Omega) \) such that
\[
F(x, u, \nabla_x u, \nabla_{xx} u) \geq 0 \quad \forall x \in \Omega, (p, X) \in J_{\Omega}^{2-} u(x)
\]
Finally, \( u \) is a viscosity solution of \( 13 \) on \( \Omega \) if it is both a viscosity subsolution and a viscosity supersolution in \( \Omega \).

The notations \( USC(\Omega) \) and \( LSC(\Omega) \) represent the sets of upper and lower semicontinuous functions on domain \( \Omega \) respectively, and \( J_{\Omega}^{2+} u(x) \) and \( J_{\Omega}^{2-} u(x) \) represents the second order "superjets" and "subjets" of \( u \) at \( x \) respectively, a completely unrestricted domain in our setting. For further details, readers may refer to [31]. For the remainder of this paper, we assume a unique nontrivial viscosity solution to \( 6 \) and \( 9 \) exists (see [26], Chapter V) and denote the unique solutions as \( \Psi^* \) and \( V^* \) respectively.

The equality constraints of \( 9 \) may be relaxed as follows
\[
1 \lambda q\Psi - \mathcal{L}(\Psi) \leq 0 \quad \Psi(x) \leq \psi(x) \quad x \in \partial \Omega. \tag{14}
\]

Such a relaxation provides a point-wise bound to the solution \( \Psi^* \), and this relaxation may be enforced via SOS programming. In particular, a solution to \( 14 \), denoted as \( \Psi_l \), is a lower bound on the solution \( \Psi^* \) over the entire problem domain.

**Theorem 15.** Given a smooth function \( \Psi_l \) that satisfies \( 14 \), then \( \Psi_l \) is a viscosity subsolution and \( \Psi_l \leq \Psi^* \) for all \( x \in \Omega \).

**Proof.** By Definition \( 14 \), the solution \( \Psi_l \) is a viscosity subsolution. Note that \( \Psi^* \) is both a viscosity subsolution and a viscosity supersolution, and \( \Psi_l \leq \Psi^* \) on the boundary \( \partial \Omega \).

Hence, by the maximum principle for viscosity solutions [31, Thm 3.3], \( \Psi_l \leq \Psi^* \) for all \( x \in \Omega \).

Similarly, the analogous relaxation
\[
1 \lambda q\Psi - \mathcal{L}(\Psi) \geq 0 \quad \Psi(x) \geq \psi(x) \quad (x) \in \partial \Omega \tag{15}
\]
gives an over-approximation of the desirability function, and its solution, denoted as \( \Psi_u \), is an upper bound of \( \Psi^* \) over domain \( \Omega \). Thus, we also have

**Theorem 16.** Given a smooth function \( \Psi_u \) that satisfies \( 15 \), then \( \Psi_u \) is a viscosity supersolution and \( \Psi_u \geq \Psi^* \) for all \( x \in \Omega \).

**Proof.** The proof is identical to the proof for Theorem 15.

Because the logarithmic transform \( 9 \) is monotonic, one can relate these bounds on the desirability function to bounds on the value function as follows

**Proposition 17.** If the solution to \( 6 \) is \( V^* \), given solutions \( V_u = -\lambda \log \Psi_u \) and \( V_l = -\lambda \log \Psi_l \) from \( 14 \) and \( 15 \) respectively, then \( V_u \geq V^* \) and \( V_l \leq V^* \).

**Proof.** Recall that \( V^* = -\lambda \log \Psi^* \). Applying Theorem 15 and 16, \( V_u \geq V^* \) and \( V_l \leq V^* \).

Although the solutions to \( 14 \) and \( 15 \) do not satisfy \( 9 \) exactly, they provide point-wise bounds to the solution \( \Psi^* \).

**B. SOS Program**

Given that relaxation \( 14 \) and \( 15 \) results in a point-wise upper and lower bound to the exact solution of \( 9 \), we construct the following optimization problem that provides a suboptimal controller with bounded residual error:
\[
\min_{\Psi_l, \Psi_u} \epsilon \tag{16}
\]
s.t. \[
1 \lambda q\Psi_l - \mathcal{L}(\Psi_l) \leq 0 \quad x \in \Omega
\]
\[
0 \leq 1 \lambda q\Psi_u - \mathcal{L}(\Psi_u) \quad x \in \Omega
\]
\[
\Psi_u - \Psi_l \leq \epsilon \quad x \in \Omega
\]
\[
0 \leq \Psi_l - \psi \leq \Psi_u \quad x \in \partial \Omega
\]
\[
\partial_x \Psi_l \leq 0 \quad x^1 \geq 0
\]
\[
\partial_x \Psi_u \geq 0 \quad x^1 \leq 0
\]
\[
\Psi_l(0) = 1
\]
where $x^i$ is the $i$-th component of $x \in \Omega$. As mentioned in Section III-A, the first two constraints result from the relaxations of the HJB equation, and the fourth constraint arises from the relaxation of the boundary conditions. The third constraint ensures that the difference between the upper bound and lower bound solution is bounded, and the last three constraints ensure that the solution yields a stabilizing controller, as will be made clear in Section IV. Note that in the optimization problem, $\Psi_u$ and $\Psi_l$ are polynomials whereby the coefficients and the degree for both are optimization variables. The term $\epsilon$ is related to the error of the approximation.

