Supplemental Material for
“Unity-Efficiency Parametric Down-Conversion via Amplitude Amplification”

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I. QUANTUM THEORY OF PARAMETRIC DOWN-CONVERSION

In this section we present analytic results for the quantum dynamics of parametric down-conversion with single-mode signal, idler, and pump beams whose joint state is evolving in the \( n \)-pump-photon subspace where \( n \leq 4 \). We start from the three-wave-mixing interaction Hamiltonian

\[
\hat{H} = i\hbar\kappa \left( \hat{a}_s^\dagger \hat{a}_i \hat{a}_p - \hat{a}_p^\dagger \hat{a}_s \hat{a}_i \right),
\]

where \( \kappa \) is assumed to be real valued, and seek solutions to the Schrödinger equation

\[
\hat{H}|\Psi(t)\rangle = \hbar |\Psi(t)\rangle, \quad \text{for } t \geq 0,
\]

subject to the initial condition \( |\Psi(0)\rangle \) on the joint state of the signal, idler, and pump. In general, this initial condition can be decomposed into components that lie within subspaces spanned by \( \{ |0,0,n\rangle, |1,1,n-1\rangle, \ldots, |n,n,0\rangle \} \) where \( |n_s,n_i,n_p\rangle \) denotes a state containing \( n_s \) signal photons, \( n_i \) idler photons, and \( n_p \) pump photons, i.e.,

\[
|\Psi(0)\rangle = \sum_{n=0}^{\infty} c_n |\Psi_n(0)\rangle, \quad \text{where } \sum_{n=0}^{\infty} |c_n|^2 = 1,
\]

and

\[
|\Psi_n(0)\rangle = \sum_{k=0}^{n} f_k^{(n)}(0)|k,k,n-k\rangle, \quad \text{with } \sum_{k=0}^{n} |f_k^{(n)}(0)|^2 = 1.
\]

Schrödinger evolution occurs independently within each of these \( n \)-pump-photon subspaces according to the following coupled ordinary differential equations:

\[
f_k^{(n)}(t) = \begin{cases} 
-\kappa \sqrt{n} f_{k+1}^{(n)}(t), & k = 0 \\
\kappa \left[ k \sqrt{n-k+1} f_{k-1}^{(n)}(t) - (k+1) \sqrt{n-k} f_{k+1}^{(n)}(t) \right], & k = 1, 2, \ldots, n-1 \\
\kappa n f_{n-1}^{(n)}(t), & k = n.
\end{cases}
\]

Equations (5) have closed-form solutions for \( n \leq 4 \), given \( \{ f_k^{(n)}(0) : 0 \leq k \leq n \} \), but the Abel-Ruffini theorem tells us that no such analytic solutions are possible for \( n \geq 5 \). In the remainder of this section we explore the implications of the closed-form solutions with respect to the efficiency of converting pump photons to signal-idler photon pairs. We assume that the \( \{ f_k^{(n)}(0) \} \) are real valued, so that the \( \{ f_k^{(n)}(t) \} \) are also real valued. (Inasmuch as our principal interest is in the initial condition \( f_k^{(n)}(0) = \delta_{k0} \), where \( \delta_{k0} \) is the Kronecker delta, there is little loss of generality in making the real-valued assumption.)

For the one-pump-photon subspace, Eqs. (5) imply that

\[
f_0^{(1)}(t) = f_0^{(1)}(0) \cos(\kappa t),
\]

\[
f_1^{(1)}(t) = f_1^{(1)}(0) \sin(\kappa t).
\]
FIG. 1: Trajectories of \( \{ f_k^{(2)}(t) \} \) obtained from Eqs. (8) for \( \phi_0 = 0 \) with \( m = 0 \) (orange), 0.4 (green), 1 (blue), 2 (red), and 4 (purple). These trajectories form circles on the unity-radius sphere that is centered at the origin, \( ( f_0^{(2)}, f_1^{(2)}, f_2^{(2)} ) = (0, 0, 0) \). The circles lie in planes that are perpendicular to the line \( f_1^{(2)} = \sqrt{2} f_2^{(2)} \) in the \( f_0^{(2)} = 0 \) plane.

