GEOMETRICAL OPTICS IN ABSORBING MEDIA

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1. *Introductory.*—The problem of propagation of rays in a medium of variable absorption and refraction has acquired some practical importance in radiotelegraphy. The radio waves penetrate into the conducting and absorbing layer of the upper atmosphere[1] and describe there a certain path. The question was brought to my attention by Dr. B. Van der Pol of Eindhoven (Holland) who asked me to give a formula by which the shape of a ray in the conducting layer could be computed. The solution given in this paper is not restricted to electromagnetic rays, but applies, equally well, to waves and rays of other types. With respect to the radio waves, one of our results is that the rays, after going up, do not bend down again and do not ever come back to earth. The rays, observed by the receiving station, as coming from the conducting layers are not primary but secondary waves produced in it by reflection. The writer is preparing a second paper dealing with the problem of space reflection in an inhomogeneous absorbing medium.

Before tackling the problem of absorption, let us reduce to its simplest terms the ordinary theory of geometrical optics in *transparent* media. As starting point, we may choose the *Fermat* principle, according to which light (sound, etc.) travels from a point A to a point B in the shortest time:

$$S = c \int_A^B dt = \text{minimum}. \tag{1}$$

If the medium has a variable index of refraction n, the velocity of propagation is given by $v = c/n$, where c represents the velocity in a standard medium (vacuum in the case of light). For this reason, we may multiply the element dt of our integral by

$$v^2n^2/c^2 = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)n^2/c^2 = 1, \tag{2}$$

obtaining

$$S = \int L dt, \tag{3}$$

with

$$L = n^2v^2/c = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)n^2/c \tag{4}$$

2. *Rays in Transparent Media.*—Our problem is now formally equivalent with a Hamiltonian dynamical problem having the Lagrangean function (4). The Lagrangean equations, representing the equations of a ray, evidently become

$$2\frac{d}{dt}(n^2\dot{x}) - \frac{\partial n^2}{\partial x}\dot{x}^2 = 0, \text{ etc.} \tag{5}$$

For the solution of these equations, we may proceed on the lines of the Hamiltonian theory. We introduce a "momentum" by the equations $p_x = \partial L/\partial\dot{x}$, etc., or

$$\bar{p} = \bar{v}n^2/c, \tag{6}$$

and a Hamiltonian

$$H = -L + \bar{v}.\bar{p} = p^2n^2 = v^2n^2/c^2. \tag{7}$$

From this point of view, equation (2) can be written as

$$H = (p_x^2 + p_y^2 + p_z^2)n^2 = 1; \tag{8}$$

it is the analog to the energy integral of dynamics with a special choice of the energy constant. We make use of a further relation from Hamilton's theory

$$\bar{p} = \nabla S, \tag{9}$$

and, substituting this into equation (8), we obtain Hamilton's partial differential equation in the form

$$(\nabla S)^2 = n^2, \tag{10}$$

which in geometrical optics is called "*equation of the iconal.*" An integral of this equation will have the form

$$S = S(x, y, z, \alpha_1, \alpha_2) + C, \tag{11}$$

where α_1 and α_2 are two non-additive integration constants. The physical meaning of the quantity S in geometrical optics is the "*phase of the wave.*" The equation

$$S(x, y, z, \alpha_1, \alpha_2) = \text{const.} \tag{12}$$

represents, therefore, a surface of equal phase. Written in differential form it becomes

$$\frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz = 0, \tag{12'}$$

showing that its characteristics

$$\frac{dx}{\partial S/\partial x} = \frac{dy}{\partial S/\partial y} = \frac{\partial z}{\partial S/\partial z}, \tag{13'}$$

which are equivalent to equations (5), are at the same time the equations of the normal trajectories: *the rays are everywhere normal to the wave surfaces.*

For obtaining the integrated form of equation (5) or (13′) we may use the *Jacobi* method which consists in differentiating (11) with respect to α_1, α_2 and equating the partials to two new constants β_1, β_2:

$$\frac{\partial S}{\partial \alpha_1} = \beta_1 \quad , \quad \frac{\partial S}{\partial \alpha_2} = \beta_2. \tag{13}$$

This is the final expression for the equations of a ray in a transparent medium.