As discussed in the review of SOS techniques, a general optimization problem involving parameterized nonnegative polynomials is not necessarily tractable. In order to solve (16) using a polynomial-time algorithm, we restrict the polynomial inequalities such that they are SOS polynomials instead of nonnegative polynomials. We therefore apply Proposition 9 to relax optimization problem (16) into

$$\min_{\Psi_l, \Psi_u, S, T} \epsilon \quad (17)$$

s.t.\hspace{1em} \begin{align*}
\frac{1}{\lambda} q \Psi_l + L(\Psi_l) - D(\Omega, S_1) &\in S[x] \\
\frac{1}{\lambda} q \Psi_u - L(\Psi_u) - D(\Omega, S_2) &\in S[x] \\
\epsilon - (\Psi_u - \Psi_l) &- D(\Omega, S_3) \in S[x] \\
B(\Psi_l, \partial \Omega, T_1) &\in S[x] \\
B(\Psi_u, \partial \Omega, T_2) &\in S[x] \\
\partial_x(\Psi_l - D(\Omega \cap \{x^i \geq 0\}, S_4) &\in S[x] \\
\partial_x(\Psi_u - D(\Omega \cap \{-x^i \geq 0\}, S_5) &\in S[x] \\
\Psi_l(0) &= 1
\end{align*}$$

where $S = (S_1, \ldots, S_5)$, $S_i \subseteq S[x]$ is defined as in Definition 12, $T = (T_1, T_2, T_3)$, and $T_i \subseteq \mathbb{R}[x]$ is defined as in Definition 13. With a slight abuse of notation, $B(\cdot) \in S[x]$ implies that each polynomial in $B(\cdot)$ is a SOS polynomial.

If the polynomial degrees are fixed, optimization problem (17) is convex and solvable using a semidefinite program via Theorem 7. The next section will discuss the systematic approach we used to solve the optimization problem. Henceforth, denote the solution to (17) as $(\Psi_u, \Psi_l, S, T, \epsilon)$.

**Remark 18.** By Definition 12, the viscosity solution is a continuous function. Consequently, the solution $\Psi^*$ is a continuos function defined on a bounded domain. Hence, $\Psi_u$ and $\Psi_l$ can be made arbitrary close to $\Psi^*$ by the Stone-Weierstrass Theorem 12 in 16. However, this guarantee is lost when $\Psi_u$ and $\Psi_l$ are restricted to be a SOS polynomials. The feasible set of the optimization problem (17) is therefore not necessarily non-empty for a given polynomial degree. One would not expect feasibility for all instances of (17) as this would imply there exists a linear stabilizing controller for any given system.

**C. Controller Synthesis**

Let $d$ be the maximum degree of $\Psi_l$, $\Psi_u$ and polynomials in $S$ and $T$, and denote $(\Psi_u^d, \Psi_l^d, S^d, T^d, \epsilon^d)$ as a solution to (17) when the maximum polynomial degree is fixed at $d$. The hierarchy of SOS programs with increasing polynomial degree produces a sequence of (possibly empty) solutions $(\Psi_u^d, \Psi_l^d, S^d, T^d, \epsilon^d)_{d \in \mathbb{Z}^+}$, where $I \subseteq \mathbb{Z}^+$. This sequence will be shown in the next section to improve, under the metric of the objective in (17).

In other words, if solutions exist for $d$ and $d'$ such that $d > d'$, then $\epsilon^d \leq \epsilon^{d'}$. Therefore, one could keep increasing the degree of polynomials in order to achieve tighter bounds on $\Psi^*$, and invaribly, $V^*$. The use of such hierarchies has become commonplace in polynomial optimization 1, 13. If at certain degree, $\epsilon^d = 0$, the solution $\Psi^*$ is found.

Once a satisfactory error is achieved or computational resources run out, the lower bound $\Psi_l$ can be used to compute a suboptimal controller. Recall that $u^* = -R^{-1}G^T \nabla_x V^*$ and $V^* = -\lambda \log \Psi^*$. The suboptimal controller $u^\epsilon$ for a given error $\epsilon$ is computed as $u^\epsilon = -R^{-1}G^T \nabla_x V_{\epsilon u}$ where $V_{\epsilon u} = -\lambda \log \Psi_l$. Even when $\epsilon$ is larger than a desired value, the solution $\Psi_l$ still satisfies conditions in Definition 5 to yield a stabilizing suboptimal controller. Next section will analyze properties of the solutions and the suboptimal controller.

**IV. ANALYSIS**

This section establishes several properties of the solutions to the optimization problem (17) that are useful for feedback control. First we show that the solutions in the SOS program hierarchy are uniformly bounded relative to the exact solutions. We next prove that the relaxed solutions to the stochastic HJB equation are SCLFs, and the approximated solution leads to a stabilizing controller. Finally, we show that the costs of using the approximate solutions as controllers are bounded above by the approximated value functions.

**A. Properties of the Approximated Desirability Function**

First, the approximation error of $\Psi_l$ or $\Psi_u$ obtained from (17) is computed relative to the true desirability function $\Psi^*$. Hence, $\Psi_u$ and $\Psi_l$ are restricted to be a SOS polynomials. The feasible set of the optimization problem (17) is therefore not necessarily non-empty for a given polynomial degree. One would not expect feasibility for all instances of (17) as this would imply there exists a linear stabilizing controller for any given system.

**Proposition 19.** Given a solution $(\Psi_u, \Psi_l, S, T, \epsilon)$ to (17) for a given degree $d$, the approximation error of the desirability function is bounded as $||\Psi - \Psi^*||_\infty \leq \epsilon$ where $\Psi$ is either $\Psi_u$ or $\Psi_l$.