It follows that single-mode down-conversion with a one-photon pump can achieve unity-efficiency conversion to a signal-idler photon pair, i.e., the initial state \( |0, 0, 1\rangle \) is completely converted to \( |1, 1, 0\rangle \) when \( t = \pi/2\kappa \). We shall see below that such unity-efficiency conversion is not possible with SPDC in the \( n \)-pump-photon subspaces for \( n = 2, 3, 4 \).

For the two-pump-photon subspace, Eqs. (5) yield the general solution

\[
\begin{align*}
    f_0^{(2)}(t) &= \frac{2}{3} \sqrt{\frac{3}{1 + 2m^2}} \left[ m + \frac{1}{2} \cos \left( \kappa \sqrt{6} t + \phi_0 \right) \right], \\
    f_1^{(2)}(t) &= \frac{1}{\sqrt{1 + 2m^2}} \sin \left( \kappa \sqrt{6} t + \phi_0 \right), \\
    f_2^{(2)}(t) &= \frac{\sqrt{6}}{3\sqrt{1 + 2m^2}} \left[ m - \cos \left( \kappa \sqrt{6} t + \phi_0 \right) \right],
\end{align*}
\]

where \( m \) and \( \phi_0 \) are determined by the initial conditions \( \{ f_k^{(2)}(0) : 0 \leq k \leq 2 \} \).

Figure 1 shows five \( \{ f_k^{(n)}(t) \} \) trajectories that were obtained from Eqs. (8) using the initial conditions \( m = 0, 0.4, 1, 2, \) and \( 4 \), all with \( \phi_0 = 0 \). The state evolution for \( f_k^{(n)}(0) = \delta_{k0} \), given by Eqs. (8) with \( m = 1 \) and \( \phi_0 = 0 \), leads to a maximum conversion efficiency

\[
\mu_2 = \max_t \sum_{k=1}^{2} k|f_k^{(2)}(t)|^2 \approx 0.89,
\]

with virtually all of the conversion being to the \( |2, 2, 0\rangle \) state, because \( |f_2^{(2)}(t_{\text{opt}})|^2 \gg |f_1^{(2)}(t_{\text{opt}})|^2 \), where \( t_{\text{opt}} \) is the interaction time that maximizes \( \mu_2 \).

Equations (5)’s solutions for the three-pump-photon subspace take the general form

\[
\begin{align*}
    f_0^{(3)}(t) &= [B_+ \cos(\omega_+ kt + \phi_1) + B_- \cos(\omega_- kt + \phi_2)], \\
    f_1^{(3)}(t) &= \frac{1}{\sqrt{3}} [B_+ \omega_+ \sin(\omega_+ kt + \phi_1) + B_- \omega_- \sin(\omega_- kt + \phi_2)], \\
    f_2^{(3)}(t) &= \sqrt{\frac{6}{73}} \left[ \cos(\omega_+ kt + \phi_1) - \cos(\omega_- kt + \phi_2) \right], \\
    f_3^{(3)}(t) &= 6 \sqrt{\frac{3}{146}} \left[ \frac{\sin(\omega_+ kt + \phi_1)}{\omega_+} - \frac{\sin(\omega_- kt + \phi_2)}{\omega_-} \right],
\end{align*}
\]

with the initial conditions \( f_k^{(3)}(0) = \delta_{k0} \), for \( k = 0, 1, 2, 3 \).
with $\omega_\pm = \sqrt{10 \pm \sqrt{73}}$ and the remaining constants being determined by the initial conditions $\{f_k^{(3)}(0)\}$. Here, the irrationality of $\omega_+ / \omega_-$ implies that the $\{f_k^{(3)}(t)\}$ evolve in an aperiodic manner. We illustrate this aperiodic behavior in Fig. 2, where we have plotted $f_0^{(3)}(t), f_3^{(3)}(t)$, and $f_L^{(3)} \equiv \sqrt{1 - |f_0^{(3)}(t)|^2 - |f_3^{(3)}(t)|^2}$ for $0 \leq t \leq 30\pi/\kappa\omega_+ \text{ and initial condition } f_0^{(3)}(0) = 1$, which corresponds to $\phi_1 = \phi_2 = 0$ and $B_\pm = (\sqrt{73} \pm 7)/2\sqrt{73}$. In this case we find that the maximum conversion efficiency to the $|3, 3, 0\rangle$ completely-converted state is $\max_t |f_0^{(3)}(t)|^2 \approx 0.40$, while the maximum conversion efficiency is

