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On the mathematical theory of globally optimal planned economic systems

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ABSTRACT Applying a theorem of non-negative matrix to an economic system, we obtain the rate of growth of the system and the ratio between the sectors of production. Accordingly, we establish a positive eigenvector method for planning economics.

Section 1. A Theorem About Matrices

We begin with a result from the theory of non-negative matrices that is of interest in its own right.

THEOREM. *Let* $\mathsf{A} = (a_{ij})$ *be an* $n \times n$ *irreducible non-negative nonsingular matrix—i.e.,* $a_{ij} \geq 0$*—denoted by* $\mathsf{A} \geq 0$ *and* $\mathbf{x}$ *be a* n*-vector* > 0*—i.e.,* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_i > 0$. *Suppose* A *is not a generalized permutation matrix—i.e.,* $\mathsf{A}^{-1} \not\geq 0$. *If* $\mathbf{x}$ *is not an eigenvector of* A, *then there exists an integer* l_0 *such that, for* $l \geq l_0$,

$$\mathbf{x}\mathsf{A}^{-l} = \mathbf{x}^{(l)} \qquad [1]$$

is a vector with components of both signs.

However, by the theorem of Perron–Frobenius, A has one and only one positive eigenvector $\mathbf{x}_0$, apart from a scalar factor, such that

$$\mathbf{x}_0\mathsf{A} = g\mathbf{x}_0, \qquad [2]$$

where g is the corresponding positive eigenvalue. In this case,

$$\mathbf{x}_0^{(l)} = x_0\mathsf{A}^{-l} = g^{-l}\mathbf{x}_0 \qquad [3]$$

is a positive vector for all l.

Proof.

A. Let us suppose that, in addition to g, the absolute value of all the other eigenvalues of A are less than g. Then

$$\lim_{l \to 0} \left(\frac{\mathsf{A}}{g}\right)^l$$

tends to a matrix of rank 1, with 1 as its only eigenvalue $\neq 0$. Therefore, we have

$$\lim_{l \to \infty} \left(\frac{\mathsf{A}}{g}\right)^l = \mathbf{u}'\mathbf{v}, \qquad \mathbf{v}\mathbf{u}' = 1, \qquad [4]$$

where $\mathbf{u}$ and $\mathbf{v}$ are n-vectors and $\mathbf{u}'$ (a column vector) is the transpose of $\mathbf{u}$. Since

$$\lim_{l \to \infty} \left(\frac{\mathsf{A}}{g}\right)^{l+1} = \frac{\mathsf{A}}{g}\mathbf{u}'\mathbf{v} = \mathbf{u}'\mathbf{v}\frac{\mathsf{A}}{g} = \mathbf{u}'\mathbf{v},$$

multiplying on the right by $\mathbf{u}'$ or on the left by $\mathbf{v}$, we have, respectively,

$$\mathsf{A}\mathbf{u}' = g\mathbf{u}', \qquad \mathbf{v}\mathsf{A} = g\mathbf{v}.$$

Therefore, $\mathbf{u} > 0$, $\mathbf{v} > 0$. We may assume $u_1 + \cdots + u_n = 1$, where $\mathbf{u} = (u_1, \ldots, u_n)$; thus, $\mathbf{u}$ and $\mathbf{v}$ are unique.

Without loss of generality, we may assume that $g = 1$ and $\mathbf{x}\mathbf{u}' = 1$, then $\mathbf{x}^{(l)}\mathbf{u}' = \mathbf{x}\mathsf{A}^{-l}\mathbf{u}' = \mathbf{x}\mathbf{u}' = 1$. Suppose $\mathbf{x}^{(l)} \geq 0$ for all l. From $\mathbf{x}^{(l)}\mathbf{u}' = 1$, we deduce that $\mathbf{x}^{(l)}$ is a bounded sequence. By the Weierstrass–Bolzano theorem, we have a subsequence l_i such that

$$\lim_{i \to \infty} \mathbf{x}^{(l_i)} = \mathbf{x}^* \geq 0.$$

Also $\mathbf{x}^*\mathbf{u}' = 1$. Hence,

$$\mathbf{x} = \lim_{i \to \infty} \mathbf{x}\mathsf{A}^{-l_i}\cdot\mathsf{A}^{l_i} = \mathbf{x}^*\cdot\mathbf{u}'\mathbf{v} = \mathbf{v};$$

i.e., $\mathbf{x}$ must be a positive eigenvector of A. The theorem follows.