**Proof.** By Corollary 15 and 16, $\Psi_l$ is the lower bound of $\Psi^*$, and $\Psi_u$ is the upper bound of $\Psi^*$. So, $\epsilon \geq \Psi_u - \Psi_l \geq 0$ and $\Psi_u \geq \Psi^* \geq \Psi_l$. Combining both inequalities, one has $\Psi_u - \Psi^* \leq \epsilon$ and $\Psi^* - \Psi_l \leq \epsilon$. Therefore, $||\Psi - \Psi^*||_\infty \leq \epsilon$ where $\Psi$ is either $\Psi_u$ or $\Psi_l$. 

**Proposition 20.** The hierarchy of SOS programs consisting of solutions to (17) with increasing polynomial degree produces a sequence of solutions $(\Psi_u^d, \Psi_l^d, S^d, T^d, \epsilon^d)$ such that $\epsilon^{d+1} \leq \epsilon^d$ for all $d$.

**Proof.** Polynomials of degree $d$ form a subset of polynomials of degree $d+1$. Thus, at a higher polynomial degree $d+1$, a previous solution at a lower polynomial degree $d$ is still a feasible solution when the coefficients for monomials with total degree $d+1$ is set to 0. Consequently, the optimal value $\epsilon^{d+1}$ cannot be smaller than $\epsilon^d$ for all $d$. 


Thus, as the polynomial degree of the optimization problem is increased, the pointwise error \( \epsilon \) is non-increasing. Therefore, one could keep increasing the degree of polynomials in order to achieve tighter bounds on \( \Psi^* \), and invariably, \( V^* \). However, \( \epsilon \) is only non-increasing as the polynomial degree is increased, and a convergence of the bound \( \epsilon \) to zero is not guaranteed.

Although the bound on the pointwise error is non-increasing, the actual difference between \( \Psi \) and \( \Psi^* \) may increase between iterations. We bound this variation as follows.

\[ \text{Corollary 21. Suppose } \left\| \Psi^d - \Psi^* \right\|_{\infty} \leq \epsilon^d \text{ and } \left\| \Psi^{d+1} - \Psi^* \right\|_{\infty} = \gamma^{d+1}. \text{ Then, } \gamma^{d+1} \leq \epsilon^d. \]

\[ \text{Proof. By Proposition 20 } \epsilon^{d+1} \leq \epsilon^d. \text{ Because } \gamma^{d+1} \leq \epsilon^{d+1}, \gamma^{d+1} \leq \epsilon^d. \]

In other words, the approximation error of the desirability function for a SOS program using \( d + 1 \) polynomial degree cannot increase such that it is larger than \( \epsilon^d \) in each step of the hierarchy of SOS programs.

\[ \text{B. Properties of the Approximated Value Function} \]

Up to this point, the analysis has focused on properties of the desirability solution. We now investigate the implications of these results upon the value function. Recall that the value function is related to the desirability via the logarithmic transform (8). Henceforth, denote the solution to (6) as \( \Psi^*(x) = \min_{u \in [0,1]} E_{\omega, u}[J(x)] = -\lambda \log \Psi^*(x) \), the solution to (17) for a fixed degree \( d \) as \( \Psi_u, \Psi_l, S, T, \epsilon \), and the suboptimal value function computed from the solution of (17) as \( V_u = -\lambda \log \Psi_l \). Only \( \Psi_l \) and \( V_u \) is considered henceforth, because \( \Psi_u \) is defined, but not \( \Psi_l \), gives an approximate value function that satisfies the properties of SCLF in Definition 3 a fact shown in the next section.

\[ \text{Theorem 22. } V_u \text{ is an upper bound of the optimal cost } V^* \text{ such that} \]

\[ 0 \leq V_u - V^* \leq -\lambda \log \left(1 - \min \left\{ 1, \frac{\epsilon}{\eta} \right\} \right) \tag{18} \]

\[ \text{where } \eta = d - \frac{\| \Psi^* \|_{\infty}}{\epsilon^d}. \]

\[ \text{Proof. By Proposition 17 } V_u \geq V^* \text{ and hence, } V_u - V^* \geq 0. \]

To prove the other inequality, by Proposition 19

\[ V_u - V^* = -\lambda \log \frac{\Psi_u}{\Psi^*} \leq -\lambda \log \frac{\Psi^* - \epsilon}{\Psi^*} \leq -\lambda \log \left(1 - \frac{\epsilon}{\eta} \right). \]

The last inequality holds because \( \Psi^* \geq e^{-\frac{\| \Psi^* \|_{\infty}}{\epsilon^d}} \) by definition in (8). Since \( \Psi_l \) is the lower bound of \( \Psi^* \), the right hand side of the first equality is always a positive number. Therefore, \( V_u \) is a point-wise upper bound of \( V^* \).

\[ \text{Corollary 23. Let } V_u^d = -\lambda \log \Psi_{l_d} \text{ and } V_u^{d+1} = -\lambda \log \Psi_{l_{d+1}}. \text{ If } V_u^d - V^* \leq \epsilon^d \text{ and } V_u^{d+1} - V^* = \gamma^{d+1}, \text{ then } \gamma^{d+1} \leq \epsilon^d. \]

\[ \text{Proof. This result is given by Corollary 21 and Theorem 22} \]

