

which will limit the fruitfulness of attempts to advance along this particular path.

¹ C. Spearman, *Abilities of Man, Their Nature and Measurement*, Macmillan, 1927; see Appendix, p. viii.

² H. T. H. Piaggio, *Brit. J. Psychol.*, **24**, 88-105 (1933) states the matter clearly (p. 104) when in discussing Kelley's tests he says: "However, in deciding what tests to retain we make the tetrad relations the criterion, and *not* preconceived notions about dissimilarity of tests; indeed in the table below we retain three forms of memory test." Now unless' memory be the general g , it would seem as though memory for words, memory for meaningful symbols and memory for meaningless symbols should have in common some general memory factor in addition to g , but they do not.

³ For notation see E. B. Wilson, these PROCEEDINGS, **19**, 768, 882 (1933), and preceding papers.

⁴ T. L. Kelley, *Crossroads in the Mind of Man: A Study of Differentiable Mental Abilities*, Stanford Univ. Press, 1928.

ON THE TEMPERATURE DEPENDENCE OF FERRO-MAGNETIC SATURATION

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Communicated November 4, 1933

1. Accurate data on the saturation intensity of magnetization at low temperatures for monocrystalline materials were recently published by Allen and Constant¹ who, moreover, show that their results fit into the following interesting law: Let the saturation intensity at the temperature T be denoted by M_{ST} , the same intensity at the absolute zero of temperature by M_{SO} and the Curie point by T_C . If the ratio M_{ST}/M_{SO} is plotted against T/T_C , the points for all ferro-magnetic crystals of the cubic system fall onto one and the same universal curve. Two other curves obtain in the case of hexagonal and of ortho-rhombic crystals.

The writer has given some time ago a theory of ferro-magnetism² which represents the continuation of previous theoretical work by Weiss,³ Heisenberg⁴ and Bloch⁵ and which leads to precisely the above results, although it was carried through in detail only for the case of simple cubic crystals. It is, therefore, interesting to see whether the formula derived in this theory for the saturation intensity of magnetization as a function of temperature quantitatively fits the data now available. This is, in fact, the case: It will be shown in section 3 that a constant left indeterminate in the present form of the theory can be chosen so as to bring it into a very good agreement with the results of Allen and Constant.

2. As was shown in the paper referred to,² the intensity of magnetic saturation is obtained by partial differentiation of a function denoted by $Z^1(H)$:

$$M_{ST} = kT \left[\frac{\partial \log Z^1(H)}{\partial H} \right]_{H=0}, \quad (1)$$

where H is the magnetic strength of field and k the Boltzmann constant. If we introduce the abbreviation

$$\tau = 2\mu H/kT, \quad (2)$$

(μ being the Bohr magneton) there follows

$$M_{ST} = 2\mu \left[\frac{\partial \log Z^1(H)}{\partial \tau} \right]_{\tau=0}. \quad (3)$$

For the function $Z^1(H)$ we gave in Eq. (38) l. c. the expression

$$\begin{aligned} \log Z^1(H) = & \frac{1}{2}N\tau + \\ & + \frac{8N}{\pi^3} \int \int \int_0^{\frac{\pi}{2}} \log \{1 + \exp[-\tau - 2\beta(\sin^2 \xi + \sin^2 \eta + \sin^2 \zeta)]\} \\ & d\xi d\eta d\zeta \end{aligned} \quad (4)$$

where β is inversely proportional to the temperature

$$\beta = \Theta/T, \quad (5)$$

Θ being a constant.

The expression (4) can be transformed by expanding with respect to $\exp[-2\beta(\sin^2 \xi + \sin^2 \eta + \sin^2 \zeta)]$ under the sign of the integral and remembering that

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \exp(n\beta \cos 2\xi) d\xi = J_0(in\beta), \quad (6)$$

where $J_0(x)$ denotes the Bessel function of the index 0 and of the argument x . We arrive, therefore, at the expression

$$\log Z^1(H) = \frac{1}{2}N\tau - N \sum_{n=1}^{\infty} (-1)^n \exp(-n\tau - 3n\beta) [J_0(in\beta)]^3/n. \quad (7)$$

According to Eq. (3) we obtain for the saturation intensity

$$M_{ST} = M_{SO} \left\{ 1 + 2 \sum_{n=1}^{\infty} (-1)^n [\exp(-n\beta) J_0(in\beta)]^3 \right\}, \quad (8)$$

where $M_{SO} = N\mu$. The numerical evaluation of this formula leads to the table of values given in table 1.