B. For the case where A is irreducible but has an eigenvalue $\lambda \neq g$ and $|\lambda| = g$, without loss of generality, according to the theorem of Perron–Frobenius, we may assume that

$$\mathsf{A} = \begin{pmatrix} 0 & \mathsf{A}_{12} & 0 & \cdots & 0 \\ 0 & 0 & \mathsf{A}_{23} & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & \mathsf{A}_{q-1,q} \\ \mathsf{A}_{q1} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence,

$$\mathsf{A}^q = \begin{pmatrix} \mathsf{A}_{12}\mathsf{A}_{23}\cdots\mathsf{A}_{q-1,q}\mathsf{A}_{q1} & 0\cdots & 0 \\ 0 & \mathsf{A}_{23}\cdots\mathsf{A}_{q1}\mathsf{A}_{12}\cdots & 0 \\ 0 & & \mathsf{A}_{q1}\mathsf{A}_{12}\cdots\mathsf{A}_{q-1,q} \end{pmatrix}.$$

Since A is nonsingular, then $n = rq$ and $\mathsf{A}_{i,i+1}$ ($1 \leq i \leq q$; put $\mathsf{A}_{q,q+1} = \mathsf{A}_{q,1}$ are $r \times r$ nonsingular matrices. If $r = 1$, A is a generalized permutation matrix. Now we assume $r > 1$.

By the theorem of Perron–Frobenius, we have one and only one positive eigenvector $\mathbf{u}$ of A such that $\mathbf{u}\mathsf{A} = g\mathbf{u}$. We break $\mathbf{u}$ into

$$\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q),$$

where $\mathbf{u}_i$ ($1 \leq i \leq q$) are r-vectors. Putting $\mathbf{u}_{q+1} = \mathbf{u}_1$, we have

$$\mathbf{u}_i\mathsf{A}_{i,i+1} = g\mathbf{u}_{i+1}, \qquad 1 \leq i \leq g.$$

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Abbreviations: PEM, positive eigenvector method; PEV, positive eigenvector.

Therefore,

$$\mathbf{u}_2 = g^{-1}\mathbf{u}_1\mathsf{A}_{12},$$
$$\mathbf{u}_3 = g^{-1}\mathbf{u}_2\mathsf{A}_{23} = g^{-2}\mathbf{u}_1\mathsf{A}_{12}\mathsf{A}_{23},$$
$$\cdots$$
$$\mathbf{u}_q = g^{1-q}\mathbf{u}_1\mathsf{A}_{12}\mathsf{A}_{23}\cdots\mathsf{A}_{q-1,q},$$

and

$$g^q\mathbf{u}_1 = \mathbf{u}_1\mathsf{A}_{12}\mathsf{A}_{23}\cdots\mathsf{A}_{q-1,q}\mathsf{A}_{q,1} \qquad (= \mathbf{u}_1\mathsf{B}_1\text{, say}).$$

