EIGENVALUE BOUNDS FOR SCHRÖDINGER OPERATORS
WITH A HOMOGENEOUS MAGNETIC FIELD

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Abstract. We prove Lieb-Thirring inequalities for Schrödinger operators with a
homogeneous magnetic field in two and three space dimensions. The inequalities
bound sums of eigenvalues by a semi-classical approximation which depends on the
strength of the magnetic field, and hence quantifies the diamagnetic behavior of the
system. For a harmonic oscillator in a homogenous magnetic field, we obtain the
sharp constants in the inequalities.

1. Introduction and main result

Lieb-Thirring inequalities [LiTh] provide bounds on the sum of negative eigenval-
ues of Schrödinger operators in terms of a phase space integral. In this paper, we
are interested in two-dimensional Schrödinger operators $H_B + V$ with a homogenous
magnetic field of strength $B > 0$. Here

$$H_B = \left(-i \frac{\partial}{\partial x_1} + \frac{B x_2}{2}\right)^2 + \left(-i \frac{\partial}{\partial x_2} - \frac{B x_1}{2}\right)^2$$

is the Landau Hamiltonian in $L^2(\mathbb{R}^2)$ and $V$ is a real-valued function. The Lieb-
Thirring inequality states that

$$\text{Tr} (H_B + V) \leq r_2 (2\pi)^{-2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (|p|^2 + V(x))^{-} \, dx \, dp$$

(1)

with the (currently best, but presumably non-optimal) constant $r_2 = \pi/\sqrt{3}$ from [DoLaLo]. Physically, the left side is (minus) the energy of a system of non-interacting fermions in an external potential $V$ and an external, homogeneous magnetic field of strength $B$, whereas the right side is $-r_2$ times a semi-classical approximation to that energy.

Physically, one expects the system to show a diamagnetic behavior, that is, to have
a higher energy in the presence of a magnetic field. This is however not reflected in (1),
which has a right hand side independent of $B$. We refer to [Fr] for further references
and a survey over this problem. Our goal in this paper is to obtain a bound similar to
(1), but with a more refined semi-classical approximation which takes $B$ into account.

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The approximation we propose is
\[ \frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m + 1)B + V(x)) \, dx. \tag{2} \]
This quantity reflects the diamagnetic behavior since
\[ \frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m + 1)B + V(x)) \, dx \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (|p|^2 + V(x)) \, dp \tag{3} \]
for every \( V \). Inequality (3) follows (even before the \( x \)-integration) from an easy convexity inequality (see Lemma 12 below). We also note that when \( B \to 0 \), by a Riemann sum argument, the quantity (2) approaches
\[ \frac{4\pi}{\rho^2} \int_0^{\infty} dE \int_{\mathbb{R}^2} (E + V(x)) \, dx = (2\pi)^{-2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (|p|^2 + V(x)) \, dp, \tag{4} \]
which is the ‘usual’ phase space integral.

While the right side of (1) (up to the constant \( r_2 \)) has the correct limiting behavior when a small parameter \( \hbar \) is introduced, it is not useful in the coupled limit \( B \to \infty \) and \( \hbar \to 0 \). This limit is physically relevant, for instance, in the study of neutron stars \[ \text{LiSoYn} \]. The magnetic quantity (2) reproduces the correct behavior in this regime. It is remarkable that this asymptotic profile is, indeed, a uniform, non-asymptotic bound. This is implicitly contained in \[ \text{LiSoYn2} \] who use, however, only an approximation of (2). Our first result is

**Theorem 1.** For any \( B > 0 \) and any \( V \) on \( \mathbb{R}^2 \) one has
\[ \text{Tr} (H_B + V) \leq \rho_2 \frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m + 1)B + V(x)) \, dx \tag{5} \]
with \( \rho_2 = 3 \).

Hence, up to the moderate increase from \( r_2 = \pi/\sqrt{3} \approx 1.81 \) to \( \rho_2 = 3 \), we have found a magnetic analogue of (1) which reflects the desired diamagnetic behavior (3). An important ingredient in our proof is a method developed recently by Rumin \[ \text{Ru} \] to derive kinetic energy inequalities; see Subsection 2.1.