$$
\mu_3 = \max_t \sum_{k=1}^3 \frac{k|f_k^{(3)}(t)|^2}{3} \approx 0.89.
$$

(11)

For the four-pump-photon subspace, we limit our attention to the behavior of $f_0^{(4)}(t)$ and $f_4^{(4)}(t)$ when the initial condition is $f_k^{(4)}(0) = \delta_{k0}$:

$$
f_0^{(4)}(t) = \frac{C}{17/41 + m^2/1168992} \left[ B_+ \cos(\omega_+ \kappa t) + B_- \cos(\omega_- \kappa t) + m \right],
$$

$$
f_4^{(4)}(t) = -\frac{6\sqrt{6}C}{17/41 + m^2/1168992} \left[ \omega_-^2 \cos(\omega_+ \kappa t) - \omega_+^2 \cos(\omega_- \kappa t) - m/24 \right],
$$

(12)

where $\omega_\pm = \sqrt{25 \pm 3\sqrt{33}}, m = 144\sqrt{33}, C = 1/246\sqrt{33}$, and $B_\pm = 51\sqrt{33} \pm 261$. Here, the maximum conversion efficiency to the full-converted $|4, 4, 0\rangle$ state is $\max_t |f_4^{(4)}(t)|^2 \approx 0.74$, which is lower than the two-pump-photon subspace’s maximum conversion efficiency to its fully-converted state but higher than that for the three-pump-photon subspace. On the other hand, the maximum conversion efficiency of the four-pump photon input is $\mu_4 \approx 0.86$, which is lower than that for both the two-pump-photon and three-pump-photon subspaces.

II. UNITY-EFFICIENCY CONVERSION IN THE LIMIT OF HIGH PUMP-PHOTON NUMBERS

In this section we provide a proof by induction that amplitude amplification can achieve an arbitrarily-close-to-unity efficiency for converting the input state $|0, 0, n\rangle$ to the completely-converted output state $|n, n, 0\rangle$ in the large-$n$ limit. We preface our induction proof by justifying the assertion that passing the input state through a length-$L_0$, type-II phase-matched $\chi^{(2)}$ crystal (equivalent to an interaction time $t_0 = L_0/\nu$) yields the UPDC procedure’s initial state

$$
|\Psi_0\rangle = \cos(\theta_g/2)|0\rangle + \sin(\theta_g/2)|1\rangle,
$$

(13)
for $0 < \theta_g \ll 1$, where $|1\rangle \equiv |n, n, 0\rangle$ and $|0\rangle$ is a normalized state satisfying $\langle 1 | 0 \rangle = 0$. For $\kappa \delta t \ll 1$, Eqs. (5) yield

$$f_{1}^{(n,0)}(\delta t) = \kappa \sqrt{n} \int_{0}^{\delta t} f_{0}^{(n,0)}(t) \, dt = \sqrt{n} \kappa \delta t,$$

$$f_{2}^{(n,0)}(\delta t) = 2\kappa \sqrt{n-1} \int_{0}^{\delta t} f_{1}^{(n,0)}(t) \, dt = \sqrt{n(n-1)} (\kappa \delta t)^2,$$

$$\vdots$$

$$f_{n-1}^{(n,0)}(\delta t) = \sqrt{n!} (\kappa \delta t)^{n-1},$$

$$f_{n}^{(n,0)}(\delta t) = \sqrt{n!} (\kappa \delta t)^n,$$

to lowest order in $\delta t$. Setting $t_0 = \delta t$ then shows that we can realize Eq. (13) with $\sin(\theta_g/2) = \sqrt{n!} (\kappa t_0)^n$.