At this point, we have shown that the lower bound of the desirability function gives us an upper bound of the suboptimal cost. More importantly, the upper bound of the suboptimal cost is not increasing as the degree of polynomial increases.

\[ \text{C. The Approximate HJB solutions are SCLFs} \]

This section shows that the approximate value function derived from the desirability approximation, \( \Psi_l \), is a SCLF.

\[ \text{Theorem 24. } V_u \text{ is a stochastic control Lyapunov function according to Definition 3} \]

\[ \text{Proof. The constraint } \Psi_l(0) = 1 \text{ ensures that } V_u(0) = -\lambda \log \Psi_l(0) = 0. \text{ Notice that all terms in } J(x, u) \text{ from (2)} \]

are positive definite, resulting in \( V^* \) being a positive definite function. In addition, by Proposition 17 \( V_u \geq V^* \). Hence, \( V_u \) is also a positive definite function. The second and third to last constraints in (17) ensures that \( \Psi_l \) is nonincreasing away from \( \epsilon^d \). Hence, \( V_u \) is nondecreasing away from the origin.

Next, we show that there exists a u such that \( L(V_u) \leq 0 \). Following (5), let

\[ u^* = -R^{-1}G^T \nabla_x V_u, \tag{19} \]

the control law corresponding to \( V_u \). Notice that from the definition of \( V_u \), \( \nabla_x V_u = -\frac{\lambda}{\Psi_l} \nabla_x \Psi_l \) and \( \nabla_{xx} V_u = \frac{\lambda}{\Psi_l^2} (\nabla_x \Psi_l)(\nabla_x \Psi_l)^T - \frac{\lambda}{\Psi_l} \nabla_{xx} \Psi_l \). So, \( u^* = \frac{\lambda}{\Psi_l} R^{-1}G^T \nabla_x \Psi_l \). Then, from (10).

\[ L(V_u) = -\frac{\lambda}{\Psi_l} (\nabla_x \Psi_l)^T (f + \frac{\lambda}{\Psi_l} GR^{-1}G^T \nabla_x \Psi_l) \]

\[ + \frac{1}{2} Tr \left( \left( \frac{\lambda}{\Psi_l} (\nabla_x \Psi_l)(\nabla_x \Psi_l)^T - \frac{\lambda}{\Psi_l} \nabla_{xx} \Psi_l \right) B \Sigma_t B \right) \]

where \( \partial_t V_u = 0 \) because \( V_u \) is not a function of time. Applying the assumption in (7) and simplifying,

\[ L(V_u) = -\frac{\lambda}{\Psi_l} (\nabla_x \Psi_l)^T f - \frac{\lambda}{2 \Psi_l^2} (\nabla_x \Psi_l)^T \Sigma_t \nabla_x \Psi_l \]

\[ - \frac{\lambda}{2 \Psi_l} Tr ((\nabla_{xx} \Psi_l) \Sigma_t). \]

From the first constraint in (17),

\[ \frac{1}{\lambda} q \Psi_l - \frac{1}{2} Tr ((\nabla_{xx} \Psi_l) \Sigma_t) \leq 0 \implies -\frac{\lambda}{\Psi_l} (\nabla_x \Psi_l)^T f \leq -q + \frac{\lambda}{2 \Psi_l} Tr ((\nabla_{xx} \Psi_l) \Sigma_t). \]

Substituting this inequality into \( L(V_u) \) and simplifying yields

\[ L(V_u) \leq -q - \frac{\lambda}{2 \Psi_l^2} (\nabla_x \Psi_l)^T \Sigma_t \nabla_x \Psi_l \leq 0 \tag{20} \]

because \( q \geq 0, \lambda > 0 \) and \( \Sigma_t \) is positive semidefinite by definition. Since \( V_u \) satisfies Definition 3 \( V_u \) is a SCLF.

\[ \text{Corollary 25. The suboptimal controller } u^* = -R^{-1}G^T \nabla_x V_u \text{ is stabilizing in probability within the domain } \Omega. \]

\[ \text{Proof. This corollary is a direct consequence of the constructive proof of Theorem 24 and Theorem 4}. \]
Corollary 26. If $\Sigma_t$ is a positive definite matrix, the suboptimal controller $u^* = -R^{-1}G^T \nabla_x V_u$ is asymptotically stabilizing in probability within the domain $\Omega$.

Proof. This corollary is a direct consequence of the constructive proof of Theorem 24 and Theorem 5. In (20), $L(V_u) < 0$ for $x \in \Omega \cup \{0\}$ if $\Sigma_t$ is positive definite. Recall that $q$ is positive definite in the problem formulation. \qed

D. Bound on the Total Trajectory Cost

We conclude this section by showing that the expected total trajectory cost incurred by the system while operating under the suboptimal controller of (12) can be bounded as follows.

Theorem 27. Given the control law $u^* = -R^{-1}G^T \nabla_x V_u$, $J_u \leq V_u \leq V^* - \lambda \log \left(1 - \min \left\{ 1, \frac{c}{\eta} \right\}\right)$ (21)

where $J_u = E_{\omega_t}[\psi_t(x_T)] + \int_0^T r(x_t, u^*_t) dt$, the expected cost of the system when using the control law, $u^*$.