Similarly as in the non-magnetic case, one might ask for the optimal value of the constant \( \rho_2 \). By the semi-classical result mentioned above one necessarily has \( \rho_2 \geq 1 \). A first result in this direction was obtained in \[ \text{FrLoWe} \] (extending previous work of \[ \text{ErLoVo} \]), where it was shown that if one takes \( V \) to be constant on a set of finite measure and plus infinity otherwise, then (5) holds with \( \rho_2 = 1 \). Our second main result is an analogous optimal bound for a harmonic oscillator.

**Theorem 2.** For any \( B > 0 \), \( \omega_1 > 0 \), \( \omega_2 > 0 \) and \( \mu > 0 \), inequality (5) holds with \( \rho_2 = 1 \) for \( V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu \).
In particular, letting $B \to 0$ and using the limit in (4) we recover the known bounds in the non-magnetic case from [dB, La2]. Even though the eigenvalues of a harmonic oscillator in a homogeneous magnetic field are explicitly known (Lemma 10), the proof of Theorem 2 relies on a delicate property of a subclass of convex functions (Lemma 14) which, we feel, could be useful even beyond the context of this paper.

**Moments of eigenvalues.** Using some by now standard techniques we derive a few consequences of Theorems 1 and 2. First, following Aizenman and Lieb [AiLi] one can replace $V$ by $V - \mu$ in (5) and integrate with respect to $\mu$ to obtain that for any $\gamma \geq 1$

$$\operatorname{Tr} (\frac{H_B + V}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))^m dx, \quad (6)$$

where $\rho_2 = 3$ for general $V$ and $\rho_2 = 1$ for $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu$. The restriction $\gamma \geq 1$ is necessary, since one easily checks that for $0 \leq \gamma < 1$ there is no constant $\rho_2$ such that (6) holds for all potentials $V$. Restricting ourselves to the quadratic case we shall show in Subsection 3.4

**Proposition 3.** For any $0 \leq \gamma < 1$ there are $B > 0$, $\mu > 0$ and $\omega_1 = \omega_2$ such that for $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu$ one has

$$\operatorname{Tr} (\frac{H_B + V}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))^m dx, \quad (7)$$

Our counterexample appears in the limit $\omega_j/B \to 0$ (with $\mu/B = 3$ fixed).

**Three dimensions.** Next, we shall show that our bounds for $d = 2$ can be applied to deduce analogous bounds for $d = 3$. This argument is in the spirit of the lifting argument from [La1, La2, LaWe]. We denote by $\hat{H}_B = H_B - \frac{\partial^2}{\partial x_3^2}$ the Landau Hamiltonian in $L^2(\mathbb{R}^3)$.

**Corollary 4.** For any $B > 0$ and any $V$ on $\mathbb{R}^3$, one has

$$\operatorname{Tr} \left( \hat{H}_B + V \right) \leq \rho_3 \frac{B}{(2\pi)^2} \sum_{m=0}^{\infty} \int_{\mathbb{R}^3 \times \mathbb{R}} ((2m+1)B + p_3^2 + V(x)) \ dx \ dp_3 \quad (8)$$

with $\rho_3 = \sqrt{3\pi}$.

**Proof.** From the operator-valued Lieb-Thirring inequality of [DoLaLo] we know that

$$\operatorname{Tr} \left( \hat{H}_B + V \right) \leq \frac{\pi}{\sqrt{3}} \int_{\mathbb{R}^2} \operatorname{Tr}_{L^2(\mathbb{R}^2)} (H_B + p_3^2 + V(\cdot, x_3)) \ dx_3 \ dp_3 \frac{2\pi}{2\pi}.$$Inequality (8) is therefore a consequence of Theorem 11. □

For the harmonic oscillator we have

**Corollary 5.** For any $B > 0$, $\omega_1 > 0$, $\omega_2 > 0$, $\omega_3 > 0$ and $\mu > 0$, inequality (8) holds with $\rho_3 = 1$ for $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 - \mu$. 
Proof. We denote by $E_j$ the eigenvalues of the one-dimensional harmonic oscillator $H = -\frac{d^2}{dx_j^2} + \omega_j^2 x_j^2$. Then, since $\hat{H}_B + V = (\hat{H}_B + V) \otimes I + I \otimes H$ with $V(x_1, x_2) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2$, we have

$$\text{Tr}_{L^2(\mathbb{R}^3)} (\hat{H}_B + \hat{V}) = \sum_j \text{Tr}_{L^2(\mathbb{R}^2)} (H_B + V + E_j - \mu).$$