At this point we begin the induction proof in earnest. We must first show that, after applying the $U^{(n)}_{\text{NGS}}$ gate to $|\Psi_0\rangle$ to obtain the state $|\Psi_1\rangle$, there is an SPDC crystal length $L_1$ (equivalent to an interaction time $t_1 = L_1/v$) which will produce

$$|\Psi_1\rangle = \cos(3\theta_g/2) |0\rangle + \sin(3\theta_g/2) |1\rangle,$$

as the first Grover iteration’s output state, with $|0\rangle$ being a normalized state satisfying $\langle 1 | 0 \rangle = 0$. Consider an interaction time $t_1'$ satisfying $\kappa t_1' \ll 1$. We have

$$f_{n-1}^{(n,0)}(t_0) = \sin(\theta_g/2)^{n-1} (n!)^{1/2n} \approx \sin(\theta_g/2) \sqrt{n/e},$$

where we have used the Stirling approximation for $n!$ and $\sin(\theta_g/2)^{(n-1)/n} \approx \sin(\theta_g/2)$ for $n \gg 1$. Using this result we find that

$$f_{n}^{(n,1)}(t_1') = -f_{n}^{(n,0)}(t_0) + \int_{0}^{t_0 + t_1'} \kappa n f_{n-1}^{(n,0)}(t) \, dt$$

$$\approx -\sin(\theta_g/2) + \int_{0}^{t_0 + t_1'} \kappa \sin(\theta_g/2) \sqrt{n/e} \, dt$$

$$= \sin(\theta_g/2)(\sqrt{n/e} \kappa t_1' - 1).$$

Because $0 < \sin(3\theta_g/2) < 3 \sin(\theta_g/2)$ for $0 < \theta_g \ll 1$, it follows that having $\kappa t_1' \ll 1$ and $\kappa t_1' \geq 4\sqrt{e/n}$ ensures there is a $t_1 < t_1'$ such that Eq. (19) holds.

Next, we assume that

$$|\Psi'_{m+1}\rangle = \cos((2m + 1)\theta_g/2) |0\rangle + \sin((2m + 1)\theta_g/2) |1\rangle,$$

for $m > 1$, is the $m$th Grover iteration’s output state, where $|0\rangle$ is a normalized state satisfying $\langle 1 | 0 \rangle = 0$. Our induction proof will be complete if we can show that

$$|\Psi'_{m+1}\rangle = \cos((2m + 3)\theta_g/2) |0\rangle + \sin((2m + 3)\theta_g/2) |1\rangle,$$

with $|0\rangle$ being a normalized state satisfying $\langle 1 | 0 \rangle = 0$, is the $(m + 1)$th Grover iteration’s output state.

Using $f_{k}^{(n,m+1)}(0) = (-1)^{k+1} f_{k}^{(n,m)}(t_m)$, which holds for $m > 1$, Eqs. (5) give us

$$f_{n-1}^{(n,m+1)}(\delta t) = \kappa \sqrt{2} (n-1) f_{n-2}^{(n,m)}(t_m) + nf_{n}^{(n,m)}(t_m) |0\rangle \kappa \delta t + f_{n-1}^{(n,m)}(t_m)$$

and

$$f_{n}^{(n,m+1)}(\delta t) = -f_{n}^{(n,m)}(t_m) + \int_{0}^{\delta t} \kappa n f_{n-1}^{(n,m+1)}(t) \, dt,$$

for $\kappa \delta t \ll 1$. Another use of Eqs. (5) with $\kappa \delta t \ll 1$ plus Eq. (27) then leads to

$$\int_{0}^{\delta t} \kappa n f_{n-1}^{(n,m+1)}(t) \, dt = \int_{0}^{\delta t} \kappa n f_{n-1}^{(n,m)}(t_m + t) \, dt + \int_{0}^{\delta t} 2\kappa^2 n^2 f_{n}^{(n,m)}(t_m) \, dt$$