$\mathbf{u}_1$ is a positive eigenvector of B_1. Suppose B_1 has another non-negative eigenvector $\mathbf{v}_1$, $\mathbf{v}_1 \neq \alpha\mathbf{u}_1$ for any α. Let g_1^q be the positive eigenvalue corresponding to $\mathbf{v}_1$. We define $\mathbf{v}_i$ by

$$\mathbf{v}_i\mathsf{A}_{i,i+1} = g_1\mathbf{v}_{i+1}, \qquad 1 \le i \le q.$$

Then $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_q)$ is another non-negative eigenvector of A. This is impossible. Since irreducibility is equivalent to the existence of a unique non-negative eigenvector (it is indeed positive), B_1 must be irreducible. B_1 has g^q as its eigenvalue and the absolute value of other eigenvalues of B_1 are less than g^q; therefore, it satisfies the requirements in *A*. Similarly, $\mathsf{B}_i = \mathsf{A}_{i,i+1}\mathsf{A}_{i+1,i+2}\cdots\mathsf{A}_{i-1,i}$, which has g^q as its eigenvalue with the positive eigenvector $\mathbf{u}_i$, also satisfies the requirements in *A*. If $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_q) \neq (\mathbf{u}_1, \ldots, \mathbf{u}_q)$, then there is an i such that $\mathbf{x}_i \neq \mathbf{u}_i$, $\mathbf{x}_i\mathsf{B}_i^{-l}$ is a vector with components of different signs for sufficiently large l by *A*. The theorem is thus proven.

Section 2. The Positive Eigenvector Method

Let us consider an economic system. Usually the products are divided into two categories: I, products necessary for reproduction activities, and II, final consumption. Let us start with a system without final consumption and with sectors 1, 2, ..., n. The production of the ith sector is measured in its own units and is denoted by x_i. Let $\mathbf{x}$ be the n-vector with components x_i.

We take a unit of time—for example, a year. We use $\mathbf{x}^{(j)}$ to denote the collection of products in the jth year. The progression of production from year to year is given by $\mathbf{x}^{(0)} \to \mathbf{x}^{(1)} \to \cdots \to \mathbf{x}^{(l)} \cdots$. If we assume that to produce a unit of the ith sector requires a_{ij} units of inputs from the jth sector, then we have

$$\mathbf{x}^{(l)} = \mathbf{x}^{(l+1)}\mathsf{A}, \qquad \mathsf{A} = (a_{ji}).$$

Consequently, we have

$$\mathbf{x}^{(l)} = \mathbf{x}^{(0)}\mathsf{A}^{-l}. \qquad [5]$$

The matrix A is called the structural matrix (or matrix of consumption coefficients) for the system. In general, A is an irreducible non-negative matrix.

Since it is impossible for $\mathbf{x}^{(l_0)}$ to have negative components, the economic meaning of the preceding theorem is apparent. If an economic system makes no technical progress (i.e., if A remains unaltered), then the system will break down sooner or later except in the case in which the initial state $\mathbf{x}^{(0)}$ is a positive eigenvector. If $\mathbf{x}^{(0)}$ is a positive eigenvector, then $\mathbf{x}^{(l)} = g^{-l}\mathbf{x}^{(0)}$ for *all* $l \ge 0$. That is to say, the production of each sector for any year is equal to the production of the previous year times factor g^{-1}.

We shall call this method the positive eigenvector method, abbreviated PEM. We will use the term PEV to denote positive eigenvector.

Table 1.

	Agriculture (A)	Manufacturing (M)	Rate of annual increase A	Rate of annual increase M
Initial	45	20	—	—
1st year	100	50	2.2	2.5
2nd year	307.7	57.7	3.08	1.15
3rd year	−532.5	1102.1	Production cannot be sustained	

Since the PEV is unique except for a scalar multiple, an economic system has one and only one "correct" proportion between different sectors.

Section 3. Adjustment

The advantage of the PEM is that we can discover the defects of an economic system early and make prompt adjustments, thus avoiding awkward situations in which we are forced to retreat at great loss. We can best illustrate the method of adjustment by an example.