According to Theorem 2 (which trivially holds for $\mu \leq 0$ as well), this is bounded from above by

$$\frac{B}{2\pi} \sum_j \sum_{m=0}^\infty \int_{\mathbb{R}^2} (2m + 1) B + V(x_1, x_2) + E_j - \mu \, dx_1 \, dx_2$$

$$= \frac{B}{2\pi} \sum_{m=0}^\infty \int_{\mathbb{R}^2} \text{Tr}_{L^2(\mathbb{R})} (H + (2m + 1) B + V(x_1, x_2) - \mu) \, dx_1 \, dx_2.$$ 

Next, we shall use that $H$ satisfies a Lieb-Thirring inequality with semi-classical constant $[\text{dB}, \text{La2}]$, that is, for any $\Lambda \in \mathbb{R}$,

$$\text{Tr}_{L^2(\mathbb{R})} (H - \Lambda) \leq \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} (p_3^2 + \omega_3^2 x_3^2 - \Lambda) \, dx_3 \, dp_3.$$ 

(This can also be seen from Lemma 12 and recalling the explicit form of the eigenvalues of $H$.) It follows that for every fixed $(x_1, x_2)$

$$\text{Tr}_{L^2(\mathbb{R})} (H + (2m + 1) B + V(x_1, x_2) - \mu) \leq \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} (p_3^2 + \omega_3^2 x_3^2 + (2m + 1) B + V(x_1, x_2) - \mu) \, dx_3 \, dp_3,$$

which proves the claimed bound.  

Remark 6. The previous proof shows that (8) with $\rho_3 = 1$ is valid for more general potentials $\hat{V}(x) = V(x_1, x_2) + v(x_3)$, where $V(x_1, x_2) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2$ and where $v$ is such that $\text{Tr}_{L^2(\mathbb{R})} \left(-\frac{d^2}{dx_3^2} + v(x_3) - \Lambda \right) \leq \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} (p_3^2 + v(x_3) - \Lambda) \, dx_3 \, dp_3$ for all $\Lambda$.

A similar argument as in the proofs of Corollaries 4 and 5 (based on the operator-valued Lieb-Thirring inequalities of [HuLaWe, LaWe]) shows that for general $V$ one has

$$\text{Tr} \left(\hat{H}_B + V\right)^\gamma \gamma \leq \rho_{3, \gamma} \frac{B}{(2\pi)^2} \sum_{m=0}^\infty \int_{\mathbb{R}^3 \times \mathbb{R}} ((2m + 1) B + p_3^2 + V(x)) \, dx \, dp_3$$

with $\rho_{3, \gamma} = 6$ if $\gamma \geq 1/2$, with $\rho_{3, \gamma} = \pi \sqrt{3}$ if $\gamma \geq 1$ and with $\rho_{3, \gamma} = 3$ if $\gamma \geq 3/2$. Moreover, in the special case of $\hat{V}(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 - \mu$, (9) holds with $\rho_3 = 1$ for $\gamma \geq 1$ and with $\rho_{3, \gamma} = 2(\gamma/(\gamma + 1))^{\gamma}$ for $0 \leq \gamma < 1$. The latter follows from the fact [FrLoWe] that

$$\text{Tr}_{L^2(\mathbb{R})} \left(-\frac{d^2}{dx_3^2} + \omega_3^2 x_3^2 - \Lambda \right) \gamma \leq 2 \left(\frac{\gamma}{\gamma + 1}\right)^\gamma \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} (p_3^2 + \omega_3^2 x_3^2 - \Lambda)\, dx_3 \, dp_3.$$
2. Proof of Theorem 1

2.1. A kinetic energy inequality. We define a piecewise affine function \( j : [0, \infty) \to [0, \infty) \) by

\[
j(\rho) = \frac{B^2}{2\pi} (L^2 + (2L + 1)r) \quad \text{if} \quad \rho = \frac{B}{2\pi} (L + r), \quad L \in \mathbb{N}_0, \quad r \in [0, 1).
\]

We note that \( j \) is continuous, increasing and convex. One has \( j(\rho) = B\rho \) if \( \rho \leq B/(2\pi) \) and \( j(\rho) \sim 2\pi\rho^2 \) if \( \rho \gg B \). The connection between this function and the right side of (5) will become clearer in the next subsection.

