An Energy-Based Model of Longitudinal Splitting in Unidirectional Fiber-Reinforced Composites

Fiber-reinforced composite materials are used in the form of laminates in numerous structural applications by taking advantage of their directional properties. Such applications are often limited by the compressive strength of the composite materials that are used. Failure modes in composite laminates are complex and are not always easily understood (e.g., [1,2]). On the other hand, unidirectional fiber-reinforced composites serve as excellent model materials for investigating the associated strength and failure issues. Unidirectional fiber-reinforced composites also have much lower compressive strength than their tensile strength for loading in the fiber direction. Therefore, the prediction of the compressive strength is a critical issue in designing composite materials and composite structures. Commonly observed failure modes in unidirectional composites under compression in the fiber direction include (i) longitudinal or axial splitting due to transverse cracking, (ii) fiber kinking (initiation and propagation of kink bands or microbuckles), and (iii) longitudinal splitting followed by fiber kinking (see for e.g., [2,3]). These failure modes are also observed under axial compression in the presence of lateral confinement. However, the mechanisms, which govern these failure modes in composites, are not completely understood. The effect of lateral confinement on compressive strength is an outstanding issue because of its relevance in developing and validating existing phenomenological failure models for composites (e.g., [4,5]). Also, in composite laminates, even under uniaxial compression, the stress state is multiaxial, and hence there is a need for models that can reliably predict their strength under multiaxial stress states. For the kinking mode of failure, a wide range of experimental, analytical, computational efforts have been undertaken (e.g., [2,3,6–9]). On the other hand, relatively little is known about longitudinal splitting due to transverse cracking. A number of researchers have observed an increase in the compressive strength with increasing lateral confinement (e.g., [10–12]). Further, from a materials design point of view, it is desirable to have models that can predict

1 Introduction

Fiber-reinforced composite materials are often observed to fail in a longitudinal splitting mode in the fiber direction under far-field compressive loading with weak lateral confinement. An energy-based model is developed based on the principle of minimum potential energy and the evaluation of effective properties to obtain an analytical approximation to the critical stress for longitudinal splitting. The analytic estimate for the compressive strength is used to illustrate its dependence on material properties, surface energy, fiber volume fraction, fiber diameter, and lateral confining pressure. The predictions of the model show good agreement with available experimental data.

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2 Energy-Based Model for Longitudinal Splitting

2.1 Problem Formulation. Consider a cylindrical specimen of an ideal unidirectional fiber-reinforced composite under lateral confining stress, \( \sigma_e \), and axial compressive stress, \( \sigma \), shown schematically in Fig. 1(a). Under this setting, compare two configurations shown in Fig. 1: (a) one is unsplit, and (b) the other is totally split in the fiber direction. Let the total potential energy density of unsplit and split specimen be \( \Pi_u \) and \( \Pi_s \), respectively. Comparing between \( \Pi_u \) and \( \Pi_s \) provides the critical axial stress for splitting under given lateral confining stress, \( \sigma_e \). The criterion for longitudinal splitting is the minimization of the total potential energy density of the specimen. In other words, when \( \Pi_s \) exceeds \( \Pi_u \), the specimen splits ([13]).

The total potential energy is computed in terms of the effective material properties as a function of the properties of fiber and matrix using the concept of representative volume element (RVE). Instead of considering the entire problem, an auxiliary problem is set up focusing on an element (RVE) which consists of a fiber surrounded by the matrix according to the volume fraction under the same strain or stress boundary condition as that of the original problem. If the specimen is macroscopically homogeneous, the average strain and stress over the RVE are the same as that of the entire specimen. In the problem under consideration, because of the random in-plane distribution of the fibers, the RVE reduces to a circular cylinder which consists of a single straight fiber of the specimen length surrounded with matrix according to the fiber volume fraction. The issues related to establishing RVEs in fiber-reinforced composites are well established (e.g., [16–18]).