Example. Our system has two sectors—1 is agriculture, 2 is manufacturing. We start with values of 45 and 20—i.e., sectors 1 and 2 have 45 and 20 units of product, respectively. We assume that, for sector 1 to produce one unit, we need 0.25 and 0.14 unit of product from sectors 1 and 2, respectively, while for sector 2 to produce one unit, we need 0.40 and 0.12 unit of product from sectors 1 and 2, respectively. The matrix of consumption coefficients is then

$$\mathsf{A} = \begin{pmatrix} 0.25 & 0.14 \\ 0.40 & 0.12 \end{pmatrix}. \qquad [6]$$

From

$$(45, 20)\ \mathsf{A}^{-l}, \qquad [7]$$

we obtain the results given in Table 1. This table shows that in the first year the situation is not bad, in the second year the balance is in danger, while in the third year the negative sign appears and the system collapses. What can we do? The negative sign appears in agriculture, so perhaps someone might suggest boosting agriculture production. No! If we do so, we will go from bad to worse! The right way is the following. We give up 0.656 unit from the agricultural sector to start with and obtain the results given in Table 2.

Table 2 shows that production increases by a multiple of 2.323 in the first 4 years but that the system then begins to lose its balance until, in the 8th year, the negative sign again

Table 2.

	Agriculture (A)	Manufacturing (M)	Rate of annual increase A	Rate of annual increase M
Initial	4.344	20	—	—
1st year	103.02	46.467	2.323	2.323
2nd year	239.37	107.95	2.323	2.323
3rd year	566.11	250.86	2.323	2.323
4th year	1292.8	582.24	2.324	2.320
5th year	2990.6	1362.9	2.313	2.340
6th year	7165.7	2998.2	2.395	2.199
7th year	13054	9754.7	1.821	3.253
8th year	89821	−23501	Production cannot be sustained	

appears. If we use $\mathbf{x}^{(0)} = (44.34397483, 20)$, then on calculation we find that the situation is stable up to the 8th year and that breakdown occurs in the 13th year. Also if we add $\pm 10^{-8}$ to the first component, the negative sign appears 1 year earlier. This fact shows the delicacy of the economic system. If we "prune" earlier, the loss will be less and the rate of growth, higher. Certainly how much we should give up is determined by the PEM. The exact initial vector should in fact be

$$\left(\frac{5}{7}\left(\sqrt{2409} + 13\right), 20\right) \approx (2.217198742, 1)$$

which is the PEV of A. The inverse of the corresponding positive eigenvalue is $\frac{5}{26}(\sqrt{2409} - 37) \doteq 2.32337189$, which is the growth multiple.

Let us go back to Table 1. If we do not take action at the beginning but do so in the 1st year, the correct vector should be

$$(100, 45.102).$$

Hence we give up $50 - 45.102 = 4.898$ manufactured units. If we do not take action in the 1st year but do so in the 2nd year, the correct vector becomes

$$(127.9, 57.7).$$

We have to give up $307.7 - 127.9 = 179.8$, more than half of the agricultural output. If we still do not take action, then the crisis will come. Therefore the earlier we take action, the less will be the loss.

Here we use the phrase "give up"—it does not mean really to "throw away into the sea." Sometimes we can use it for trade. In this case, we could export agricultural output and import manufactured units to keep

$$(45 - \alpha):(20 + \beta) = 2.2172:1;$$

i.e.,

$$2.2172 \times (20 + \beta) = 45 - \alpha.$$

If the market prices are A and B dollars for one unit of agricultural and manufactured output, respectively, then we have another equation

$$\alpha \mathsf{A} = \beta \mathsf{B}.$$

We solve the simultaneous equations with unknowns α, β, whereby the system is kept in the PEV position with a balanced balance of payments.

We took trade as an example, but there are many ways in which we can achieve balance. The most important principle is to consider how best to benefit the people, so as to raise their standard of living.