Theorem 7. Let \( 0 \leq \gamma \leq 1 \) be a density matrix on \( L^2(\mathbb{R}^2) \) with finite kinetic energy. Then

\[
\text{Tr } H_B \gamma \geq 3 \int_{\mathbb{R}^2} j(\rho_{\gamma}(x)/3) \, dx,
\]

where \( \rho_{\gamma}(x) = \gamma(x,x) \).

It is easy to see that \( 3j(\rho/3) \geq (1/3)j(\rho) \) for all \( \rho \geq 0 \), and therefore we also have

\[
\text{Tr } H_B \gamma \geq (1/3) \int_{\mathbb{R}^2} j(\rho_{\gamma}(x)) \, dx.
\]

Proof. The first part of our proof follows the method introduced by Rumin [Ru]. We define \( j_R : [0, \infty) \to [0, \infty) \) by

\[
j_R(\rho) = B\rho + 2B \sum_{k=1}^{\infty} \left( \sqrt{\rho} - \sqrt{\frac{Bk}{2\pi}} \right)^2.
\]

We note that \( j_R \) is differentiable and convex, \( j_R(\rho) = B\rho \) if \( \rho \leq B/(2\pi) \) and \( j_R(\rho) \sim 2\pi\rho^2/3 \) if \( \rho \gg B \). We shall first show that

\[
\text{Tr } H_B \gamma \geq \int_{\mathbb{R}^2} j_R(\rho_{\gamma}(x)) \, dx.
\]

In the second part of our proof (see Lemma 8) we show that \( j_R(\rho) \geq 3j(\rho/3) \) for all \( \rho \geq 0 \).

For the proof of (10) we write

\[
\text{Tr } H_B \gamma = \int_0^{\infty} \text{Tr } (P^E \gamma) \, dE = \int_{\mathbb{R}^2} \int_0^{\infty} \rho^E_{\gamma}(x) \, dE \, dx,
\]

where \( P^E \) is the spectral projection of \( H_B \) corresponding to the interval \([E, \infty)\) and where \( \rho^E_{\gamma}(x) = (P^E \gamma P^E)(x,x) \). It is well-known that

\[
(1 - P^E)(x,x) = \frac{B}{2\pi} \#\{m \in \mathbb{N}_0 : (2m + 1)B < E\}.
\]

The same clever use of the triangle inequality as in [Ru] leads to the pointwise lower bound

\[
\rho^E_{\gamma}(x) \geq \left( \sqrt{\rho_{\gamma}(x)} - \sqrt{\frac{B}{2\pi} \#\{m \in \mathbb{N}_0 : (2m + 1)B < E\}} \right)^2.
\]
Inserting this bound in (10) we obtain
\[
\text{Tr} \, H_B \gamma \geq \int_{\mathbb{R}^2} \left( \int_0^B \rho_\gamma(x) \, dE \right) + \sum_{k=1}^{\infty} \int_{(2k-1)B}^{(2k+1)B} \left( \sqrt{\rho_\gamma(x)} - \frac{\sqrt{Bk}}{2\pi} \right)^2 \, dE \, dx
\]
\[
= \int_{\mathbb{R}^2} j_R(\rho_\gamma(x)) \, dx.
\]
This completes the proof of (10) and also, by Lemma 8 below, the proof of the theorem.

Lemma 8. \( j_R(\rho) \geq 3 \, j(\rho/3) \) for all \( \rho \geq 0 \).

Proof. We are going to prove that
\[
j_R(3\rho) \geq 3 \, j(\rho).
\]
Note that this is an equality for \( \rho \leq B/(6\pi) \). Moreover, since the left side of (12) is convex and the right side linear for \( \rho \leq B/(2\pi) \), we conclude that (12) holds for all \( \rho \leq B/(2\pi) \).