For reasons which will be given in section 4, we have included in this table also some values of the function $\log Z^2(0)/N$ which were computed with the help of the formula

$$\log Z^2(0) = N[0.693 - 1.5\beta + 0.1875\beta^2 - 0.0293\beta^4 + \dots]. \quad (9)$$

3. The characteristic temperature Θ which appears in our formulas and tables does not coincide with the Curie point T_C . The writer adduced general consideration (l. c.) which tended to show that Θ must be pro-

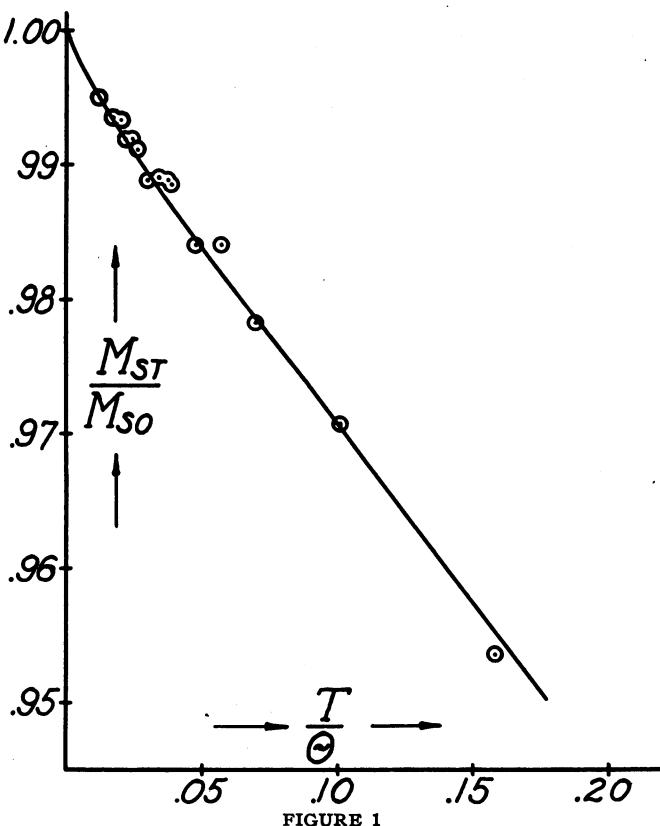


FIGURE 1

portional to T_C and even roughly estimated the factor of proportionality putting it at, approximately, $2/3$. It must be admitted, however, that this argument did not constitute a mathematical proof and that the theory in its present state does not permit to settle the question (compare section 4). It will be, therefore, better to regard Θ as an empirical constant and to ask whether the experimental evidence supports the view that it is proportional to the Curie point. In order to test this, we give

in figure 1 the curve which results from our formula (8) and from table 1 if we put $\Theta = 0.81 T_C$. As in Allen and Constant's graph, M_{ST}/M_{SO} is plotted here against $(T/T_C)^2$ and the circles in our figure represent the experimental results of these authors within the accuracy with which we could infer them from their diagram.

It will be seen that the agreement is very satisfactory within the whole range of temperatures covered by the experimental data, i.e., up to 0.4 of the Curie point. Since our theory works with approximations which are justifiable for small T/Θ but become more and more questionable as the temperature rises, we cannot expect it to hold much above this range. Because of these limitations of the theory, it is unable to describe the conditions in the vicinity of the Curie point, so that the connection between Θ and T_C remains theoretically undeterminate. For this reason, the empirical confirmation of the proportionality between Θ and T_C receives a particular significance. There exists at present a general concurrence of opinion that the ferromagnetic substances consist of per-

TABLE I

β	T/Θ	M_{ST}/M_{SO}	$\log Z^1(0)/N$	$\log Z^2(0)/N$
∞	0	1.000	0.000	
10	0.100	0.997		
8	0.125	0.995		
6	0.167	0.993		
5	0.200	0.990		
4	0.250	0.986	0.009	
3	0.333	0.978		
2.5	0.400	0.969		
2.00	0.500	0.953	0.026	
1.00	1.000	0.839	0.090	
0.90	1.111	0.811	0.106	
0.80	1.250	0.769	0.126	
0.60	1.667	0.668	0.187	-0.15
0.50	2.000		0.23	-0.012
0.40	2.500		0.28	0.12
0.20	5.000		0.44	0.40
0.00	∞		0.69	0.69

manently magnetized microcrystalline units or "blocks." The question arises, therefore, whether the disappearance of ferro-magnetic properties characterizing the Curie point is due to the sudden loss of permanent magnetization by these blocks (primary Curie point) or to the difficulty of aligning the permanently magnetized blocks above a certain temperature (secondary Curie point). The quantity M_{ST} is, simply, the intensity of permanent magnetization of the primary blocks and, according to the experimental results of Allen and Constant, it is determined by T_C even at very low temperatures. It is hard to see what the behavior of M_{ST}

in the neighborhood of the absolute zero can have to do with a secondary Curie point, and the conclusion must, therefore, be that in the case of the substances measured by these authors the temperature T_C refers to the primary Curie point of the blocks themselves, a result which can be strongly supported by other lines of reasoning.