3. *Absorbing Media.*—In the accurate theory of wave optics, the state of a medium is controlled by the equation

$$\nabla^2 u + k^2 n^2 u = 0 \tag{14}$$

($k = 2\pi/\lambda$, where λ is the wave-length in the standard medium). In an absorbing medium, the index of refraction is complex $n^2 = \kappa + i\sigma$. If n is constant (homogeneous medium) the simplest solution of equation (14), corresponding to a plane wave, is

$$u = \exp. \, [i(a_1 x + a_2 y + a_3 z) - (b_1 x + b_2 y + b_3 z)], \tag{15}$$

where the coefficients a and b must satisfy the single condition

$$(a_1 + ib_1)^2 + (a_2 + ib_2)^2 + (a_3 + ib_3)^2 = n^2 = \kappa + i\sigma. \tag{16}$$

The meaning of expression (15) is that we obtain the state of the optical field in a point x, y, z, at the time t, if we multiply (15) by the time factor $\exp(-ikct)$ and take the real part of the product. We see that we have to distinguish here two families of surfaces: the planes $a_1 x + a_2 y + a_3 z =$ const. are *surfaces of equal phase*, the planes $b_1 x + b_2 y + b_3 z =$ const. are *surfaces of equal amplitude*, or of equal intensity. The two families are in general distinct and do not coincide.

In the more general case of an inhomogeneous medium, n is a function of the coördinates. The method of geometrical optics applies to the case when n changes appreciably only in distances which are long compared with the wave-length λ. In a small volume, whose linear size is but a few λ, n remains practically constant and it is permissible to use the expression (15) considering in it a, b to be functions of x, y, z. We arrive so at the fundamental expression of a wave in geometrical optics of absorbing media

$$\begin{cases} u = \exp(ikS_1 - kS_2) = \exp(ikS) \\ S = S_1 + iS_2 \end{cases} \tag{17}$$

S_1 and S_2 are two functions of x, y, z. Equation $S_1(x, y, z) =$ const. represents the surfaces of equal phase, $S_2(x, y, z) =$ const. represents the surfaces of equal amplitude.

We derive a partial differential equation for the complex quantity S by substituting in the usual way[2] the approximate solution (16) into equation (14) and neglecting terms of lower degree in k than k^2. We obtain

$$(\nabla S)^2 = n^2. \tag{10'}$$

Formally this equation is identical with (10), but both S and n are now complex numbers. This relation, therefore, contains two simultaneous equations for the determination of S_1 and S_2. An integral of the form (11) determines both the surfaces of equal phase

$$S_1 = Re\, S(x, y, z, \alpha_1, \alpha_2) = \text{const.}, \tag{11'}$$

and those of equal intensity

$$S_2 = Im\, S(x, y, z, \alpha_1, \alpha_2) = \text{const.}, \tag{11''}$$

where the symbols *Re* and *Im* denote the real and imaginary parts of the expressions following them

We have seen that the variation principles (1) and (3) are mathematically equivalent with (10). However, it is hardly desirable to use them in the theory of absorbing media, because the quantities t and v lose their simple physical meaning. In fact, the phase S is now complex and, therefore, can no longer be interpreted as time. It is, therefore, best to base the whole theory on equation (10′) of the iconal and not to make use of Fermat's principle.

4. *Rays in an Absorbing Medium.*—In a transparent medium, the only set of surfaces characterizing a wave motion are the surfaces of constant phase. The direction of a ray is always normal to these surfaces. We have seen in the preceding section that waves in absorbing media are characterized by two sets. Since the time of Poynting's theorem, we understand under a ray the curve in which the energy travels. In the case of electromagnetic waves, we can determine the instantaneous direction of the energy flow by means of Poynting's vector. To fix our ideas, let us consider the two-dimensional case, in which the index of refraction n and the optical field are independent of the coördinate z. Let us, further, assume that the electric vector is parallel to $z(E_x = E_y = 0)$. For E_z, we substitute the approximate expression (16) of geometrical optics which we write in its real form

$$E_z = \exp(-kS_2) \cos k(S_1 - ct). \tag{18}$$

According to Maxwell's equations the magnetic field is given by $H_z = 0$,

$$\mu H_x = \exp(-kS_2)\left[\frac{\partial S_1}{\partial y}\cos k(S_1 - ct) + \frac{\partial S_2}{\partial y}\sin k(S_1 - ct)\right],$$

$$\mu H_y = -\exp(-kS_2)\left[\frac{\partial S_1}{\partial x}\cos k(S_1 - ct) + \frac{\partial S_2}{\partial x}\sin k(S_1 - ct)\right].$$