Section 4. The PEM and the Economic System

The example of *Section 3* and the theorem of *Section 1* indicate the importance of the PEM. Not only category I sectors but also category II sectors have to be maintained in the PEV position. The capacity vector should also be in the PEV position. By capacity, we mean that, during the jth year, because of the limitation of facilities, materials, and energy resources of ith sector, we cannot produce more than $\xi_i^{(j)}$ units of output. The capacity vector $\xi^{(j)} = (\xi_i^{(j)}, \ldots, \xi_n^{(j)})$ is then the constraint on the system.

However, we have to look at the problem dynamically. The exact data cannot be achieved. The coefficients of consumption may vary.* Just like an old clock, after we adjust it to the right time, we have to repeatedly adjust it at certain time intervals. The PEV is considered as an instantaneous position (what a big instant, it is a year!). We then have to adjust the plan every year (and sometimes semiannual or seasonal checks are needed).

Section 5. The Measure of Deviation from the PEV

The adjustment is not so simple if an economic system is in a very bad condition (chaos). However, if the PEM is recognized, the following technique will also be helpful.

To measure the deviation of a vector $\mathbf{x}$ from the PEV position $\mathbf{u}$, we consider

$$d^2 = \min_{\alpha}(\mathbf{x} - \alpha\mathbf{u})(\mathbf{x} - \alpha\mathbf{u})'$$

where p' denotes the transposed matrix (or vector). It is not difficult to prove that

$$d = d(\mathbf{x}) = [\mathbf{x}\mathbf{x}' - (\mathbf{u}\mathbf{x}')^2/\mathbf{u}\mathbf{u}']^{1/2} \text{ for } \alpha = \mathbf{u}\mathbf{x}'/\mathbf{u}\mathbf{u}'. \quad [8]$$

We can think of d as the measure of deviation of $\mathbf{x}$ from the PEV.

We can then use the difference

$$x_i - u_i\mathbf{u}\mathbf{x}'/\mathbf{u}\mathbf{u}' \quad [9]$$

to determine how much we should modify the level of production of the ith sector.

Note that the necessary and sufficient condition for $\mathbf{x}$ to be an eigenvector is $d = 0$. It suggests that the smaller the value of d the better.

Section 6. Method of Adjustment

To give up some product or to close some factories are ways of obtaining equilibrium. There is a method, known in the theory of non-negative matrices, that is helpful for making adjustments. Let

$$\tau_j = \frac{\sum_{i=1}^{n} x_i a_{ij}}{x_j} \quad [10]$$

and $L = \min_{1 \le j \le n} \tau_j$, $M = \max_{1 \le j \le n} \tau_j$. If $L = M$, then the economic system is in equilibrium. If $L < M$, the method can narrow the gap between L and M. We can then find, under present conditions, a better proportion between different sectors than the existing one.

Planners should always keep in mind the capacity constraint $x_i \le \xi_i$. Even the simple formula of Leontief $y = x(I - A)^{-1}$ would make no sense without capacity constraints. Any plan or modification of a plan cannot violate these constraints. If $x_j \not\le \xi_j$ for some j, then we should focus our attention on enlarging capacity. Whatever is required to increase the capacity of ξ_j should be taken into account in the plan. If we cannot meet the requirements necessary for enlargement, then for the time being, we cannot use the proportions suggested by the PEV. The adjustment suggested in this section should then be used.

*Let $\lambda(\mathsf{A})$ be the eigenvalue of the PEV of A. If $\mathsf{A} \ge \mathsf{B}$, then $\lambda(\mathsf{A}) \ge \lambda(\mathsf{B})$. Therefore, any improvement in technology will increase the rate $1/\lambda(\mathsf{A})$ of growth of the system.

Section 7. Systems with Final Consumption

At first sight, the system with final consumption would seem to be more difficult. However, mathematically, if the irreducible part is analyzed carefully, the remaining part can be treated easily.