2.2 Energy Criterion for Longitudinal Splitting

2.2.1 Total Potential Energy of Unsplit Specimen. The total potential energy density of the unsplit specimen, \( \Pi_u \), is the same as the elastic energy density. Hence, under stress (traction) boundary condition, \( \Pi_u \) is given as follows:

\[
\Pi_u = \frac{1}{V} \int \left( \frac{1}{2} \varepsilon(x) : C(x) : \varepsilon(x) - \sigma(x) : \varepsilon(x) \right) dx
\]

\[
= \frac{1}{V} \int \left( - \frac{1}{2} \sigma(x) : S(x) : \sigma(x) \right) dx = \frac{1}{2} \sigma : S^* : \sigma
\]

(2.1)

where \( V \) is the volume of the RVE, \( C(x) \) and \( S(x) \) are the fourth-order elasticity and compliance tensors at point \( x \), respectively, \( \varepsilon(x) \) is the strain field, \( \sigma(x) \) is the stress field, and \( \sigma \) is the volumetric average stress tensor over \( V \) which corresponds to the prescribed stress on the boundary of the specimen. \( S^* \) is the effective compliance tensor of the unsplit specimen.

Because of the unidirectional reinforcement of the fibers, the specimen is transversely isotropic. Besides, the cartesian coordinates, \( x_1, x_2, \) and \( x_3 \) directions are also the principal directions. Therefore, to evaluate \( \Pi_u \), we need only four independent effective moduli, namely, the longitudinal Young’s modulus, \( E^s_1 \), Poisson ratio, \( \nu^e_{21} \), the plane strain bulk modulus, \( K^e_{23} \) and the shear modulus, \( G^e_{23} \). Using the cylindrical RVE introduced before, effective elastic moduli of the unidirectional composite for random in-plane distribution of fibers, \( E^s_1, \nu^e_{21}, K^e_{23} \), and the upper and lower bounds for \( G^e_{23} \) have been obtained by Hashin and Rosen [16]. Since the lower bound corresponds to the macro-stress prescribed problem, the lower bound for \( G^e_{23} \) is used here.

The expressions for the moduli tensor and related elasticity constants are shown in the Appendix in terms of the elastic constants of the fiber and the matrix as well as their volume fractions.

The average stress-strain relation for the RVE is given as follows:

\[
\bar{\sigma}_{11} = C^s_{11} \bar{e}_{11} + C^s_{12} \bar{e}_{22} + C^s_{13} \bar{e}_{33}
\]

\[
\bar{\sigma}_{22} = C^s_{22} \bar{e}_{11} + C^s_{22} \bar{e}_{22} + C^s_{23} \bar{e}_{33}
\]

\[
\bar{\sigma}_{33} = C^s_{23} \bar{e}_{11} + C^s_{23} \bar{e}_{22} + C^s_{23} \bar{e}_{33}
\]

(2.2)

The prescribed stress boundary conditions are

\[
\bar{\sigma}_{11} = -\sigma - \bar{\sigma}_{22} = \bar{\sigma}_{33} = -\sigma_e \quad \bar{\sigma}_{12} = \bar{\sigma}_{13} = \bar{\sigma}_{23} = 0
\]

(2.3)

where \( \sigma \) and \( \sigma_e \) are the magnitudes of the axial stress and the lateral confinement. Compressive stress components are assumed to be negative. The total potential energy density for the unsplit specimen, \( \Pi_u \), is a quadratic form of the compressive stress, \( \sigma \),

\[
\Pi_u = \frac{1}{2} \left[ \begin{array}{c} \sigma \\ \sigma_e \\ \sigma_e \end{array} \right] \left[ \begin{array}{ccc} C^s_{11} & C^s_{12} & C^s_{13} \\ C^s_{12} & C^s_{22} & C^s_{23} \\ C^s_{13} & C^s_{23} & C^s_{33} \end{array} \right] \left[ \begin{array}{c} \sigma \\ \sigma_e \\ \sigma_e \end{array} \right] - \frac{1}{2} \sigma \sigma_e
\]

(2.4)

2.2.2 Total Potential Energy of Split Specimen. Under the same boundary condition as that of the unsplit specimen (2.3) and assuming that each RVE splits at the boundary of the matrix and the fiber, i.e., the split is caused by an interfacial crack (delamination), the split RVE can be regarded as two columns, consisting of either the fiber or the matrix. Such a simplifying assumption enables gaining insights into the strength of composites. The elastic energy density of the RVE after splitting, \( E_s \), is given by

\[
E_s = \frac{1}{V} \int \left( -\frac{1}{2} \sigma(x) : S^*(x) : \sigma(x) \right) dx
\]

\[
= \frac{1}{2} \sigma : S^* : \sigma
\]

(2.5)

where \( S^* \) is the effective compliance tensor of the split specimen, \( \sigma_f, \sigma_m \) are volume fractions of fiber and matrix, respectively. The matrix volume fraction \( \sigma_m \) is assumed throughout to be \( (1-v_f) \).