The components of Poynting's vector are

$$P_x = -cE_zH_y, \quad P_y = cE_zH_x, \quad P_z = 0 \tag{19}$$

and the direction of it is given by the ratio

$$P_y/P_x = -H_x/H_y \tag{20}$$

In the case of transparent media, $S_2 = 0$ and this ratio becomes independent of time. Poynting's vector has in every point of space a constant orientation, and this is the reason why we can compute the curves of energy flow. It is not so *in the case of absorbing media,* because here Poynting's vector oscillates: we cannot deduce the energy path from this vector and must look for some other method for doing this.

From the consideration of a single infinite wave, it is impossible to determine either the velocity or the direction of the flow of energy, as we have no way of identifying the individual energy elements. To determine the velocity, Lord Rayleigh used a group of wave frequencies. We shall try to use a group of waves of different orientations, in order to determine the path of the energy. This can be made in the most convincing way by considering the properties of a limited pencil of rays.

5. *A Pencil of Rays.*—In order to produce a pencil of rays, the experimental physicist uses a diaphragm. The theoretical one can do the same by means of Huyghens' principle. The expression for the field of a pencil in air, which, originating in an infinitely removed source, has passed through a rectangular diaphragm is[3]

$$u = u_0 \int_{-a}^{+a}\int_{-b}^{+b} \exp\left[ik(z' + \alpha x' + \beta y')\right] d\alpha d\beta. \tag{21}$$

In this expression α and β are small numbers whose squares and higher powers are neglected. This is permissible if we observe the pencil at a large distance from the diaphragm (Fraunhofer diffraction). We see that u represents the superposition of an infinitely large number of plane waves whose wave normals include different small angles with the z' axis.

The choice of coördinates in expression (21) is a special one: z' has the direction of the axis of the pencil, but by a proper adjustment of the complex constant u we can choose the origin at any distance from the diaphragm. Let us generalize our equation by a rotation of coördinates going over to the new system x, y, z. If we denote the substitution angles $\cos(xx') = l_{xx'}$, etc., the phases S_0 of the individual waves in air will have the expression

$$\begin{aligned} S_0 = (\alpha l_{xx'} + \beta l_{xy'} + l_{xy'})x + (\alpha l_{yx'} + \beta l_{yy'} + l_{yz'})y \\ + (\alpha l_{zx'} + \beta l_{zy'} + l_{zz'})z. \end{aligned} \tag{22}$$

Now we let our pencil pass into a second absorbing homogeneous medium, selecting as the surface of discontinuity the xy-plane. Each of the constituent plane waves of integral (21) will be refracted into a wave of the type (15) in the second medium. We can, therefore, make use of the well-known rules for the refraction of plane waves,[4] according to which the coefficients of x and y remain unchanged, while the coefficient of z must be computed from relation (16). This leads to the following formula for the second medium

$$\begin{aligned} S' &= (\alpha l_{xx'} + \beta l_{xy'} + l_{xz'})x + (\alpha l_{yx'} + \beta l_{yy'} + l_{yz'})y \\ &+ \sqrt{n^2 - (\alpha l_{xx'} + \beta l_{xy'} + l_{xz'})^2 - (\alpha l_{yx'} + \beta l_{yy'} + l_{yz'})^2}\, z. \end{aligned} \tag{23}$$

The expression for the pencil in the second absorbing medium will be

$$u = u_0 \int_{-a}^{+a} \int_{-b}^{+b} \exp\,(ikS')\, d\alpha d\beta. \tag{24}$$

Strictly speaking, not only the phase S_0 but also the amplitude u_0 is changed in the process of refraction. But since our plane waves impinge all at about the same angle, α and β being very small, this will involve only effects which are immaterial for our purpose.