$$= f_{n}^{(n,m)}(t_m + \delta t) - f_{n}^{(n,m)}(t_m) + (n \kappa \delta t)^2 f_{n}^{(n,m)}(t_m).$$
Substituting this result into Eq. (28), we have that

\[
f^{(n,m+1)}_n(\delta t) = -2f^{(n,m)}_n(t_m) + f^{(n,m)}_n(t_m + \delta t) + (n\kappa \delta t)^2 f^{(n,m)}_n(t_m) \geq f^{(n,m)}_n(t_m)\left[(n\kappa \delta t)^2 - 2\right],
\]

where \(f^{(n,m)}_n(t_m) = \sin(2m + 1)\theta_g/2 > 0\), and the continuity of the Schrödinger evolution plus \(\kappa \delta t \ll 1\) ensures that \(f^{(n,m)}_n(t_m + \delta t) > 0\). Now we see that

\[
f^{(n,m+1)}_n(\delta t) \geq \sin((2m + 3)\theta_g/2)
\]

prevails if

\[
\kappa \delta t \geq \sqrt{\frac{2 + \sin(2m + 3)\theta_g/2}{\sin(2m + 1)\theta_g/2}},
\]

and this can be satisfied with \(\kappa \delta t \ll 1\) if

\[
n \gg \sqrt{\frac{2 + \sin(2m + 3)\theta_g/2}{\sin(2m + 1)\theta_g/2}}.
\]

Because \(0 < \sin(3\theta_g/2) \leq 3 \sin(\theta_g/2)\) for \(0 < \theta_g \ll 1\), and \(\sin(2m + 1)\theta_g/2\) is monotonically decreasing with increasing \(m\), the preceding condition on \(n\) is met if \(n \gg \sqrt{5}\). So, choosing \(n\) large enough we can find a \(t_{m+1}\) that provides the amplitude amplification needed to complete the induction proof. Thus, with \(M\) being the largest integer satisfying \((2M + 1)\theta_g < \pi\), we can get a \(\sin^2[(2M + 1)\theta_g]\) conversion efficiency, from the input state \(|0, 0, n\rangle\) to the fully-converted state \(|n, n, 0\rangle\), and this conversion efficiency can be made arbitrarily close to unity for small enough \(\theta_g\). Furthermore, choosing \(\theta_g \approx 1/\sqrt{n}\), for \(n \gg 1\), we have that \(M\) is \(O(\sqrt{n})\), as expected for Grover search.

### III. Grover-Search Example: Two-Pump-Photon Subspace

Here we supplement the large-\(n\) proof from Sec. II by presenting an explicit demonstration of complete conversion to the fully-converted state for the two-pump-photon subspace. In particular, using Eqs. (8), we show that the four-step UPDC procedure described in the main paper realizes complete conversion in a single Grover iteration.

I. **Initialization:** The input state \(|0, 0, 2\rangle\), shown as the blue dot in Fig. 3, undergoes a duration-\(t_0\) interaction in the type-II phase-matched \(\chi^{(2)}\) crystal to yield, via Eqs. (8) with \(m = 1\) and \(\phi_0 = 0\), the UPDC procedure’s initial state

\[
|\Psi_0\rangle = \frac{2}{3}\left[1 + \frac{1}{2}\cos(\sqrt{6}t_0)\right]|0, 0, 2\rangle + \frac{1}{\sqrt{3}}\sin(\sqrt{6}t_0)|1, 1, 1\rangle + \frac{\sqrt{2}}{3}\left[1 - \cos(\sqrt{6}t_0)\right]|2, 2, 0\rangle.
\]

In order to achieve complete conversion in a single Grover iteration, we choose \(t_0 = 0.976/\sqrt{6}\), which leads to \(|\Psi_0\rangle\) being the purple dot in Fig. 3 obtained from duration-\(t_0\) evolution around the red circle from the blue dot in that figure.