Henceforth we shall assume that \( \rho \geq B/(2\pi) \) and we write \( 3\rho = (B/2\pi)(K + s) \) with \( K \in \mathbb{N} \) and \( s \in [0,1) \). If \( K = 3L + m \) with \( L \in \mathbb{N} \) and \( m \in \{0,1,2\} \), then the lemma says that
\[
K + s + 2 \sum_{k=1}^{K} \left( \sqrt{K + s} - \sqrt{k} \right)^2 \geq 3 \left( L^2 + \frac{1}{3}(2L + 1)(m + s) \right).
\]
We expand the square on the left side and insert \( L = (K - m)/3 \) on the right side. This shows that the assertion is equivalent to
\[
K + s + 2K(K + s) - 4\sqrt{K + s} \sum_{k=1}^{K} \sqrt{k} + K(K + 1) \geq \frac{1}{3}K^2 + \frac{2}{3}Ks + s + R,
\]
for \( K \in \mathbb{N} \) and \( s \in [0,1) \), where \( R = -\frac{1}{3}m^2 - \frac{2}{3}ms + m \). Since the inequality has to be true for any \( m \in \{0,1,2\} \), we can replace \( R \) by its maximum over these \( m \) (with fixed \( s \)), that is, by \( (2/3)(1-s) \). Thus (12) is equivalent to
\[
4K^2 + (3 + 2s)K - 6\sqrt{K + s} \sum_{k=1}^{K} \sqrt{k} - 1 + s \geq 0.
\]
The proof is straightforward for \( K = 1 \) and we may therefore assume that \( K \geq 2 \). By the concavity of the square root we have
\[
\frac{\sqrt{k} + \sqrt{k+1}}{2} \leq \int_k^{k+1} \sqrt{t} \, dt.
\]
Summing this from \( k = 1 \) to \( k = K - 1 \) we get
\[
\sum_{k=1}^{K} \sqrt{k} \leq \int_1^{K} \sqrt{t} \, dt + \frac{1 + \sqrt{K}}{2} = \frac{2K^{3/2}}{3} + \frac{K^{1/2}}{2} - \frac{1}{6}.
\]
This shows that
\[
4K^2 + (3 + 2s)K - 6\sqrt{K} + s \sum_{k=1}^{K} \sqrt{k} - 1 + s \\
\geq 4K^2 + (3 + 2s)K - \sqrt{K(K + s)(4K + 3)} + \sqrt{K + s} - 1 + s \\
= \frac{sK((4s - 12)K - 9)}{4K^2 + (3 + 2s)K + \sqrt{K(K + s)(4K + 3)}} + \sqrt{K + s} - 1 + s.
\]

In the quotient on the right side we estimate the numerator from below by 
\[-3sK(4K + 3)\] and the denominator from below by \(4K^2 + 3K + K(4K + 3) = 2K(4K + 3)\). Thus the quotient is bounded from below by \(-3s/2\), and we conclude that
\[
4K^2 + (3 + 2s)K - 6\sqrt{K} + s \sum_{k=1}^{K} \sqrt{k} - \frac{3R}{2} \geq \sqrt{K + s} - 1 - \frac{s}{2}.
\]

The right side is easily seen to be positive for \(K \geq 2\) and \(s \in [0, 1)\), and this concludes the proof. \(\square\)

2.2. **Proof of Theorem** \(\mathbb{I}\) In this section we are going to deduce Theorem \(\mathbb{I}\) from Theorem \(\mathbb{V}\). We define
\[
p(v) := -\frac{B}{2\pi} \sum_{m=0}^{\infty} ((2m + 1)B + v)_-
\]
for \(v \in \mathbb{R}\). This is a convex, decreasing and non-positive function. The key observation is that this \(p\) is the Legendre transform of the function \(j\) from the previous subsection, that is,
\[
p(v) = \inf_{\rho \geq 0} (j(\rho) + v\rho).
\]
This can be verified by elementary computations.

In order to prove Theorem \(\mathbb{I}\) we apply Theorem \(\mathbb{V}\) to get the estimate
\[
\text{Tr} (H_B + V) \gamma \geq \int_{\mathbb{R}^2} \left(3j(\rho_\gamma(x)/3) + V(x)\rho_\gamma(x)\right) dx
\]
for any \(0 \leq \gamma \leq 1\). According to \((13)\) this is bounded from below by \(3 \int_{\mathbb{R}^2} p(V(x)) \, dx\). For \(\gamma\) equal to the projection corresponding to the negative spectrum of \(H_B + V\) we obtain the assertion of Theorem \(\mathbb{I}\)

**Remark** 9. Similar arguments show that Theorem \(\mathbb{V}\) can be deduced from Theorem \(\mathbb{I}\). Indeed, since \(j\) is convex it is its double Legendre transform. By \((13)\) we obtain
\[
j(\rho) = \inf_{v \in \mathbb{R}} (p(v) + v\rho).
\]
By the variational principle and Theorem \(\mathbb{I}\) we can estimate for any \(0 \leq \gamma \leq 1\) and any \(V\)
\[
\text{Tr} H_B \gamma \geq -\text{Tr} (H_B + V)_- + \int_{\mathbb{R}^2} V(x)\rho_\gamma(x) \, dx \geq \int_{\mathbb{R}^2} \left(3p(V(x)) + V(x)\rho_\gamma(x)\right) dx.
\]
According to (14) this is bounded from below by $3 \int_{R^2} j(\rho, (x)/3) \, dx$, and this shows Theorem 7.