6. *Equation of a Ray.*—To be consistent, we have to expand expression (23) with respect to α and β and to retain terms of zero and first order only. If we substitute the result into (24), we can carry out the integration and we obtain

$$u = u_0 \exp\,(ikS)\, \frac{\sin kA}{kA}\, \frac{\sin kB}{kB}. \tag{25}$$

$$\left\{ \begin{aligned} S &= l_{xz'}x + l_{yz'}y + (a_3 + ib_3)z, \\ A &= l_{xx'}\left(x - \frac{l_{xz'}}{a_3 + ib_3}\, z\right) + l_{yx'}\left(y - \frac{l_{yz'}}{a_3 + ib_3}\, z\right), \\ B &= l_{xy'}\left(x - \frac{l_{xz'}}{a_3 + ib_3}\, z\right) + l_{yy'}\left(y - \frac{l_{yz'}}{a_3 + ib_3}\, z\right). \end{aligned} \right. \tag{26}$$

The axis of the pencil is there where the absolute value of the complex expression (25) has its maximum. We find the maximum in the routine way, multiplying (25) by its complex conjugate, differentiating the product with respect to x and y, and making the two partials equal to nothing. The result of these operations is

$$x/l_{xz'} = y/l_{yx'} = Re(z/(a_3 + ib_3)).$$

As

$$l_{xz'} = \partial S/\partial x \quad , \quad l_{yz'} = \partial S/\partial y \quad , \quad a_3 + ib_3 = \partial S/\partial z,$$

the last equation can be also put into the form

$$Re\left[\frac{x}{\partial S/\partial x} = \frac{y}{\partial S/\partial y} = \frac{z}{\partial S/\partial z}\right]. \tag{27}$$

This is the equation of a ray in a homogeneous medium. If the medium is inhomogeneous, we make use of the principle of geometrical optics, according to which we still can apply the results found for homogeneous media for an infinitesimal length of the ray. Let us use curvilinear orthogonal coördinates given by the line element

$$ds^2 = Udu^2 + Vdv^2 + Wdw^2. \tag{28}$$

Our equation becomes

$$Re\left[\frac{Udu}{\partial S/\partial u} = \frac{Vdv}{\partial/S\partial v} = \frac{Wdu}{\partial S/\partial w}\right]. \tag{29}$$

It must be noted that, in view of our derivation, this equation applies only to a choice of coördinates in which

$$\partial S_2/\partial u = \partial S_2/\partial v = 0. \tag{30}$$

This can be always reached by selecting the surfaces of equal intensity (S_2 = const.) as one set (w = const.) of our orthogonal mesh system. In most cases occurring in applications conditions (30) are satisfied automatically. One usually considers systems where absorption and refraction are produced by the same physical cause, so that the surfaces κ = const. and σ = const. coincide. Thus they give the appropriate coördinates for the solution of equation (10′) and, at the same time, they turn out to be identical with S_2 = const.

As equations (29) represent the characteristics of (10′) in curvilinear coördinates, we again may use the Jacobi method and obtain

$$Re\, \partial S(u, v, w, \alpha_1, \alpha_2)/\partial\alpha_1 = \beta_1\ ,\ Re\, \partial S(u, v, w, \alpha_1, \alpha_2)/\partial\alpha_2 = \beta_2, \tag{31}$$

provided that constants α_1, α_2 are real.

7. *Examples of Application.*—Let us consider a stratified medium, i.e. a medium in which the index of refraction is a function of the coördinate z only; let, further, our rays lie in the (x, z)-plane. The problem then becomes two-dimensional and we have from (10′) the equation