The fiber and the matrix are assumed to be isotropic and the compliance tensor of fiber and matrix, \( S_f, S_m \), can be expressed in terms of their respective Young’s moduli \( (E^s_1, E^s_2) \) and Poisson’s ratios \( (\nu^f, \nu^m) \). Therefore, the elastic energy density for the split specimen, \( E_s \), is given as a quadratic form of the axial compressive stress, \( \sigma \)
where $P_1$, $P_2$, and $P_3$ are expressed in terms of the elastic constants of the materials

$$P_1 = \frac{1}{2} \left( \frac{v_f}{E_f} + \frac{v_m}{E_m} - \frac{1}{E_1^c} \right), \quad P_2 = \frac{v_f}{E_f} + \frac{v_m}{E_m} - \frac{v_{21}^2}{E_1^c},$$

where $G_c = \frac{1}{2} \frac{v_f}{E_f} + \frac{v_m}{E_m} - \frac{1}{E_1^c}$. For a given confining pressure $\sigma_c$ and surface energy density $\gamma$, $\sigma_f \geq \sigma_c$, hence, $\sigma_c$ is taken as the critical stress $\sigma_c^*$. Letting $\sigma_c = 0$ in (3.1), the critical stress without confinement, i.e., the unconfined longitudinal compressive strength for the composite can be obtained:

$$\sigma_c^*|_{\sigma_c=0} = 2 \left( \frac{2 \gamma f}{a} \right)^{1/2} \left( \frac{1}{E_f} + \frac{1}{E_m} \right)^{-1/2}. \quad (3.2)$$

Equation (3.2) shows that unconfined strength is proportional to the square root of surface energy and inversely proportional to the square root of fiber diameter. This result indicates that for a given volume fraction, all other things remaining unchanged, composites with larger fiber diameter are more susceptible to axial splitting than smaller diameter fibers. Since $E_f > E_m$ in usual fiber-reinforced composites, $v_m/E_m > v_f/E_f$ and $E_1^c = E_m^c = E_1^c$ hold. Based on these evaluations, (3.2) can be simplified as follows:

$$\sigma_c^*|_{\sigma_c=0} = 2 \left( \frac{2 \gamma f}{a} \right)^{1/2} \left( \frac{1}{E_f} + \frac{1}{E_m} \right)^{-1/2} \quad (3.3)$$

Examining the quadratic form of the energy surface, $\Phi(\sigma, \sigma_c) = \Pi_a - \Pi_f$, for a constant surface energy density $\gamma$, and assuming that the longitudinal (fiber direction) compliance is smaller than the lateral (transverse) compliance in the composite (typical for most fiber reinforced composites) the following inequality holds:

$$\frac{d \sigma_c^*}{d \sigma_c} < 1 \quad (3.4)$$

subject to the constraints

$$\sigma_f > \sigma_c \quad (3.5a)$$

and

$$d \Phi = \frac{\partial \Phi}{\partial \sigma} d \sigma + \frac{\partial \Phi}{\partial \sigma_c} d \sigma_c = 0. \quad (3.5b)$$

The first constraint (3.5a) corresponds to axial compression and the second constraint (3.5b) corresponds to the equi-potential line. From (3.4), one can conclude that if the splitting failure is governed by the principle of minimum total potential energy and the surface energy density $\gamma$ is a constant, the slope of the relationship between compressive strength and confining pressure, i.e., $\sigma_c^*$ versus $\sigma_c$, cannot exceed unity. Even if the surface energy density $\gamma$ is an increasing function of confining pressure $\sigma_c$, the inequality (3.4) holds at least for small $\sigma_c$. The effect of lateral confinement and material properties on the compressive strength of composites can be investigated by using (3.1).