II. **Sign flip on the marked state:** Applying the \(U^{(2)}_{\text{NSG}}\) gate to the \(|\Psi_0\rangle\) obtained with \(t_0 = 0.976/\sqrt{6}\) yields

\[
|\Psi_1\rangle = \frac{2}{3}\left[1 + \frac{1}{2}\cos(0.976)\right]|0, 0, 2\rangle + \frac{1}{\sqrt{3}}\sin(0.976)|1, 1, 1\rangle - \frac{\sqrt{2}}{3}\left[1 - \cos(0.976)\right]|2, 2, 0\rangle,
\]

which corresponds to transitioning from the purple dot on the red circle to the green dot on the blue circle in Fig. 3.

III. **Rotation toward the marked state:** Using the \(|\Psi_0\rangle\) obtained with \(t_0 = 0.976/\sqrt{6}\) as the input to a duration-\(t_1\) interaction in a \(\chi^{(2)}\) crystal implies that the initial conditions Eqs. (8) use for that evolution are \(m = 1/2\) and \(\phi_0 = 0.626\). With those initial conditions Eqs. (8) now give us

\[
|\Psi_1'\rangle = \frac{\sqrt{2}}{3}\left[1 + \cos(\sqrt{6}t_1 + \phi_0)|0, 0, 2\rangle + \frac{\sqrt{2}}{\sqrt{3}}\sin(\sqrt{6}t_1 + \phi_0)|1, 1, 1\rangle + \frac{1}{3}\left[1 - 2\cos(\sqrt{6}t_1 + \phi_0)\right]|2, 2, 0\rangle.
\]

To obtain complete conversion we choose \(t_1 = (\pi - \phi_0)/\sqrt{6}\), which reduces \(|\Psi_1'\rangle\) to \(|2, 2, 0\rangle\), as shown by the red dot in Fig. 3 obtained from duration-\(t_1\) evolution around the blue circle in that figure.
FIG. 3: 3D plot showing the UPDC procedure in the two-pump-photon subspace that realizes unity-efficiency conversion from the \(|0, 0, 2\rangle\) input state, shown as the blue dot, to the \(|2, 2, 0\rangle\) final state, shown as the red dot, in a single Grover iteration. The UPDC procedure’s initial state \(|\Psi_0\rangle\), prepared by passing the input state through a type-II phase-matched \(\chi^{(2)}\) crystal for an interaction time \(t_0 = 0.976/\kappa\sqrt{6}\), is shown by the purple dot that is obtained by evolution around the red circle from the blue dot. Sign flip on the marked state (\(|0, 0, 2\rangle\)) transforms the \(|\Psi_0\rangle\) state to \(|\Psi_1\rangle\), which is indicated by the green dot. Rotation toward the marked state by passing \(|\Psi_1\rangle\) through a type-II phase-matched \(\chi^{(2)}\) crystal for an interaction time \(t_1 = (\pi - 0.626)/\kappa\sqrt{6}\) leads to evolution around the blue circle to \(|\Psi'_1\rangle\) indicated by the red dot, which is the desired output state \(|2, 2, 0\rangle\).

IV. Termination: Complete conversion having been achieved, the UPDC procedure’s Grover iterations terminate after a single iteration.

Figure 4 is a schematic for realizing the two-pump-photon UPDC procedure’s Steps I through IV using the nondeterministic NSG proposed in Ref. [1]. The corresponding schematic for the deterministic-NSG version of two-photon-pump UPDC is the Level 1 unit cell in Fig. 5.