Proof. By Itô’s formula,

$$dV_u(x_t) = L(V_u(x_t)) dt + \nabla_x V_u(x_t) B(x_t) d\omega_t,$$

where $L(V)$ is defined in (11). Then,

$$V_u(x_t) = V_u(x_0, 0) + \int_0^t L(V_u(x_s)) ds + \int_0^t \nabla_x V_u(x_s) B(x_s) d\omega_s.$$ (22)

Given that $V_u$ is derived from polynomial function $\Psi_t$, the integrals are well defined, and we can take the expectation of (22) to get

$$E[V_u(x_t)] = V_u(x_0, 0) + E \left[ \int_0^t L(V_u(x_s)) ds \right]$$

whereby the last term of (22) drops out because the noise is assumed to have zero mean. The expectations of the other terms return the same terms because they are deterministic. From (20),

$$L(V_u) \leq -q - \frac{\lambda}{2} (\nabla_x \Psi_t)^T \Sigma_t \nabla_x \Psi_t$$

$$= -q - \frac{1}{2} (\nabla_x V_u)^T G R^{-1} G^T (\nabla_x V_u)$$

$$= -q - \frac{1}{2} (u^*_t)^T R u^*_t$$

where the first equality is given by the logarithmic transformation and the second equality is given by the control law $u^* = -R^{-1}G^T \nabla_x V_u$. Therefore,

$$E_{\omega_t}[V_u(x_T)] = V_u(x_0) + E_{\omega_t} \left[ \int_0^T L(V_u(x_s)) ds \right]$$

$$\leq V_u(x_0) - \int_0^T q(x_s) + \frac{1}{2} (u^*_s)^T R u^*_s ds$$

$$= V_u(x_0) - J(x_0, u^*) + E_{\omega_t}[\phi(x_T)]$$

where the last equality is given by (2). Therefore, $V_u(x_0) - J(x_0, u^*) \geq E_{\omega_t}[V_u(x_T) - \phi(x_T)]$. By definition, $V_u(x_T) \geq \phi(x_T)$ for all $x_T \in \Omega$. Thus, $E_{\omega_t}[V_u(x_T) - \phi(x_T)] \geq 0$. Consequently, $V_u(x_0) - J(x_0, u^*) \geq 0$, and $V_u(x_0) \geq J(x_0, u^*)$.

Theorem 22 gives the second inequality in the theorem. \qed

V. Extensions

This section briefly summarizes some extensions of the basic framework to a few related problems.

A. Linearly Solvable Approximations

The approach presented in this paper would appear up to this point to be limited to systems that are linearly solvable, i.e., those that satisfy condition (7). However, the proposed methods may be extended to a system which does not satisfy these conditions by approximating the system with one that is linearly solvable. One example is to introduce stochastic forcing into an otherwise deterministic system.

We first construct a comparison theorem between HJB solutions to systems that share the same general dynamics, but with differing noise covariance. This comparison allows for the approximated value function of one system to bound the value function for another, providing pointwise bounds, and indeed SCLFs, for those that do not satisfy (7).

Proposition 28. Suppose $V^{o^*}$ is the solution to the HJB equation (6) with noise covariances $\Sigma_a$, and $V^b$ is a supersolution to (6) with identical parameters except the noise covariance $\Sigma^b$ where $\Sigma_b - \Sigma_a \geq 0$, then $V^b \geq V^{o^*}$ for all $x \in \Omega$.

Proof. From (31) Def. 2.2, $V$ is a viscosity supersolution to the HJB equation (6) with noise covariance $\Sigma$ if it satisfies

$$0 \leq -q - (\nabla_x V)^T f + \frac{1}{2} (\nabla_x V)^T G R^{-1} G^T (\nabla_x V)$$

$$- \frac{1}{2} Tr ((\nabla_{xx} V) B \Sigma B^T).$$ (23)

Since $\Sigma_b - \Sigma_a \geq 0$ the following trace inequality holds,

$$Tr ((\nabla_{xx} V^a) B \Sigma_a B^T) \geq Tr ((\nabla_{xx} V^o) B \Sigma_o B^T).$$

Therefore, we have the inequality

$$0 \leq -q - (\nabla_x V^b)^T f + \frac{1}{2} (\nabla_x V^b)^T G R^{-1} G^T (\nabla_x V^b)$$

$$- \frac{1}{2} Tr ((\nabla_{xx} V^b) B \Sigma_b B^T)$$

$$\leq -q - (\nabla_x V^b)^T f + \frac{1}{2} (\nabla_x V^b)^T G R^{-1} G^T (\nabla_x V^b)$$

$$- \frac{1}{2} Tr ((\nabla_{xx} V^b) B \Sigma_a B^T)$$

which implies that $V^b$ is in fact a viscosity supersolution to the system with noise covariance $\Sigma^b$ (i.e., $V^b$ satisfies (23) for $\Sigma^b$). As $V^b$ is a supersolution to the system with parameter $\Sigma^o$, then $V^b \geq V^{o^*}$. \qed

A particular class of such approximations arises from a deterministic HJB solution, which is not linearly solvable, but is approximated by one that is linearly solvable. Consider a deterministic system of the form

$$dx_t = (f(x_t) + G(x_t)u_t) dt$$ (24)
with cost function
\[ J(x,u) = \phi(x_T) + \int_0^T q(x_t) + \frac{1}{2} u_t R u_t \, dt \] (25)
where \( \phi, q, R, f, G \), and the state and input domains are defined as in the stochastic problem in Section II-B. Then, the HJB equation is given by
\[ 0 = q + (\nabla_x V)^T f - \frac{1}{2} (\nabla_x V)^T G R^{-1} G^T (\nabla_x V) \] (26)
and the optimal control is given by \( u^* = -R^{-1} G^T \nabla_x V \).

