EFFECTS OF MAGNET NON-LINEARITIES ON BETATRON OSCILLATION FREQUENCIES FOR THE 300 BEV PROTON SYNCHROTRON

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I. Introduction

The purpose of this note is to study the effects on betatron frequencies -- in particular, their variation with particle energy -- due to non-linear magnet fields. The idea is to determine whether non-linearities can be introduced deliberately in such a way that \( \nu \), the number of betatron wavelengths in the circumference, is independent of particle energy for energies near the equilibrium energy. The result is that non-linearities of a magnitude such that the field index \( n \) varies by about 1 per cent per cm will suffice.

II. Equations of Motion

The differential equations describing horizontal and vertical betatron oscillations are

\[
\frac{d^2x}{ds^2} + \left[ k^2 \left( \frac{2p}{p} - 1 \right) + \frac{e}{p} \frac{dB}{dx} \right] x = -k \left( \frac{p}{p} - 1 \right) \\
\quad - \left( k^3 \frac{p}{p} + 2k \frac{e}{p} \frac{dB}{dx} + \frac{e}{2p} \frac{d^2B}{dx^2} \right) x^2 + \ldots. \tag{1}
\]

\[
\frac{d^2z}{ds^2} - \frac{e}{p} \frac{dB}{dx} z = (2k \frac{e}{p} \frac{dB}{dx} + \frac{e}{p} \frac{d^2B}{dx^2}) xz + \ldots. \tag{2}
\]

respectively. \( s \) measures distance along the equilibrium orbit, in a counter-clockwise direction when viewed from above. \( x \) and \( z \) are the departures from the equilibrium orbit, outward and downward, respectively. The magnetic field in the orbit plane is assumed to be

\[ B(x) = B + x \frac{dB}{dx} + \frac{1}{2} x^2 \left( \frac{d^2B}{dx^2} \right) + \ldots \]

in a downward (+ z) direction. Note that \( B, dB/dx \) and \( d^2B/dx^2 \) are constants, measured at \( x = 0 \). \( k \) denotes the curvature of the equilibrium
orbit, traversed by a particle of momentum $p_0$; $p$ denotes the actual particle momentum. Balancing magnetic and centrifugal forces gives

$$ B = \frac{p_0}{e} \quad (3) $$

We shall now summarize briefly the linear theory of oscillations about the equilibrium orbit$^1$; we set $p = p_0$ and drop the non-linear terms on the right of Eqs. (1) and (2). Both equations are then of the form

$$ \frac{d^2y}{ds^2} + K(s) y = 0 \quad (4) $$

In an alternating-gradient synchrotron $K(s)$ will be periodic with some period $L$, which may be the circumference $C$ or some fraction of $C$. The general solution of (4) may be written

$$ y(s) = A \ v(s) \ \cos \left[ \psi(s) + \delta \right] \quad (5) $$

where $A$ and $\delta$ are arbitrary constants, $v(s + L) = v(s)$, $\psi(s + L) = \psi(s) + \mu$ and

$$ \frac{d\psi}{ds} = \frac{1}{w^2} \quad (6) $$

$\mu$ is the betatron phase shift in $L$. The quantity $w^2(s)$ is generally called $\beta(s)$; according to Eq. (6), it represents the local betatron (reduced) wavelength.

The number of betatron oscillations in the machine circumference is

$$ n = \frac{N \mu}{2\pi} \quad (7) $$

$^1$ A good review of the theory is given by Courant and Snyder, Annals of Physics 3, 1 (1958).
where $N$ is the number of identical sections; $C = NL$.

Now consider a small perturbation $\delta(s)$ in $K(s)$; that is, consider instead of Eq. (4) the equation

$$\frac{d^2 y}{ds^2} + \left[ K(s) + \delta(s) \right] y = 0 \quad (8)$$

It can be shown\textsuperscript{1,2} that, to first order in $\delta$, the change in $\nu$ is given by

$$\Delta \nu = \frac{1}{l_m} \oint \delta(s) \beta(s) ds$$

where $\oint$ denotes an integral around the entire machine circumference.

### III Effects of Synchrotron Oscillations

Suppose now that the particle momentum $p$ differs slightly from the equilibrium momentum $p_0$. The equilibrium orbit is then shifted outward by an amount $X(s)$. We now write

$$x(s) = X(s) + \xi(s) \quad (9)$$

where $\xi$ describes the rapid horizontal betatron oscillations about the slowly varying orbit $X$. Substituting Eq. (9) into (1) and (2), we obtain the differential equations

$$\frac{d^2 \xi}{ds^2} + \left[ k^2 (1 - 2 \frac{\Delta p}{p_0}) + \frac{e}{p} \frac{dB}{dx} + 2X \left( k^3 \frac{p_0}{p} + 2k \frac{e}{p} \frac{dB}{dx} \right) \right] \xi = 0 \quad (10)$$

\textsuperscript{2) L. S. Smith, Internal Report LS-4, Brookhaven National Laboratory.}
where we have linearized in the small quantities \( \xi \) and \( z \).