### 3.2 Model Predictions

Examining the functional form shown in (3.1) and (3.2), important parameters for longitudinal splitting can be identified as $\gamma/a$, $v_f$, and $\sigma_c$. To investigate the dependence of compressive strength on each of these parameters and compare the effect of each parameter, parametric studies have been performed. In the present parametric study, two different types of commonly used fiber-reinforced composite are investigated to illustrate the dependence of compressive strength on material properties. These materials are a unidirectional E-glass/vinyl ester composite (indicated as “G/VE” in the figures) and a unidirectional carbon/epoxy composite (indicated as “C/ER” in the figures). Experimental data and material properties for these materials are available in the literature ([11,15]). The relevant...
material properties including those of the fiber and the matrix as well as the radius of the fibers for these composites are shown in Table 1. Surface energy density $\gamma$'s shown in Table 1 are obtained by calibration to the corresponding experimental data for unconfined compressive strength. Figure 2 shows the compressive strength of two types of composite for different $\gamma/a$ and $\sigma_c$ (0 and 100 MPa) with fixed fiber volume fraction $v_f = 60$ percent. One can observe a strong dependence of compressive strength on $\gamma/a$ (proportional to $\sqrt{\gamma/a}$) and relatively weak dependence on $\sigma_c$. Also, the compressive strength seems to be almost insensitive to the choice of the material for a given value of $\gamma/a$. Small values of $\gamma/a$ correspond to low interfacial energy (weak interface) and/or large diameter fibers, whereas large values of $\gamma/a$ correspond to large interfacial energy (tough interface) and/or small diameter fibers. The unconfined compressive strengths of E-glass/vinylester composite and carbon/epoxy composite with $v_f = 60$ percent are 667 MPa [15] and 1.5 GPa [11], respectively. Based on experimental observations, if the $\sigma_c$ is identical, the carbon/epoxy composite appears to be stronger than the E-glass/vinylester composite. However, the strong dependence on $\gamma/a$ plays a significant role here. Suppose $\gamma$ is of the same order for both composites, fiber radii $a$ for E-glass/vinylester composite and carbon/epoxy composite are 12.1 $\mu$m and 3.4 $\mu$m, respectively (see Table 1). This results in $\gamma/a$ for the carbon/epoxy composite to be approximately four times as that of the E-glass/vinylester composite. Figure 3 shows unconfined compressive strength (i.e., $\sigma_c = 0$) as a function of $\gamma/a$ and $v_f$. For a given $\gamma/a$, effect of $v_f$ on compressive strength is much stronger than that of the material properties. This observation together with the insensitivity of the strength to the choice of the material observed in Fig. 2 has the following implication. The compressive strength of the unidirectional fiber-reinforced composite is relatively insensitive to the magnitude of the material properties of each constituent, i.e., fiber and matrix. Instead, the degree of anisotropy introduced by combining the materials with different material properties is an important factor in the determination of compressive strength. Longitudinal splitting can be considered to be the process in which excessive stored elastic energy due to the heterogeneity and anisotropy can be released through the formation of new surfaces. The importance of anisotropy has been evidenced in this parametric study.

Compressive strength for different $v_f$ and $\sigma_c$ with fixed $\gamma/a$ is shown in Fig. 4. Based on experimental observations, $\gamma/a = 1.32 \times 10^7$ J/m$^3$ and $\gamma/a = 4.17 \times 10^7$ J/m$^3$ are used for E-glass/vinylester and carbon/epoxy, respectively, as the best fitting values for the model prediction of their unconfined compressive strength ([11,15]). It is again seen that if the same values for $\gamma/a$ were used, the compressive strength for both materials are close to each other as expected from previously shown parametric studies (Figs. 2 and 3). In this case, the difference between the results for two different levels of confinement $\sigma_c = 0$ MPa $\sigma_c = 100$ MPa is small and nearly constant for all values of $v_f$ shown here. This shows that the effect of $\sigma_c$ on compressive strength is much weaker than that of $v_f$ and is relatively insensitive for a given $v_f$.

### 3.3 Comparison With Experiments

To verify the validity of the energy-based model for longitudinal splitting, the compressive strengths predicted by the present model are compared with the experimental results obtained for E-glass/vinylester and carbon/epoxy composites. Uniaxial compression tests on unidirectional fiber-reinforced E-glass/vinylester composite with different fiber volume fraction ranging from 0 percent to 60 percent were performed by Waas et al. [15]. For carbon/epoxy composites, compression tests on unidirectional fiber-reinforced composites under superposed hydrostatic confinement have been performed.