**Corollary 29.** Let \( V^* \) be the value function that solves (26), and \( V^{**} \) be the upper bound solution obtained from (17) where all parameters are the same as (26) and \( \Sigma_t \) is not zero. Then, \( V^{**} \) is an upper bound for \( V^* \) over the domain (i.e., \( V^* \leq V^{**} \)).

**Proof.** A simple application of Proposition 28 where \( \Sigma_t \) takes the form of a zero matrix, gives \( V^* \leq V^{**} \).

Interestingly, using the solution from (17) and the transformation \( V_u = -\log \Psi_t \), the suboptimal controller \( u^* = -R^{-1} G^T \nabla_x V_u \) is a stabilizing controller for the deterministic system (24) if a simple condition is satisfied. This fact is shown using the Lyapunov theorem for deterministic systems introduced next (5).

**Definition 30.** Given the system (24) and cost function (25), a control Lyapunov function (CLF) is a proper positive definite function \( V \in C^1 \) on a compact domain \( \Omega \cup \{0\} \) such that
\[ V(0) = 0, \quad V(x) \geq \mu(x) \quad \forall x \in \Omega \setminus \{0\} \]
\[ \exists u(x) \text{ s.t. } (\nabla_x V)^T (f + Gu) \leq 0 \quad \forall x \in \Omega \setminus \{0\} \] (27)
where \( \mu \in K \).

**Theorem 31.** [5, Thm. 2.5] Given a system (24) and cost function (25), if there exists a CLF \( V \) and a \( u \) satisfying Definition 30 then the controlled system is stable, and \( u \) is a stabilizing controller. Furthermore, if \( (\nabla_x V)^T (f + Gu) < 0 \) for all \( x \in \Omega \setminus \{0\} \), the controlled system is asymptotically stable, and \( u \) is an asymptotically stabilizing controller.

Verifying that the controller \( u^* = -R^{-1} G^T \nabla_x V_u \) is in fact stabilizing and that \( V_u \) is a CLF may be seen as follows.

**Corollary 32.** Given the controller \( u^* = -R^{-1} G^T \nabla_x V_u \), if
\[ Tr \left( (\nabla_x V_u)^T B \Sigma_x B^T \right) \geq 0 \quad \forall x \in \Omega \setminus \{0\}, \]
then \( u^* \) is a stabilizing controller for (24). If
\[ Tr \left( (\nabla_x V_u)^T B \Sigma_x B^T \right) > 0 \quad \forall x \in \Omega \setminus \{0\}, \]
then \( u^* \) is an asymptotically stabilizing controller for (24).

**Proof.** Recall that from the proof of Theorem 24 all conditions in Definition 30 are satisfied by \( V_u \) except (27). To show that \( V_u \) satisfies (27), rearrange (6) to yield the following
\[ (\nabla_x V_u)^T (f + Gu^*) = (\nabla_x V_u)^T f - (\nabla_x V_u)^T G R^{-1} G^T (\nabla_x V_u) \leq -q - \frac{1}{2} (\nabla_x V_u)^T G R^{-1} G^T (\nabla_x V_u) \]
\[ -\frac{1}{2} Tr \left( (\nabla_x V_u)^T B \Sigma_x B^T \right). \]

Recall that \( q \) and \( R \) are positive definite. If \( Tr \left( (\nabla_x V_u)^T B \Sigma_x B^T \right) \geq 0 \) for all \( x \in \Omega \setminus \{0\} \), then \( (\nabla_x V_u)^T (f + Gu^*) \leq 0 \) implying that \( V_u^* \) is a CLF and \( u^* \) is a stabilizing controller by Theorem 31. Furthermore, if \( Tr \left( (\nabla_x V_u)^T B \Sigma_x B^T \right) > 0 \) for all \( x \in \Omega \setminus \{0\} \), \( u^* \) is an asymptotically stabilizing controller.

The trace condition in Corollary 32 is easily enforced in (17) by adding one extra constraint in the optimization problem. Thus, the optimization problem (17) can also produce a CLF for the corresponding deterministic system, with analytical results from the Section IV including a priori trajectory suboptimality bounds (Theorem 27), inherited as well.

**B. Robust Controller Synthesis**

The proposed technique may be extended to incorporate uncertainty in the problem data. Assume there exists unknown coefficients \( a \in \mathcal{H} \) in \( f(x), G(x), B(x) \), where \( \mathcal{H} \subset \mathbb{R}^k \), \( \mathcal{H} = \{ a \mid g_i(a) \geq 0, g_i(a) \in \mathbb{R}\}, i \in \{1, 2, \ldots, d\} \) is a basic closed semialgebraic set describing the uncertainty set of \( a \). The problem data is then defined by the expressions \( f(x,a), G(x,a), B(x,a) \) for \( x \in \Omega \), and \( a \in \mathcal{H} \). In this case, the uncertain parameters may be considered as additional domain variables, defined over their own compact space.