Equations (10) and (11) are now of the form (8). The unperturbed 
\( K(s) \) for \( p = p_o \) are

\[
K = \begin{cases} 
  k^2 + \frac{e}{p_o} \frac{dB}{dx} & \text{(horizontal oscillations)} \\
  -\frac{e}{p_o} \frac{dB}{dx} & \text{(vertical oscillations)} 
\end{cases}
\]

so that the horizontal and vertical perturbations \( \delta(s) \) are

\[
\delta_h = -2k^2 \frac{\Delta p}{p_o} - \frac{\Delta p}{p_o} \frac{e}{p_o} \frac{dB}{dx} + 2X (k^3 + 2k \frac{e}{p_o} \frac{dB}{dx} + \frac{e}{2p_o} \frac{d^2B}{dx^2})
\]

\[
\delta_v = \frac{\Delta p}{p_o} \frac{e}{p_o} \frac{dB}{dx} - X (2k \frac{e}{p_o} \frac{dB}{dx} + \frac{e}{p_o} \frac{d^2B}{dx^2})
\]

It is conventional to define \( n = -\frac{1}{kB} \frac{dB}{dx} = -\frac{e}{p_o k^2} \frac{dB}{dx} \). This gives

\[
\delta_h = \frac{\Delta p}{p_o} (n - 2)k^2 + 2X \left[ k^5 (1 - 2n) + \frac{e}{2p_o} \frac{d^2B}{dx^2} \right]
\] \hspace{1cm} (12)

\[
\delta_v = -n k^2 \frac{\Delta p}{p_o} - X (-2n k^3 + \frac{e}{p_o} \frac{d^2B}{dx^2})
\] \hspace{1cm} (13)

Two simplifications can now be made. In the first place, \( |n| \gg 1 \) (for the 300 Bev machine, \( |n| \sim 10^4 \)). In the second place, we can use
the large value of $v$ ($\sim 40$) as follows: Smith$^2$ gives the approximation

$$x(s) \approx \frac{\Delta p}{p_0} \left\langle \frac{\beta^{3/2}}{} \right\rangle \beta^{1/2}(s)$$

(14)

where $\left\langle \cdot \right\rangle$ denotes an average over $s$ around the circumference. Therefore, omitting factors of order one,

$$x \sim \frac{\Delta p}{p_0} k \beta^2$$

From Eqs. (6) and (7),

$$\beta \sim \frac{c}{2\pi \nu} \frac{R}{\nu}$$

where $R$ is the machine radius $\frac{c}{2\pi}$. Therefore

$$x \sim \frac{\Delta p}{p_0} \frac{kdR^2}{\nu^2} \sim \frac{\Delta p}{p_0} \frac{R}{\nu^2} \sim \frac{\Delta p}{p_0} \frac{1}{kv^2}$$

(15)

The large value of $v$, coupled with (15), enables us to omit the $X k^3$ terms in Eqs. (12) and (13). Thus, (12) and (13) become

$$\delta_h = n k^2 \frac{\Delta p}{p_0} + \frac{k}{B} \frac{\partial^2 B}{\partial x^2} x$$

$$\delta_v = -n k^2 \frac{\Delta p}{p_0} - \frac{k}{B} \frac{\partial^2 B}{\partial x^2} x$$

We made use of Eq. (3) to simplify the last term in each.

The change in effective gradient length with orbit position produces a significant shift in $v$ for the Brookhaven AGS, and we shall therefore include this effect. According to Smith$^2$, this contributes to $\left(\delta_v \delta_h \right)$ a term
where \( \delta(s)_{\text{end}} \) consists of a delta function at each end of every magnet.

We may summarize by observing that there are three perturbing terms in \( \delta_h \) (and \( \delta_v \)), due to momentum error, magnet non-linearity, and variation in gradient length. Also, note that \( \delta_h = -\delta_v \) in this approximation.

IV Numerical Estimates

Since no definite magnet lattice has been arrived at for the 300 Bev machine, we have adopted some tentative and approximate numbers. We shall assume

\[
R = 1300 \text{ m} \\
k = (1000 \text{ m})^{-1} \\
n = \pm 10^4 \\
N = 30 \\
v = 40
\]

Each of the 30 "superperiods" contains 32 magnets, 16 radially focussing (F) and 16 radially defocussing (D). Each magnet is 7 meters long. The phase shift per unit length is

\[
\frac{2\pi v}{C} = \frac{v}{R} \approx 3 \times 10^{-2} \text{ m}^{-1}
\]

so that \( \beta \approx 33 \text{ m} \). We shall assume that \( \beta \) oscillates between 20 m and 50 m; more precisely, we assume
For vertical oscillations the situation is reversed; $\beta_v = 20$ m in F magnets and 50 m in D magnets.