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**Table 1** Material properties of fiber and matrix and geometry of fiber

<table>
<thead>
<tr>
<th>Fiber</th>
<th>E$_f$ (GPa)</th>
<th>$\nu_f^2$</th>
<th>$\nu_f$</th>
<th>$\alpha$ (µm)</th>
<th>E$_m$ (GPa)</th>
<th>$\nu_m$</th>
<th>$\gamma$ (J/m$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-Glass/Vinylester</td>
<td>72.4$^{(a)}$</td>
<td>0.2</td>
<td>0.1-0.6</td>
<td>12.1$^{(a)}$</td>
<td>3.69$^{(a)}$</td>
<td>0.38$^{(a)}$</td>
<td>110,210</td>
</tr>
<tr>
<td>Carbon/Epoxy</td>
<td>260$^{(b)}$</td>
<td>0.2</td>
<td>0.36</td>
<td>3.4$^{(b)}$</td>
<td>1.63$^{(b)}$</td>
<td>0.34$^{(b)}$</td>
<td>140</td>
</tr>
<tr>
<td>Carbon/Epoxy</td>
<td>234$^{(b)}$</td>
<td>0.2</td>
<td>0.6</td>
<td>3.4$^{(b)}$</td>
<td>4.28$^{(b)}$</td>
<td>0.34$^{(b)}$</td>
<td>140</td>
</tr>
</tbody>
</table>

$^{(a)}$Waas et al. [15]; $^{(b)}$Perry and Wronski [11]; $^{(c)}$Parry and Wronski [11]; $^{(d)}$assumed

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![Fig. 2 Effect of surface energy and lateral confinement on compressive strength (G/VE stands for E-Glass/vinylester and C/ER stands for carbon/epoxy)](image1)

![Fig. 3 Effect of surface energy and fiber volume fraction on unconfined compressive strength ($\sigma_c=0$) (G/VE stands for E-Glass/vinylester and C/ER stands for carbon/epoxy)](image2)
by Weaver and Williams [10] and Parry and Wrons \[11\]. The input parameters for the model prediction including material properties, fiber radius, and surface energy of the material used in their experiments have been shown in Table 1.

Comparison between the model prediction and experimental results by Waas et al. \[15\] provides the measure of the validity of the present model with respect to changing \(v_f\). Experimental results for the unconfined compressive strength from Waas et al. \[15\] are shown in Fig. 5. Examining the trend in compressive strength, one can observe a dip between \(v_f = 30\%\) and \(v_f = 40\%\). Based on this observation, analysis is performed for two groups of data sets. One is for low \(v_f\), i.e., \(v_f \leq 30\%\), the other is for high \(v_f\), i.e., \(v_f \geq 40\%\). Only the difference in these analyses is the input parameter for the surface energy \(\gamma\). The values of the surface energy which enable the model predictions to show good agreement with experimental results are \(\gamma = 210\, \text{J/m}^2\) for the low \(v_f\) data set and \(\gamma = 110\, \text{J/m}^2\) for the high \(v_f\) data set. In the present model, \(\gamma\) has been assumed to be the surface energy associated with delamination between the fiber and the matrix. The surface energy associated with the creation of new surfaces in the matrix has been neglected. In the case of high \(v_f\), surface energy associated with matrix failure is negligible since the average distance between fibers is small and the area of the surface created by matrix failure is much smaller than the one created by interface (fiber-matrix) debonding. On the other hand, as the fiber volume fraction decreases, the average distance between fibers increases and the surface energy associated with matrix failure becomes no longer negligible, which results in the increase of total surface energy. Also, the nonlinearity of the matrix for vinylester \([15]\) which is important at low volume fractions of the fiber has been neglected in the present analysis. The increase in surface energy associated with matrix failure is consistent with the requirement for larger surface energy \(\gamma\) for lower \(v_f\). Further work towards quantification of fracture energies as a function of volume fraction in fiber reinforced composites is needed. The model predictions for the matrix-dominated region and the fiber-interface dominated region can be regarded, respectively, as upper and lower bound for compressive strength of the composite.