3. Proof of Theorem 2

3.1. The spectrum of $H_B + V$. The explicit form of the eigenvalues of $H_B + \omega^2 |x|^2$ was discovered in [Fo]. We include an alternative derivation of this result, which is also valid in the non-radial case.

Lemma 10. For any $B > 0$ and $\omega_1, \omega_2 > 0$ the operator $H_B + \omega_1^2 x_1^2 + \omega_2^2 x_2^2$ has discrete spectrum and its eigenvalues, including multiplicities, are given by

$$B \left( a_+\left(\frac{\omega_1}{B}, \frac{\omega_2}{B}\right)(2k + 1) + a_-$$(\frac{\omega_1}{B}, \frac{\omega_2}{B})(2l + 1) \right), \quad k, l \in \mathbb{N}_0,$$

where

$$a_\pm(\sigma_1, \sigma_2) = \sqrt{\frac{1}{2} \left( 1 + \sigma_1^2 + \sigma_2^2 \pm \sqrt{(1 + \sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2 \sigma_2^2} \right)}.$$

Remark 11. It will be important for our analysis below that

$$a_-(\sigma) a_+(\sigma) = \sigma_1 \sigma_2,$$

which is easily checked.

Proof. By means of the gauge transform $e^{-iBx_1 x_2/2}$ we see that $H_B + V$ is unitarily equivalent to the operator

$$-\frac{\partial^2}{\partial x_1^2} + \left( -i \frac{\partial}{\partial x_2} - Bx_1 \right)^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2,$$

which, in turn, by a partial Fourier transform with respect to $x_2$, is unitarily equivalent to

$$-\frac{\partial^2}{\partial x_1^2} + (x_2 - Bx_1)^2 + \omega_1^2 x_1^2 - \omega_2^2 \frac{\partial^2}{\partial x_2^2}.$$

After scaling $x_2 \mapsto \omega_2 x_2$ this becomes the non-radial harmonic oscillator $-\Delta + x^t Ax$ with the matrix

$$A = \begin{pmatrix} B^2 + \omega_1^2 & -B \omega_2 \\ -B \omega_2 & \omega_2^2 \end{pmatrix}.$$

The eigenvalues of $A$ are $B^2 a_+\left(\omega_1/B, \omega_2/B\right)^2$ and $B^2 a_-\left(\omega_1/B, \omega_2/B\right)^2$. Using the eigenvectors of $A$ as basis in $\mathbb{R}^2$, we obtain a direct sum of two one-dimensional harmonic oscillators with frequencies $Ba_+$ and $Ba_-$, respectively. This leads to the stated form of the eigenvalues. \qed

According to Lemma 10 and a simple computation, (5) with $\rho_2 = 1$ is equivalent to

$$\sum_{k,l \geq 0} \left( \mu - Ba_+\left(\frac{\omega_1}{B}, \frac{\omega_2}{B}\right)(2k + 1) - Ba_-\left(\frac{\omega_1}{B}, \frac{\omega_2}{B}\right)(2l + 1) \right) \leq \frac{B}{4\omega_1 \omega_2} \sum_{m \geq 0} \left( \mu - (2m + 1)B \right)_+$$
with $a_\pm$ given by (15). Setting $\Lambda = \mu/B$, $\sigma_j = \omega_j/B$ and $a_\pm = a_\pm(\sigma)$ and substituting (16) we can rewrite the desired inequality as

$$
\sum_{k,l \geq 0} (\Lambda - a_+ (2k + 1) - a_- (2l + 1))_+ \leq \frac{1}{4a_-a_+} \sum_{m \geq 0} (\Lambda - (2m + 1))^2_+ ,
$$

and this is what we shall prove.