The formulation is

$$\mathbf{x}^{(l)} - \eta^{(l)} = \mathbf{x}^{(l+1)}\mathsf{A}, \qquad l = 0, 1, 2, \ldots, \qquad [11]$$

where $\eta^{(l)}$ is the final consumption vector and consists of government expenditure (including social welfare), construction, exports, depreciation, and such

Let $\beta^{(l)}$ be a vector defined by

$$\eta^{(l)} = \beta^{(l+1)}\mathsf{A} - \beta^{(l)}, \qquad l = 0, 1, 2, \ldots \qquad [12]$$

recurrently with undetermined $\beta^{(0)}$. By using Eqs. **11** and **12**, we obtain

$$\mathbf{x}^{(l)} + \boldsymbol{\beta}^{(l)} = [\mathbf{x}^{(l+1)} + \boldsymbol{\beta}^{(l+1)}]\mathsf{A}.$$

If we choose $\beta^{(0)}$ so that $\mathbf{x}^{(0)} + \beta^{(0)}$ is the PEV, then we have

$$\mathbf{x}^{(l)} + \boldsymbol{\beta}^{(l)} = g^{-l}[\mathbf{x}^{(0)} + \boldsymbol{\beta}^{(0)}]. \qquad [13]$$

The problem is now reduced to the calculation of $\beta^{(l)}$ from $\eta^{(l)}$. By induction, we can prove that

$$\boldsymbol{\beta}^{(l+1)} = [\eta^{(0)} + \boldsymbol{\beta}^{(0)}]\mathsf{A}^{-(l+1)} + \boldsymbol{\eta}^{(1)}\mathsf{A}^{-l} \cdots \boldsymbol{\eta}^{(l)}\mathsf{A}^{-1}.$$

If the ηs are the PEV, the calculation is particularly simple. Note that we have to examine the positiveness of $\mathbf{x}^{(l)}$.

Section 8. Prices

Abstractly, in our economic system we take a currency unit $\mathbb{B}$. The price of each unit of the jth sector is $q_j\mathbb{B}$. Then collectively we have a price vector $\mathbf{q}$. The total value of $\mathbf{x}^{(0)}$ is equal to

$$\mathbf{x}^{(0)}\mathbf{q}'.$$

If $\mathbf{x}^{(0)}$ is the PEV of A, then

$$\mathbf{x}^{(1)}\mathbf{q} = \mathbf{x}^{(0)}\mathsf{A}^{-1}\mathbf{q}' = g^{-1}\mathbf{x}^{(0)}\mathbf{q}'.$$

In general

$$\mathbf{x}^{(l)}\mathbf{q}' = g^{-l}\mathbf{x}^{(0)}\mathbf{q}'.$$

Therefore, the rate of growth of total value is equal to g^{-1}, which is independent of the choice of currency.

We shall now talk about economic prices. However, we should take care not to mix these prices with market prices. We shall indicate a way of calculating prices in terms of $\mathbb{B}$.

Let $\mathbf{q}$ be the price vector we are looking for. Since, to produce one unit in the ith sector requires a_{ij} units from the jth sector ($1 \le j \le n$), the cost of producing one unit in the ith sector is equal to

$$\sum_{j=1}^{n} a_{ij}q_j.$$

We denote this cost by $q_i^{(1)}$. If the price of each sector varies by the same multiple λ, then we have

$$\mathsf{A}\mathbf{q}' = \lambda\mathbf{q}',$$

from which we deduce that $\mathbf{q}'$ is the right PEV of A and $\lambda = g$. Therefore, the price vector in the lth year is equal to

$$q^l\mathbf{q}_0, \qquad [14]$$

where $\mathbf{q}_0$ is the initial price vector (which is a right-hand PEV of A).

This is of course far from the actual market price because we have not considered the final consumption, competition, and monopolies.

All three of the characteristic properties of the theory about irreducible non-negative matrices—i.e., the biggest positive eigenvalue, left-hand PEV, and right-hand PEV—have economic meanings—i.e., the inverse of the rate of growth, the proportion between different sectors, and costs (prices).