The experimental result shows considerable scatter for \(v_f \geq 40\%\). In general, the interfacial toughness is highly dependent on local conditions such as size/orientation of initial imperfection, mode mixity, and bonding (interface strength and toughness). As a result, the interface properties vary more than the material properties of each constituent of composite, i.e., fiber and matrix. The fracture energy of fiber-reinforced composites \(\left(G_c\right)\) depends strongly on the local mode mixity \([19]\). Therefore, for the case of low \(v_f\), the scatter in compressive strength is relatively small since the matrix plays a significant role in determining the surface energy associated with splitting. On the other hand, since the surface energy associated with fiber/matrix debonding is dominant for high \(v_f\), the local interfacial conditions play a significant role in determining the compressive strength. This results in a large scatter of the compressive strength for composites with high \(v_f\) as seen from the experimental results in Fig. 5.

Comparison between the model prediction and experimental results by Weaver and Williams \[10\] (WW) and Parry and Wronski \[11\] (PW) provides a measure of the validity of the present model with respect to the confining pressure, \(\sigma_c\). To the best knowledge of the authors, WW and PW are the most widely accepted reliable experimental data regarding compressive failure of unidirectional fiber-reinforced composites under superposed hydrostatic confinement including detailed discussion on failure modes. Although some specimen geometry dependence of failure mode is reported in PW and short specimens used in WW show end effect, their experiments are convincing enough to regard longitudinal splitting as the dominant failure mode under weak lateral confinement. The critical stress \(\sigma^*\) is plotted against the confining pressure \(\sigma_c\) in Fig. 6 (WW for \(0 \leq \sigma_c \leq 150\, \text{MPa}\)) and in Fig. 7 (PW for \(0 \leq \sigma_c \leq 300\, \text{MPa}\)). In the experiments by PW, for higher confining pressure \((\sigma_c > 150\, \text{MPa})\), the slope of \(\sigma_c\) versus \(\sigma^*\) graph is steeper than those for lower confining pressure as seen in Fig. 7.
(iii) The effect of confining pressure on compressive strength is relatively weak.

The model prediction has been compared with the experimental results and showed good agreement. This agreement supports the validity of the present method for the analysis of longitudinal splitting (delamination failure) in unidirectional fiber-reinforced composites.

The assumption of a constant $\gamma$ would predict longitudinal splitting at all levels of confinement and with markedly lower strength than experimentally observed ones at high confining pressures. Beyond certain confining pressure, longitudinal splitting is completely suppressed and the failure mode translates to kink banding ([10–12]). In order to illustrate the effect of increasing fracture surface energy $\gamma$ with increasing pressure, $\gamma$ is assumed to depend on $\sigma_c$ as follows:

$$\gamma = \gamma_0 \left[ 1 + \alpha_n \left( \frac{\sigma_c}{\sigma_0} \right)^n \right]$$  \hspace{1cm} (4.1)

where $\gamma_0$ is surface energy for $\sigma_c = 0$, $\sigma_0^n$ is the unconfined compressive strength, $n$ is the confining pressure hardening exponent, and $\alpha_n$ is a positive dimensionless parameter corresponding to the exponent $n$. For $n \neq 0$ in (4.1), $\gamma$ increases as $\sigma_c$ increases and this results in nonlinear dependence of model prediction of compressive strength on $\sigma_c$. In this case, the inequality (3.4) for the slope of $\sigma^*$ versus $\sigma_c$ being less than unity holds at least for small $\sigma_c$.

The dependence of $\gamma$ on $\sigma_c$ (4.1) can be viewed to reflect the increase in the energy release rate $G_c$ as the local mode-mixity for interface cracking changes from mostly mode I to mode II ([19]) with increasing confinement.

The model predictions of compressive strength for the carbon/epoxy composite used by Parry and Wronski [11] for the cases $n = 2$ and $n = 4$ in (4.1) are shown in Fig. 7. Input parameters for the model predictions are $\gamma_0 = 140 \text{ J/m}^2$, $\sigma_0^n = 1.5 \text{ GPa}$, $\alpha_2 = 15.58$, and $\alpha_4 = 823.6$. Comparison between the cases of $n = 2$ and $n = 4$ shows that as the exponent $n$ increases, the curvature of the failure envelope can be increased and as a result, the model prediction for longitudinal splitting stays close to experimental result in wider range of confinement than the prediction based on smaller $n$ and exceeds the experimental value at high confining pressures where formation of kink bands, instead of longitudinal splitting, is observed in experiments. This observation implies that if $\gamma$ increases as a function of $\sigma_c$ and its dependence on $\sigma_c$ is strong, i.e., exponent $n$ is large, longitudinal or axial splitting can be observed up to certain levels of confinement and is suppressed at high levels of confinement where other failure modes such as kink band formation should be considered.