The existence of another, higher Curie point was inferred by Forrer⁶ from the form of the susceptibility curve above the lower, directly measurable Curie point and Bitter⁷ interprets it as the secondary point in the above sense. If this interpretation is correct, the present theory is unable to explain the remarkable fact that Forrer's upper Curie point lies in all cases pretty close to the lower. However, in the writer's opinion, the existence of this upper Curie point is not beyond question because it is based on an extrapolation by means of a theoretical formula of uncertain validity. Only if the secondary Curie point happens to lie lower than the primary it can be observed in a direct and unambiguous way. Perhaps this case is realized for cobalt-iron alloys,⁸ as suggested by Forrer.

It goes without saying that for crystals of the non-cubical system both the form of the law (8) and the connection between Θ and T_C will be different. We expect, however, that for low temperatures, the first change will be smaller than the second, so that the formula (8) will still retain a certain value as an approximation.⁹

4. The physical reason for the existence of a primary Curie point must be, of course, that for low temperatures the blocks are found mainly in quantic states associated with a large magnetic moment while, from a certain temperature on, the unmagnetized states have the higher probability. In our theory the criterion for this change of dominance is when the functions we denote by $Z^1(0)$ and $Z^2(0)$ become numerically equal. Using rather crude and inaccurate approximations for these functions, we estimated (l. c.) that this equality takes place at a temperature of the order of $3\Theta/2$ and that even the present form of the theory (which is admittedly inaccurate for temperatures that are not low) gives the Curie point almost quantitatively. We are compelled to take this statement back: It will be seen from table 1 that $Z^2(0)$ remains for all temperatures below $Z^1(0)$. This means that the terms neglected in the theory make themselves felt already at temperatures much below the Curie point, so that the decline of the theoretical curve of saturation intensities is considerably slower than that of the experimental.¹⁰

It will be useful to enumerate here the main simplifications which are contained in Bloch's and in our theory. Perhaps, the most important inaccuracy is that inherent in the use of Slater's equation which is derived by treating the antisymmetrical wave functions in determinant form, as if they were orthogonal, while, in reality, they are only approximately orthogonal. It is very hard to estimate the error introduced by this

omission and to say whether its correction will lead only to a new interpretation of the constants of our equations or to an actual change of their form. Another simplification is the neglect of polar and excited states: While there is reason to assume that their influence is small, its exact estimate is still lacking. Further inaccuracies were discussed in our preceding paper: The use of Bloch's periodicity condition in place of the actual border conditions, the restriction to interactions between adjacent atoms and to first order perturbations. With these omissions the mathematical side of the theory becomes simple and elegant and, as the preceding sections show, it is sufficient to represent the actual conditions at low temperatures. However, the prospects of carrying this treatment to a higher degree of approximation are not very favorable. The mathematical simplicity is lost and the theory becomes rather cumbersome because one has to increase the accuracy in so many different directions. For determining the Curie point one should, perhaps, try a different approach to the problem, for instance, that recently outlined by L. Brillouin¹¹ although it seems that this method, too, is difficult to be carried through without considerable idealizations.

¹ R. I. Allen and F. W. Constant, *Phys. Rev.*, **44**, 228 (1933).

² P. S. Epstein, *Phys. Rev.*, **41**, 91 (1932).

³ P. Weiss, *Phys. Zeits.*, **9**, 361 (1908).

⁴ W. Heisenberg, *Zeits. Physik*, **49**, 619 (1928).

⁵ F. Bloch, *Ibid.*, **61**, 206 (1930).

⁶ R. Forrer, *Journ. Phys. et le Rad.*, **1**, 49 (1930).

⁷ F. Bitter, *Phys. Rev.*, **37**, 91 (1931).

⁸ A. Preuss, Thesis of Zurich (cited by Forrer).

⁹ We infer this from the fact that our formula (8) with a suitably chosen constant represents well the saturation curves obtained by Weiss and his pupils for heterocrystalline materials.

¹⁰ The formula given by Bloch (note 5) leads to a still much slower decline than ours.

¹¹ L. Brillouin, *Journ. Phys. et le Rad.*, **3**, 373, 565 (1932); **4**, 1 (1933).

A DUALITY IN INTERPOLATION TO ANALYTIC FUNCTIONS BY RATIONAL FUNCTIONS

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Communicated October 24, 1933

The following examples make clear a certain duality in interpolation to analytic functions by rational functions; reversal of the rôles of prescribed poles and points of interpolation reverses the region or regions of convergence, provided the bounding curves remain unchanged.

Ia. If the function $f(z)$ is analytic for $|z| \leq R$, then the sequence of