Like the input–output theory of Leontief (1), the PEM is a method that sees the forest but not the trees. We have to use many other methods to link the general plan with subordinate plans.

According to the price vector in Eq. **14**, it would seem that prices ought to become lower and lower. However, the real situation is often exactly the opposite. The reason for this is that final consumption is not included above and that inflation has been ignored. If we use the matrix A as obtained from item *iii* of *Section 9*, then the price thus computed would be nearer to to the total cost price. However, if we use $\mathbb{B}$ as the numeraire, then each country can have a currency as some rate of $\mathbb{B}$. For the system with currency $\mathbb{B}$, the price vector would be decreasing. Let $\mathbf{q}^{(0)}$ be the initial price vector and $\mathbf{q}^{(1)}$ be the price vector for the next year. The net profit is

$$\mathbf{f} = \mathbf{q}^{(0)} - \mathbf{q}^{(1)} = \mathbf{q}^{(0)}\mathbf{q}^{(0)}\mathsf{A}' = \mathbf{q}^{(0)}(\mathsf{I} - \mathsf{A}')$$

and

$$\mathbf{q}^{(0)} = \mathbf{f}(\mathsf{I} - \mathsf{A}')^{-1};$$

this is a dual formula to Leontief's famous formula.

Section 9. Some Remarks Concerning Economics

First, from the PEM, we deduce that

$$1/g = x_1^{(l+1)}/x_1^{(l)} = \cdots = x_n^{(l+1)}/x_n^{(l)},$$

which means that each sector varies by the same rate. More generally, if each sector has its own rate of change, then

$$\lambda_1\rho = x_1^{(l+1)}/x_1^{(l)}, \cdots, \lambda_n\rho = x_n^{(l+1)}/x_n^{(l)}$$

and, without loss of generality, we may assume that

$$(\lambda_1 + \cdots + \lambda_n)/n = 1.$$

From Eq. **15** we can easily deduce that

$$\rho\mathbf{x}^{(l)}\,\Lambda = \mathbf{x}^{(l+1)} = \mathbf{x}^{(l)}\mathsf{A}^{-1},$$

where $\Lambda = [\lambda_1, \ldots, \lambda_n]$. In this way, the new problem is reduced to the original one with $\Lambda\mathsf{A}$ instead of A.

Similar methods can be used for the price vector.

Second: there are three methods of obtaining consumption coefficients.

(*i*) Statistical methods.

(*ii*) Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the input vector; x_{ji} units from x_i

are input into the jth sector, that is

$$x_i = \sum_{j=1}^{n} x_{ji}.$$

Let $\mathbf{y} = (y_1, \ldots, y_n)$ be the output vector. We define

$$a_{ji} = x_{ji}/y_j.$$

(*iii*) Let $\mathbf{y} - \eta$ be the net output and define

$$a_{ji} = x_{ji}/(y_j - \eta_j).$$

Each method is more convenient than the preceding one. *ii* may cover the waste and loss in transportation and such. *iii* covers much more.

Third: how does one make a modified input–output table and calculate the PEV? How does one compute the year in which balance begins to be lost and the year when economic crisis strikes? The answers to these questions though available are too detailed to be included here.

Because of limited space, I have not been able to put in too many details. In particular, a wide range of miscellaneous application experiences has been accumulated over the last two decades in conjunction with thousands of other workers in different sectors of the economy. Nevertheless, these experiences are vital in forming the links between the "forest" and the "trees" down below. The aim of this paper is simply to prepare one of the stepping stones for others. Although I am anxious to do something about economic planning, my own limited knowledge of economics puts me in a position where I may face one of the failures of my academic life.