3.2. Two inequalities for convex functions. For the proof of (17) we shall need

**Lemma 12.** Let $\phi$ be a non-negative convex function on $(0, \infty)$ such that $\int_0^{\infty} \phi(t) \, dt$ exists. Then

$$
\sum_{k=0}^{\infty} \phi(k + \frac{1}{2}) \leq \int_0^{\infty} \phi(t) \, dt.
$$

**Proof.** Indeed, by the mean-value property of convex functions $\phi(k + \frac{1}{2}) \leq \int_k^{k+1} \phi(t) \, dt$ for each $k$. Now sum over $k$. \qed

**Remark 13.** The proof also shows that $\sum_{k=0}^{K-1} \phi(k + \frac{1}{2}) \leq \int_0^{K} \phi(t) \, dt$ for each integer $K$. This observation will be useful later.

The inequality from **Lemma 12** is sufficient to prove a sharp Lieb-Thirring inequality in the non-magnetic case, but for the proof of our Theorem we need a more subtle fact about convex functions. We note that by the previous lemma $h \sum_{k=0}^{\infty} \phi(h(k + \frac{1}{2})) \leq \int_0^{\infty} \phi(t) \, dt$ for any $h > 0$. Moreover, $h \sum_{k=0}^{\infty} \phi(h(k + \frac{1}{2})) \to \int_0^{\infty} \phi(t) \, dt$ as $h \to 0$ by the definition of the Riemann integral. The key for proving our sharp result is that, for a certain subclass of convex functions, this limit is approached monotonically. More precisely, one has

**Lemma 14.** Let $\phi$ be a non-negative convex function on $(0, \infty)$ such that $\int_0^{\infty} \phi(t) \, dt$ exists. Assume that $\phi$ is differentiable and that $\phi'$ is concave. Then the sum

$$
h \sum_{k=0}^{\infty} \phi(h(k + \frac{1}{2}))
$$

is decreasing in the parameter $h > 0$.

We emphasize that without assumptions on $\phi'$ the inequality

$$
\sum_{k=0}^{\infty} \phi(k + \frac{1}{2}) \leq h \sum_{k=0}^{\infty} \phi(h(k + \frac{1}{2}))
$$

is not true for all $h < 1$. Indeed, take for instance $\phi(t) = (1 - t)_+$ and $h \geq 2/3$.

In the proof of this lemma we shall make use of the following well-known fact about convex functions: If $\psi$ is a non-negative convex function on $(0, \infty)$ such that $\int_0^{\infty} \psi(t) \, dt$ exists, then $\psi(t) = \int_0^{\infty} (T - t)_+ \, d\mu(T)$ for some non-negative measure $\mu$. Indeed, it is known that the left-sided derivative $\partial_- \psi$ exists everywhere on $(0, \infty)$ and satisfies $\psi(b) - \psi(a) = \int_a^b \partial_- \psi(t) \, dt$ for $0 < a < b < \infty$. Moreover, $\partial_- \psi$ is
increasing and left-continuous, and therefore there is a non-negative measure \( \mu \) such that \( \partial_- \psi(b) - \partial_- \psi(a) = \mu([a, b)) \). Since \( \lim_{t \to \infty} \psi(t) = \lim_{t \to \infty} \partial_- \psi(t) = 0 \), we have by Fubini’s theorem
\[
\psi(t) = -\int_t^\infty \partial_- \psi(a) \, da = \int_t^\infty \left( \int_{\chi_{[a,\infty)}(T)} d\mu(T) \right) \, da = \int_0^\infty (T-t}_+ d\mu(T),
\]
as claimed.

**Proof.** By the fact recalled above (with \( \psi = -\phi’ \)) we have \( \phi(t) = \int_0^\infty (T-t}_+^2 d\mu(T) \) for a non-negative measure \( \mu \). Hence it suffices to prove the lemma for \( \phi(t) = (T-t}_+^2 \) with \( T > 0 \). We have to prove that \( \sum_{k=0}^{\infty} \left( \phi(h(k+\frac{1}{2})) + h(k+\frac{1}{2}) \phi’(h(k+\frac{1}{2})) \right) \leq 0 \), which for our \( \phi \) reads
\[
\sum_{k=0}^{\infty} ((S - 2k - 1)_+^2 - 2(2k + 1)(S - 2k - 1)_+) \leq 0
\]
with \( S = 2T/h \). Choose \( K \in \mathbb{N}_0 \) such that \( 2K + 1 \leq S < 2K + 3 \). Then the left side above equals
\[
\sum_{k=0}^{K} ((S - 2k - 1)_+^2 - 2(2k + 1)(S - 2k - 1)_+) = \sum_{l=0}^{K} (S^2 - 4S(2k + 1) + 3(2k + 1)^2)
\]
\[
= (K + 1) (S^2 - 4S(K + 1) + (2K + 1)(2K + 3))
\]
\[
= (K + 1)(S - 2K - 1)(S - 2K - 3).
\]
This is clearly non-positive for \( 2K + 1 \leq S < 2K + 3 \), thus proving the claim. \( \square \)