Uncertainty of this form may be incorporated naturally into the optimization problem (17). Define the monomial set \( \mathcal{X} = \{ a_\alpha x^\beta \}_{\alpha \in \mathbb{N}_0, \beta = 1, \ldots, m} \). The optimization variables corresponding to the polynomials in \( S \) and \( T \) are then constructed out of \( \mathcal{X} \) as
\[ p(x,a) = \sum_{\alpha=1}^n \sum_{\beta=1}^m c_{\alpha,\beta} a_\alpha x^\beta. \]
Note that \( \Psi_U \) and \( \Psi_T \) are not themselves functions of \( a \). The uncertainty set \( \mathcal{H} \) is incorporated by defining a compact domain \( \mathcal{M} = \Omega \times \mathcal{H} \), that takes the product of the original problem domain and the uncertainty set. The resulting optimization problem is therefore
\[ \min_{\Psi_U, \Psi_T, S, T} \epsilon \] (28)
s.t.
\[ -\frac{1}{\lambda} q \Psi_T + L(\Psi_T, a) - D(\mathcal{M}, S_1) \in S[x,a] \]
\[ -\frac{1}{\lambda} q \Psi_U - L(\Psi_U, a) - D(\mathcal{M}, S_2) \in S[x,a] \]
\[ - \epsilon - (\Psi_U - \Psi_T) - D(\mathcal{M}, S_3) \in S[x,a] \]
\[ B(\Psi_U - D(\mathcal{H}, S_4), \partial \Theta, T_1) \in S[x,a] \]
\[ B(\Psi_U - D(\mathcal{H}, S_5), \partial \Theta, T_2) \in S[x,a] \]
\[ B(\Psi_U - D(\mathcal{H}, S_5), \partial \Theta, T_3) \in S[x,a] \]
\[ B(\Psi_U - D(\mathcal{H}, S_5), \partial \Theta, T_3) \in S[x,a] \]
\[ \partial \Theta, \Psi_U - D(\Omega \cap \{ x^i \geq 0 \}, S_7) \in S[x,a] \]
\[ \partial \Theta, \Psi_U - D(\Omega \cap \{ -x^i \geq 0 \}, S_8) \in S[x,a] \]
\[ \Psi_T(0) - 1 - D(\mathcal{H}, S_9) \in S[a] \]
\[ - \Psi_T(0) + 1 - D(\mathcal{H}, S_{10}) \in S[a] \]
where \( S = \{ S_1, \ldots, S_{10} \} \), and \( T = \{ T_1, T_2, T_3 \} \). The operator \( L \) now depends on the variable \( a \). The resulting solutions to the optimizations (28), and the upper bound suboptimal value functions \( V_u = -\lambda \log \Psi_U \), are found for all \( a \in \mathcal{H} \), with
the control law \( u^* = -R^{-1}G^T \nabla_x V_u \) now stabilizing for the entirety of the uncertainty set \( \mathcal{H} \). Similar techniques have been studied previously for Lyapunov analysis, e.g., [34] Ch. 4.3.3.

C. Path Planning

Although our study up to this point has emphasized the use of the approximate solutions for stabilization, their use is more general. As studied in [23], the methods of this paper may also be used to construct controllers for path planning problems.

In a path planning problem, given a dynamical system of the form (1) with cost function (2), the goal is to move from a particular state to a goal state while minimizing the cost function (3). This problem is almost the same as the stabilization problem except the last three inequalities in (17) that ensure stability to the origin are omitted. Indeed, for general path planning problems the value function isn’t expected to have the Lyapunov function’s convex-like geometry. Unfortunately, without the aforementioned constraints, which provides strong guarantees upon trajectory behavior, the above results do not hold, such as Theorem 27 and other results guaranteeing trajectory convergence or trajectory suboptimality.

D. Non-Polynomial Systems

The development of this work has been limited to nonlinear systems governed by polynomial functions. A number of avenues exist for incorporating non-polynomial nonlinearities. The most straightforward approach is to simply project the approximate solutions for stabilization, their use is more limited to nonlinearities to a polynomial basis. As polynomials are universal approximators in non-polynomial functions to a polynomial basis. As polynomials are universal approximators in non-polynomial functions to a polynomial basis. As polynomials are universal approximators in non-polynomial functions to a polynomial basis. As polynomials are universal approximators in non-polynomial functions to a polynomial basis.

VI. NUMERIC EXAMPLES

This section studies the computational characteristics of this method using two examples – a scalar system and a two-dimensional system. In the following problems, the optimization parser YALMIP [35] was used in conjunction with the semidefinite optimization package MOSEK [36]. In both examples, the continuous system is integrated numerically using Euler integration with step size of 0.005s during simulations.

A. Scalar Unstable System

Consider the following scalar unstable nonlinear system

\[
\dot{x} = \left( x^3 + 5x^2 + x + u \right) \, dt + d\omega
\]

(29)

on the domain \( x \in \Omega = \{ x \mid -1 \leq x \leq 1 \} \). The noise model considered is Gaussian white noise with zero mean and variance \( \Sigma = 1 \). The goal is to stabilize the system at the origin. We choose the boundary at two ends of the domain to be \( \Psi(-1) = 20e^{-10} \) and \( \Psi(1) = 20e^{-10} \). At the origin, the boundary is set as \( \Psi(0) = 1 \). We set \( q = x^2 \), and \( R = 1 \). In the one dimensional case, the origin, which is a boundary, divides the domain into two partitions, \( x \leq 0 \) and \( x \geq 0 \). Because of the natural division of the domain, the solutions for both domains can be represented by smooth polynomial respectively, and solved independently. The simulation is terminated when the trajectories enter the interval \([-0.005, 0.005]\) centered on the origin.