Section 10. Some Remarks Concerning Mathematics

1. Markov chains are an important concept in the theory of probability. However, in applications, they are often used in dubious ways. For example, some statisticians determine a Markov matrix A from the data of just a few years. They then use

$$\mathbf{p}^{(m)} = \mathbf{p}^{(0)}\mathsf{A}^m, \qquad m = 0, 1, 2, \ldots$$

to predict future behavior, where $\mathbf{p}$ is a probability vector—i.e., the vector $\mathbf{u} = (1, \ldots, 1)$, $\mathbf{pu}' = 1$ and $\mathsf{A}\mathbf{u}' = \mathbf{u}'$. The theorem of *Section 1* shows that they really need to ask themselves when the chain starts. If, for some reason, they have started at $m = -l_0 + 1$, they ought to answer the following question: what happens at the year $-l_0$?

Similar problems occur in nuclear fission.

2. Let $\mathrm{K}(x, y)$ be a positive kernel and define inductively

$$\mathrm{K}_l(x, y) = \int_a^b \mathrm{K}(x, t)\mathrm{K}_{l-1}(t, y)dt.$$

Then, for any $g(x) > 0$, there exists an l_0 such that

$$g(x) = \int_a^b \mathrm{K}_l(x, y)f(y)dy, \qquad l > l_0$$

is insoluble for positive f except when $g(x)$ is a positive eigenfunction of $\mathrm{K}(x, y)$—i.e.,

$$\lambda g(x) = \int_a^b \mathrm{K}(x, y)g(y)dy.$$

Note that the relevant class of functions is not stated precisely here.

3. The mapping

$$\mathbf{y} = \frac{\mathbf{x}\mathsf{A}}{\sigma(\mathbf{x}\mathsf{A})}$$

where $\sigma(\mathbf{x}) = \Sigma_{i=1}^{n} x_i$, maps the compact simplex

$$x_1 + \cdots x_n = 1, \qquad x_i \geq 0$$

into itself. If we use Brouwer's fixed point theorem, we can deduce only the existence of a fixed point, because Brouwer's theorem applies under very general conditions.

In our case, the fixed point of our mapping is unique and interior to the simplex. Furthermore, no limiting cycle can exist. This is the topological meaning of our theorem.

Franklin (2) summarized this approach in a few sentences: "From Walras to Debreu (3), economists talk about equilibrium. First, the economy is described by a state variable $\mathbf{x}$ in an appropriate mathematical space. When something happens in the economy, the state $\mathbf{x}$ goes to a state $\mathbf{f}(\mathbf{x})$. An equilibrium is a state that stays fixed: $\mathbf{x} = \mathbf{f}(\mathbf{x})$." In the present article, the state is the proportion of products in different sectors. PEV is the equilibrium.

Instead of Brouwer's theorem, which others have used, I have used the Perron–Frobenius theorem and the theorem of *Section 1* in formulating our case. We not only obtained the equilibrium but also its uniqueness. Furthermore, it was proved that, if we are not at the equilibrium position, the system will become unbalanced and crisis will result.

4. Mathematicians like to fill up the gaps and to generalize theorems. In the theorem of *Section 1*, we discussed the primitive A in the first part and the irreducible A in the second part. I now give an example for reducible A. Let $\alpha > 0$, $\beta > 0$, $\mathbf{x} = (x, 1)$ and

$$\mathsf{A} = \begin{pmatrix} 1 & 0 \\ \alpha/\beta & 1/\beta \end{pmatrix}.$$

Then

$$(x, 1)\mathsf{A}^{-l} = (x, 1)\begin{pmatrix} 1 & 0 \\ \dfrac{-\alpha(1-\beta^l)}{1-\beta} & \beta^l \end{pmatrix}$$
$$= [x - \alpha(1-\beta)^{-1}, 0] + \beta^l[\alpha(1-\beta)^{-1}, 1].$$

For $\beta > 1$, we have no x such that $(x, 1)\mathsf{A}^{-l} > 0$ for all l. Therefore, there is no way to plan the system without crisis. For $\beta < 1$ and $x > \alpha/(1 - \beta)$ will be a set of dimension greater than one, each of which satisfies $(x, 1)\mathsf{A}^{-l} > 0$ for all l.

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