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = n^2(z), \tag{32}$$

whose solution is

$$S = \alpha x + \int \sqrt{n^2 - \alpha^2}\, dz. \tag{33}$$

To fix our ideas, we shall apply this to the case of radiowaves. Since these waves start in the non-absorbing lower layers of the atmosphere α is real. For the rays, we obtain (31)

$$Re \int \frac{dz}{\sqrt{n^2 - \alpha^2}} = \frac{x}{\alpha}, \tag{34}$$

or in differential form

$$\frac{dx}{dz} = Re \frac{\alpha}{\sqrt{n^2 - \alpha^2}}. \tag{35}$$

In the case of the index of refraction decreasing with height, there is a fundamental difference between transparent and absorbing media. (1) *If the medium is transparent,* the ratio dz/dx decreases with height until it becomes equal to zero for $n = \alpha$. The rays describe an arc, reaching their maximum height at this elevation, and bend back to earth. This is the well-known phenomenon of total reflection. (2) *If the medium is absorbing* ($n^2 = \kappa + i\sigma$), the ratio dz/dx has always the same sign. The ray goes up monotonically without ever bending round and coming down. The rays which come down from absorbing layers are not totally, but partially reflected waves. As long as the absorption is small ($\sigma \ll \kappa$), there is not much difference between this case and the preceding one: The primary rays become almost horizontal and produce a strong reflection only in this part of their path, while the secondary, reflected rays have here almost the same direction as the primary ones.

It seems, however, that in the upper layers of the atmosphere, the conduction and absorption are quite considerable.[5] A further investigation is, therefore, desirable dealing with the intensity and path of reflected rays.

Formula (35) contains the law of refraction for rays passing from a homogeneous transparent medium (n_0) into a homogeneous absorbing medium ($n^2 = \kappa + i\sigma$). If the angle of incidence is β_0 (angle between ray and normal to the surface of discontinuity in transparent medium), the geometrical meaning of α is $\alpha = n_0 \sin \beta_0$. The corresponding angle in the second medium is given by

$$tg\beta = \alpha \sqrt{\frac{\kappa - \alpha^2 + \sqrt{(\kappa - \alpha^2)^2 + \sigma^2}}{2(\kappa - \alpha^2)^2 + 2\sigma^2}}. \tag{36}$$

This formula shows that the deviation from the ordinary law of refraction is noticeable only then when the absorption in a thickness equal to one wave-length is appreciable.

[1] The conducting layer of the upper atmosphere is usually referred to as the "*Heavi-*

side layer." The priority in postulating it belongs, however, to Prof. A. E. Kennelly whose paper on this subject appeared in March, 1902 (*Electrical World and Engineer*, p. 473). *Heaviside's* article was published in December, 1902 (*Encyclopaedia Brittannica*, Vol. **33**, p. 215) and was written in June, 1902, according to his own testimony (*Electromagnetic Theory*, Vol. **3**, p. 335).

[2] E. g., A. Sommerfeld und I. Runge, *Ann. Physik*, **35**, p. 290, 1911.

[3] E. g., M. v. Laue, *Enzyklopädie Math. Wissenschaften*, Vol. **5**, p. 424, 1915.

[4] W. Wien, *ibid.*, p. 130.

[5] P. O. Pedersen, *Proc. Inst. Radio Engineers*, **17**, p. 1750, 1929.

RECENT PROGRESS IN THE DUAL THEORY OF METALLIC CONDUCTION

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In 1921 I published[1] a "Summary" of the numerical results tentatively arrived at by means of the dual theory of conduction, especially the values of the "characteristic constants" of the various metals dealt with. Recently I have revised these numerical estimates, availing myself of some new data furnished by the International Critical Tables and applying the results of more mature reflection upon the problem with which I am engaged. The present paper may be regarded as a revision of the "Summary" above referred to. It is not a complete statement of all the features and applications of the dual theory in its present state of development, but it undertakes to show what measure of success this theory has attained in dealing numerically with the relations between electrical conduction, thermal conduction, the Thomson effect and the Peltier effect, in eighteen metals, including two alloys.

The "characteristic constants" of a metal, according to this theory, are the z, q, C, C_1, C_2, λ'_c and s contained in the following equations:

$$n = zT^q, \tag{1}$$

$$(\kappa_f \div \kappa) = C + C_1t + C_2t^2 \tag{2}$$

and

$$\lambda' = \lambda'_c + skT. \tag{3}$$

In (1) n is the number of free electrons contained in unit "free space" within a metal at temperature T. By "free space" is meant space in which the free electrons can move as gas particles—that is, the whole volume of the metal less that part from which the free electrons are excluded by the atoms and ions. How much the "free space" differs from the total volume I cannot at present undertake to say, but the distinction