3.3. **Proof of Theorem 2.** We have to prove (17). By Lemma 12 for any \( \gamma \)
\[
\sum_{l \geq 0} (\Lambda - a_+(2k + 1) - a_-(2l + 1))_+ \leq \int_0^\infty (\Lambda - a_+(2k + 1) - 2a_-t}_+ \, dt
\]
\[
= \frac{1}{4a_-} (\Lambda - a_+(2k + 1))_+^2.
\]
A simple computation shows that \( a_+ = a_+(\sigma) \geq 1 \), and hence by Lemma 14
\[
a_+ \sum_{k \geq 0} (\Lambda - a_+(2k + 1))_+^2 \leq \sum_{k \geq 0} (\Lambda - (2k + 1))_+^2.
\]
The previous two inequalities imply the desired (17). \( \square \)

3.4. **Proof of Proposition 3.** Given \( 0 \leq \gamma < 1 \), we want to find \( \omega_1 = \omega_2 \) and \( B \) such that the reverse inequality (7) holds. We may assume \( \gamma > 0 \) in the following. (The case \( \gamma = 0 \) can be treated similarly, or one may use the argument of Aizenman and Lieb mentioned in the introduction to conclude that a counterexample for \( \gamma = \gamma_0 \) implies one for all \( \gamma < \gamma_0 \).)
By the same computation that lead to (17) we see that (7) can be written as

$$\sum_{k,l \geq 0} (\Lambda - a_+ (2k + 1) - a_- (2l + 1))_+^\gamma > \frac{1}{2(\gamma + 1)a_- a_+} \sum_{m \geq 0} (\Lambda - (2m + 1))_+^{\gamma + 1}$$

with $\Lambda = \mu/B$, $\sigma_j = \omega_j/B$ and $a_\pm = a_\pm(\sigma)$. We will let $\omega_1 = \omega_2$ and use the notation $t = \sigma^2$. One can show that $a_+ = 1 + t + O(t^2)$ and $a_- = t + O(t^2)$ as $t \to 0+$. We now choose $\Lambda = 3$ and recall that $a_+ = a_+(\sigma) \geq 1$. This gives us the inequality

$$2(\gamma + 1)a_- a_+ \sum_{l \geq 0} (3 - a_+ - a_- (2l + 1))_+^{\gamma} - 2^{\gamma + 1} > 0,$$

which may be written as

$$(\gamma + 1)a_-^{\gamma + 1} a_+ \sum_{l \geq 0} (x - l)_+^{\gamma} - 1 > 0$$

with $x = (3 - a_+ - a_-)/(2a_-)$. Since $x = t^{-1}(1 + O(t))$ as $t \to 0+$, we may choose $\sigma$ so that $x$ is an integer. In this case we may use the concavity of $y^\gamma$ and Remark 13 to bound

$$\sum_{l \geq 0} (x - l)_+^{\gamma} = \sum_{l=1}^x \int_{l/2}^{x + l/2} t^{\gamma} dt = \frac{1}{\gamma + 1} \left( (x + 1/2)^{\gamma + 1} - (1/2)^{\gamma + 1} \right).$$

This shows that

$$(\gamma + 1)a_-^{\gamma + 1} a_+ \sum_{l \geq 0} (x - l)_+^{\gamma} \geq a_+ \left( (a_- x + a_-/2)^{\gamma + 1} - (a_-/2)^{\gamma + 1} \right)$$

$$= a_+ \left( ((3 - a_-)/2)^{\gamma + 1} - (a_-/2)^{\gamma + 1} \right)$$

$$= (1 + t + O(t^2)) \left( 1 - t/2 + O(t^2) \right)^{\gamma + 1} + O(t^{\gamma + 1})$$

$$= 1 + \frac{1 - \gamma}{2} t + O(t^{\gamma + 1}).$$

Since this is strictly larger than 1 for sufficiently small $t$, we have proved our claim. □

**References**


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