![Fig. 1. The desirability function of system (29) for varying polynomial degree. The true solution is the black curve.](image)

![Fig. 2. Computational results of system (29). (a) Convergence of the objective function of (17) as the degree of polynomial increases. The approximation error for \( x \leq 0 \) is denoted as \( \epsilon_l \) and the approximation error for \( x \geq 0 \) is denoted as \( \epsilon_r \). (b) Sample trajectories using controller computed from optimization problem (17) with different polynomial degrees starting from six randomly chosen initial points. (c) The comparison between \( J_u \) and \( V_u \) for different polynomial degrees whereby \( J_u \) is the expected cost and \( V_u \) is the value function computed from optimization problem (17). The initial condition is fixed at \( x_0 = -0.5 \).](image)
The desirability functions that result from solving \cite{17} for varying polynomial degrees are shown in Figure 1. The true solution is computed by solving the HJB directly in Mathematica \cite{37}. The kink at the origin is expected because the HJB PDE solution is not necessarily smooth at the boundary, and in this instance the origin is a zero-cost boundary.

The approximation error $\epsilon$ for both partitions is shown in Figure 2(a) for increasing polynomial degree. As seen in the plots, the approximation improves as the polynomial degree increases. Polynomial degrees below 14 are not feasible, hence this data is absent in the plots. The suboptimal solution converges faster for $x > 0$ than for $x < 0$ when the degree of polynomial increases because the true solution for $x > 0$ has a simple quadratic-like shape that can be easily represented as a low degree SOS function.

Figure 2(b) shows sample trajectories using the controller computed from optimization problem (17) for different polynomial degrees. The controllers are stabilizing for six randomly chosen initial points. Unsurprisingly, the suboptimal solutions with low pointwise error result in the system converging towards the origin faster.

To compare between $J_\nu$ and $V_\nu$, a Monte Carlo experiment is illustrated in Figure 2(c). For each polynomial degree that is feasible, the controller obtained from $\Psi_\nu$ in optimization problem (17) is implemented in 30 simulations of the system subject to random samples of Gaussian white noise with $\Sigma_x = 1$. The initial condition is fixed at $x_0 = -0.5$. In the figure, $V^u \geq J^u$ as expected, and the difference between the two decreases with increasing $d$.

**B. Two Dimensional System**

In the following example, we demonstrate the power of this technique on a 2-dimensional system. Consider a nonlinear 2-dimensional problem example with following dynamics:

$$
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{bmatrix} = \left(2 \begin{bmatrix} x^3 - x^3 - x + xy \\
y^3 - y^3 - y + yx 
\end{bmatrix} + \begin{bmatrix} x u_1 \\
y u_2 
\end{bmatrix} \right) dt + \begin{bmatrix} x d\omega_1 \\
y d\omega_2 
\end{bmatrix}. 
$$

(30)

The goal is to reach the origin at the boundary of the domain $\Omega = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$. The control penalty is $R = I_{2\times2}$, and state cost is $q(x) = x^2 + y^2$. The boundary conditions for the sides at $x = 1$, $x = -1$, $y = 1$, and $y = -1$ are set to $\phi(x, y) = 5$, while at the origin, the boundary has cost $\phi(0, 0) = 0$. The noise model considered is Gaussian white noise with zero mean and an identity covariance matrix.

The approximated desirability functions and their corresponding value functions are shown in Figure 3 for different degrees of polynomial. The solutions are shown for half of the domain $x \in [0, 1]$ in order to get a view of the gaps between the upper and lower bound solutions. When the polynomial degree is 20, the upper and lower bound solutions are numerically identical in many regions. Figure 4(a) shows the convergence of the objective function of optimization problem (17) as the degree of polynomial increases. There is no data below degree of 10 because the optimization problem is not feasible in these cases. As shown in Figure 4(b), sample trajectories starting from six different initial points shows that...
the controllers computed from $\Psi_I$ for various degrees arrive at the origin. The trajectory is considered at the origin if it is within a distance of 0.01 from the origin.

Similar to the scalar example, a Monte Carlo experiment is performed to compare between $J_u$ and $V_u$. For each polynomial degree that is feasible, the controller obtained from $\Psi_I$ in optimization problem (17) is implemented in 30 simulations of the system subject to random samples of Gaussian white noise with $\Sigma_u = I_{2 \times 2}$. The initial condition is fixed at $x_0 = (0.5, 0.5)$. Figure 4(c) shows the comparison between $J_u$ and $V_u$ for different polynomial degrees whereby $J_u$ is the expected cost and $V_u$ is the value function computed from $\Psi_I$ in optimization problem (17). As expected, $V_u \geq J_u$.

VII. CONCLUSION

This paper has proposed a new method to solve the linear HJB of an optimal control problem for stochastic nonlinear systems via SOS programming. Analytical results provide guarantees on the suboptimality of trajectories when using the approximate solutions for controller design. Consequently, one can synthesize a suboptimal stabilizing controller to stochastic nonlinear dynamical systems.

As is commonly seen when using SOS programming, the numerics of the SDP may be cumbersome in practice. There are a number of avenues for future work aimed at improving the practical performance. First, the monomials of the polynomial approximation can be chosen strategically in order to decrease computation time while achieving high accuracy. A promising future direction is the synthesis of the work presented here with that of [38], wherein the curse of dimensionality is avoided via the strategic choice of basis functions. To improve the numerical conditioning of these optimization techniques, a domain partitioning technique is studied in [24], wherein the alternating direction method of multipliers is used to enable both parallelization and a solution representation that varies in resolution over the domain.

In addition, there exists a growing body of literature towards increasing the numeric stability and scalability of SOS techniques, for example [39] and [40]. The incorporation of these techniques into the present work is under investigation.

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