FIG. 4: Schematic for the two-pump-photon UPDC procedure using a nondeterministic NSG. Dotted lines separate the procedure’s Steps I through IV, whose descriptions were given earlier in Sect. III. The PDC blocks are parametric down-converters and the NSG block is the nondeterministic nonlinear sign gate from Ref. [1]. The PBS blocks are polarization beam splitters. One directs the signal photons emerging from the first PDC into the NSG, and the other recombines the signal photons emerging from the NSG with idler and pump photons at the input to the second PDC. The upper and lower ancilla rails entering the NSG are prepared in the single-photon Fock state and the vacuum state—here denoted 1 and 0—and an array of beam splitters within the NSG block (omitted here for simplicity) performs the unitary transformations described in Ref. [1] for nondeterministic NSG realization. Thus, when the first PDC’s input is in the \(|0, 0, 2\rangle\) state and the detector (DET) counts one photon, then the output signal-idler-pump joint state will be \(|2, 2, 0\rangle\).
IV. DUAL-FOCK STATE GENERATION VIA CASCADED TWO-PUMP-PHOTON UPDC

An interferometer whose two input ports are illuminated by the dual-Fock state $|n, n\rangle$ enjoys a quadratic improvement in phase-sensing precision over a coherent-state system of the same average photon number, thus achieving Heisenberg-limited performance [2]. The signal and idler outputs from SPDC, however, are in a thermal distribution of $|n, n\rangle$ states that eradicates this advantage [3]. We show in this section that cascaded two-pump-photon UPDC can produce a particular class of large-$n$ dual-Fock states. In Sec. II we proved that large-$n$ dual-Fock states can be generated, in principle, via $n$-pump-photon UPDC, but that approach requires $U^{(n)}_{NSG}$ gates for which there is no known deterministic realization, and their nondeterministic realization has $O(1/n)$ success-probability scaling. More generally, the state-of-the-art proposal for preparing a large-$n$ dual-Fock state is nondeterministic [4]. Generating a particular class of large-$n$ dual-Fock states via cascaded two-pump-photon UPDC, on the other hand, is a deterministic procedure if its UPDC elements employ $U^{(2)}_{NSG}$ gates realized with nonlinear optics.

Figure 5 shows a $K$-level version of our cascaded two-pump-photon SPDC scheme for generating dual-Fock states. Its fundamental building block is a unit cell comprised of a $t_0 = 0.976/\kappa\sqrt{6}$ interaction time, type-II phase-matched $\chi^{(2)}$ crystal, a deterministic $U^{(2)}_{NSG}$ gate, a $t_1 = (\pi - 0.626)/\kappa\sqrt{6}$ interaction time, type-II phase-matched $\chi^{(2)}$ crystal, a polarization beam splitter, and two quantum-state frequency converters. From Sec. III we know that sandwiching the $U^{(2)}_{NSG}$ gate between a unit cell’s two down-conversion crystals will take a two-photon pump at frequency $\omega_p$ and convert it to two pairs of orthogonally-polarized signal and idler photons at frequencies $\omega_s$ and $\omega_i$, respectively. The signal and idler photons are separated into distinct spatial modes by the polarization beam splitter, after which they individually enter quantum-state frequency converters [5–9]. The frequency converters perform 100%-efficiency conversion of their two-photon inputs to two-photon outputs at the pump frequency and in the polarization needed for pumping the next cascade level’s down-conversion crystals. The final level in a $K$-level cascade, however, does not use polarization beam splitters or quantum-state frequency converters. Its outputs are $2^{K-1}$...
spatial modes each containing a $|2, 2, 0\rangle$ signal-idler-pump state, making $|2, 2, 0\rangle \otimes 2^{K-1}$ the joint state of these spatial modes.

The preceding $2^{K-1}$ signal-idler outputs from the $K$th cascade level can now be combined into a single spatial mode by the following delay-and-switch procedure. Suppose that these outputs are all in a common temporal mode, $\psi(t)$, that is time limited to $|t| \leq T/2$. For $1 \leq \ell \leq 2^{K-1}$, we delay the $\ell$th spatial mode by $\ell \Delta T$, where $\Delta T > T$. We then use an optical switch yard to coherently combine the $2^{K-1}$ delayed signal-idler beams into a single spatial mode containing $2^K$ signal photons and $2^K$ idler photons. For applications in which only polarization—not temporal mode—matters, the single spatial-mode we have created with our delay-and-switch procedure will be in the $|2^K, 2^K\rangle$ state, where the first and second entries denote the signal-frequency, signal-polarization photon number and idler-frequency, idler-polarization photon number, respectively.