For $X(s)$ we shall make the approximation (14). For our assumed lattice

\[
\langle k \beta^{3/2} \rangle \approx \frac{1}{1.3} \cdot \frac{1}{1000} \cdot \frac{20^{3/2} + 50^{3/2}}{2}
\]

\[
\approx 0.17 \, m^{1/2}
\]

Thus

\[
X(s) \approx \begin{cases} 
0.76 \frac{\Delta p}{p_0} \, m \, \text{in D magnets} \\
1.20 \frac{\Delta p}{p_0} \, m \, \text{in F magnets}
\end{cases}
\]

We shall now separately estimate the three effects referred to in the previous section:

(a) Variation in gradient length -

\[
\Delta \nu_h = \frac{1}{4\pi} \phi \beta_h(s) \delta_h(s) \, ds
\]

\[
= \frac{1}{8\pi} \left| \frac{dG}{dx} \right|^2 \left| \frac{dL_s}{dx} \right| \sum \Sigma X(s) \beta_h(s)
\]

A reasonable estimate for \left| \frac{dL_s}{dx} \right| seems to be 0.5. Also
\[ \sum_{\text{ends}} x(s) \beta_h(s) \approx 960 \left[ (0.76) \times (20) + (1.20) \times (50) \right] \frac{\Delta p}{p_o} \]

\[ \approx 7.2 \times 10^4 \frac{\Delta p}{p_o} \ m^2 \]

Therefore

\[ \Delta v_h \approx \frac{-1}{8\pi} \left( \frac{10^4}{10^6} \right) (0.5) (7.2 \times 10^4) \frac{\Delta p}{p_o} \]

\[ \approx -14.3 \frac{\Delta p}{p_o} \]

For vertical oscillations

\[ \Delta v_v = \frac{-1}{8\pi} \left| n \right| k^2 \left| \frac{dg}{dx} \right| \sum_{\text{ends}} x(s) \beta_v(s) \]

Estimating as before

\[ \sum_{\text{ends}} x(s) \beta_v(s) \approx 960 \left[ (0.76) \times (50) + (1.20) \times (20) \right] \frac{\Delta p}{p_o} \]

\[ \approx 6.0 \times 10^4 \frac{\Delta p}{p_o} \ m^2 \]

and

\[ \Delta v_v \approx +11.9 \frac{\Delta p}{p_o} \]

(b) Momentum error -

\[ \Delta v_h = \frac{1}{4\pi} \frac{\Delta p}{p_o} \int n k^2 \beta_h \, ds \]
\[
\phi n k^2 A \Delta h = \frac{10^h}{10^6} (480)(7)(20 - 50)
\]

\[\approx -1010\]

so that

\[\Delta \nu_h \approx -80 \frac{\Delta P}{P_0}\]

A similar calculation gives

\[\Delta \nu_v \approx -80 \frac{\Delta P}{P_0}\]

(c) Magnet non-linearity -

\[\Delta \nu_h = \frac{1}{4\pi} \int_0^1 \frac{k}{B} \frac{d^2 B}{dx^2} \times A \Delta h \, ds\]

Let us assume

\[\frac{1}{k_B} \frac{d^2 B}{dx^2} = \begin{cases} A & \text{in D magnets} \\ B & \text{in F magnets} \end{cases}\]

Then

\[\Delta \nu_h \approx \frac{(480)(7)}{4\pi \times 10^9} \frac{\Delta P}{P_0} \left[(0.76)(20)A + (1.20)(50)B\right]\]

\[\Delta \nu_h \approx (4.1a + 16.1b) \frac{\Delta P}{P_0}\]
where we have set
\[ a = 10^{-6} \, A, \quad b = 10^{-6} \, B \]

Similarly
\[ \Delta \nu_v \approx -\frac{(480)(7)}{4\pi \times 10^{-9}} \frac{\Delta p}{p_0} \left[(0.76)(50)A + (1.20)(20)B\right] \]

\[ \Delta \nu_v \approx (-10.2 \, a - 6.4 \, b) \frac{\Delta p}{p_0} \]

V Optimum Amount of Non-linearity

Collecting the numerical results of the previous section, we have
\[ \Delta \nu_h = (-14.3 - 80 + 4.1 \, a + 16.1 \, b) \frac{\Delta p}{p_0} \]

\[ \Delta \nu_v = (11.9 - 80 - 10.2 \, a - 6.4 \, b) \frac{\Delta p}{p_0} \]

We have exhibited explicitly the variation in gradient length effect, the momentum error effect, and the magnet non-linearity effects, in that order. Note that here, in contrast to the Brookhaven AGS, the first effect is quite small compared to the second.

If we choose
\[ a = 0.7 - 13.0 = -12.3 \]
\[ b = 0.7 + 8.3 = +9.0 \] (16)

the effects cancel and \( \nu_h \) and \( \nu_v \) are independent of \( \frac{\Delta p}{p_0} \). In Eq. (16) the gradient length and momentum error contributions are presented separately, in that order.
Note that
\[ \frac{\Delta n}{\Delta x} = -\frac{1}{kB} \frac{dB}{dx^2} \]
\[ = -kA \quad \text{or} \quad -kB \]
\[ = -10^3a \quad \text{or} \quad -10^3b \]

Thus the result is that
\[ n = \begin{cases} 
10,000 + 123x_{cm} & \text{in D magnets} \\
-10,000 - 90x_{cm} & \text{in F magnets}
\end{cases} \]

where \( x_{cm} \) is simply \( x \) expressed in centimeters.

These non-linearities are clearly achievable; the accuracy with which they can be maintained would appear to be a more serious problem.