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Appendix

Following Hashin and Rosen [16], the expression for the effective moduli of the unidirectional fiber composite ($x_1$-fiber direction) $E_1^e$, $v_2^e$, $K_{23}^e$, and $G_{23}^e$ are given below:

$$E_1^e = \frac{(1-v_2^m)E_m(D_1-D_2F_1) + E_f(D_2-D_4F_2)}{E_m(D_1-D_2) + E_f(D_2-D_4)},$$  

$$v_2^e = \frac{v_f E_f L_1 + v_m E_m L_2 v_m}{v_f E_f L_3 + v_m E_m L_2},$$  

$$K_{23}^e = K_m \left( \frac{1+2v_m v_f}{K_m + K_v m(v_f + 2v_m)} \right);$$  

$$G_{23}^e = \frac{K_m (1+2v_m v_f)}{K_m + K_v (v_f + 2v_m)}.$$  

4 Conclusions and Discussion

An energy-based model has been developed for predicting the compressive strength of unidirectional fiber-reinforced composites which fail by longitudinal (axial) splitting. The following conclusions are based on the analytic results (3.1) and (3.2):

(i) The critical stress for longitudinal splitting is proportional to $\sqrt{\gamma_0}$ and this parameter is the most dominant term in the determination of the compressive strength of fiber-reinforced composites. According to the present model, composites with larger fracture energy and small fiber diameters would result in higher strength.

(ii) The degree of the anisotropy plays a significant role and the effect of fiber volume fraction appears only in this context in influencing the compressive strength.
\[ G_{23} = G_m \left[ 1 + \frac{2(1 - \nu_m)}{1 - 2\nu_m} u_f A_4 \right] \] (lower bound)

where \( D_1 = 1 - \nu_f \), \( D_2 = (1 + \nu_f)/(1 + \nu_m + \nu_m) \), \( D_4 = 2\nu_m^2 u_f/\nu_m \), \( D_5 = 2\nu_m^2 u_f/\nu_m \), \( F_1 = \frac{\nu_m u_f^n}{\nu_f u_f^n E_m} \), \( F_2 = \frac{\nu_f}{\nu_m} F_1 \), \( L_1 = 2\nu_f(1 - \nu_m^2)u_f + \nu_m(1 + \nu_m)u_m \), \( L_2 = \nu_f(1 - \nu_f - 2\nu_f^2) \), \( L_3 = 2(1 - \nu_m^2)u_f + (1 + \nu_m)u_m \), \( A_4 = 2(G_f - G_m)(2\nu_m - 1)[G_m(4\nu_f - 3)(u_f^2 - 1) - G_f((4\nu_m - 3)u_f^2 - 1)][G_m(4\nu_f - 3)(u_f^2 - 1) - 2G_fG_m(-5 + 6\nu_m - 4\nu_f + 6u_f^2 - 4\nu_m\nu_f^3 + (3 - 2\nu_m)\nu_f^4 + 2\nu_f(3 - 4\nu_m + 4\nu_m - 6u_f^2 + 4\nu_m\nu_f^3 - u_f^4) + G_f^2(3 + 4\nu_f - 6u_f^2 + 4\nu_f^3)(3 - 6\nu_m + 4\nu_m^2) + (3 - 4\nu_m)u_f^2 - \nu_m^2) \]

\( E_f \), \( \nu_f \), \( u_f \) and \( E_m \), \( \nu_m \), \( u_m \) are the Young’s moduli, Poisson’s ratios, and the volume fractions of the fiber and the matrix, respectively.

The elastic moduli \( C^y_{11}, C^y_{22}, C^y_{23} \) are expressed using \( E_1^y \), \( \nu_1^y \), \( K_{23}^y \), and \( G_{23}^y \) given above:

\[ C_{11}^y = E_1^y + 4\nu_1^y\nu_2^y K_{23}^y \]
\[ C_{22}^y = 2\nu_2^y K_{23}^y \]
\[ C_{23}^y = K_{23}^y + G_{23}^y \]
\[ C_{23}^y = K_{23}^y - G_{23}^y \]

References