Lectures on

STATISTICAL MECHANICS

by

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errors and lack of clarity are the sole responsibility of the note takers.

Bibliography:

1. P. & T. Ehrenfest - The Conceptual Foundations of the Statistical Approach in Mechanics - Cornell Press.
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3. P. R. Halmos - Lectures on Ergodic Theory - Chelsey
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The purpose of these lectures will be to go over some questions on the foundations of the subject in order to reveal the present state of knowledge. We will find out the kind of thing Boltzmann did and said and then what the problems still to be settled are. The first part will, therefore, be historical.

Outline:

1. Elementary Kinetic Theory:

   This will show the type of assumptions that are involved. Of importance here is Boltzmann's H theorem which concerns the approach to equilibrium.

2. Ergodic Theory:

   Several of the original problems are now solved and these will be discussed.

3. Introduction to models of classical fluids: Experimental work with computers on hard sphere gas indicates that it condenses into a solid.

4. Existence questions in classical and quantum statistical mechanics.

5. Along the way some practical transport theory will be thrown in.

Historical Sketch:

185[illegible] Clausius Maxwell: This period is dominated by the "Hypothesis of Molecular Chaos." It resulted in calculation of viscosity, diffusion and thermal conductivity; Maxwell-Boltzmann distribution.

1872 - Boltzmann: Announced the H-Theorem which was supposed to give an explanation of the 2nd Law. Soon after there appeared criticisms of this theorem dominated by the names of:

Loschmidt Reversibility

Poincaré } Recurrence Paradox or Wiedereinwand
Zermelo

Boltzmann then formulated a statistical interpretation which evaded these objections and also gave a statistical explanation of irreversibility

1931-1932 - von Neumann & Birkhoff produced their Ergodic Theorems

They succeeded in showing that time averages existed and under certain assumptions equalled phase space averages. Their assumption is now called: Irreducibility of Flow or Metric Transitivity. However, not many flows could be shown to be of this kind.

1958 Kolmogorov introduced the important concept of "Entropy per unit time" and what is now called a K-system. Further work showed every K-system is ergodic and has positive entropy per sec.

1963 Sinai succeeded in showing that hard spheres in a box form a K-system.

In this lecture we will derive certain useful equations.

1. Virial Equation of State - Clausius 1870.

This will illustrate the use of the Hypothesis of Molecular Chaos.

Physically we have a box of volume V containing N particles interacting with each other via a potential and with the walls via a potential..

$$m_i \frac{d^2\vec{x}_i}{dt^2} = \vec{F}_i \qquad i = 1,\ldots,n \tag{1.1}$$

$$\vec{F}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{\nabla}_i V_{ij}(x_i - x_j) - \vec{\nabla}_i \underset{\uparrow}{V}(x_i) \tag{1.2}$$

effect of the wall.

Multiply (1.1) with $\vec{x}_i$ and sum

$$\sum_{i=1}^{n} m_i \vec{x}_i \frac{d^2\vec{x}_i}{dt^2} = \sum_{i=1}^{n} \vec{x}_i \cdot \vec{F}_i \tag{1.3}$$

Now average over the time interval (o,t)

$$\therefore \frac{1}{\tau}\int_0^\tau dt \sum_{i=1}^{n} m_i \vec{x}_i(t)\cdot\frac{d^2\vec{x}_i}{dt^2}(t) = \frac{1}{\tau}\left[\sum_{i=1}^{n} m_i \vec{x}_i(t)\cdot\frac{d\vec{x}_i}{dt}(t)\Big|_0^\tau\right] - \frac{1}{\tau}\int_0^\tau dt \sum_{i=1}^{n} m_i \left(\frac{d\vec{x}_i}{dt}\right)^2(t)$$

We now assume that $\vec{x}_i$ and $\frac{d\vec{x}_i}{dt}$ are bounded and pass to the limit $\tau \to \infty$.

Then we get:

$$\overline{K.E.} = \text{Virial} = -\tfrac{1}{2} \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau dt \sum_{i=1}^{n} \vec{x}_i \cdot \vec{F}_i \tag{1.4}$$

We still don't know $\overline{K.E.}$ and $\frac{1}{\tau}\int_0^\tau dt \sum \vec{x}_i \cdot \vec{F}_i$ have limits separately. i.e., we only know that their difference has a limit. So this is a formal derivation. Later by use of the Ergodic Theorem we will show that this limit does exist.

Now split the potential V = V particle + V wall where

$$\overline{-\frac{1}{2}\sum_{i=1}^{n} \vec{x}_i \cdot \vec{F}_{i\,wall}} = \overline{+\frac{1}{2}\sum_{i=1}^{n} \vec{x}_i \cdot \vec{\nabla}_i V(x_i)}$$

By the assumption of molecular chaos, each term in the sum gives the same contribution, namely

$$\frac{1}{2}\,\frac{1}{n}\int_{surface} x.ds\,P$$

where P is the pressure.

[illegible]

$$\int_{\text{walls}} \mathbf{x} \cdot d\mathbf{s} = 3\mathcal{V} \qquad \mathcal{V} \text{ is the volume}$$

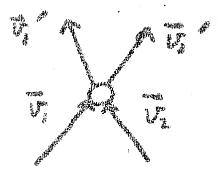
Hence:

$$p\mathcal{V} = \frac{2}{3}\,\overline{\text{K.E.}} - \frac{2}{3}\,\overline{V\ \text{particle}} \qquad (8\ )$$

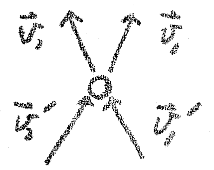
2. Maxwell-Boltzmann Distribution

We shall use detailed balance in the following form: Collisions and Inverse Collisions occur at the same rate in equilibrium. We shall only consider binary collisions. Inverse Collisions are defined by the diagram below

Collision: C

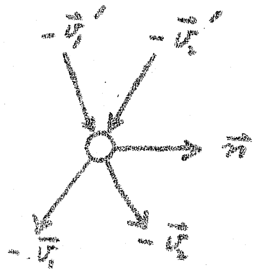


Inverse Collision: IC



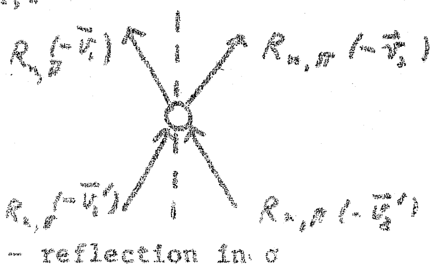
We use time-inversion T, space reflection P, and rotation $R_{\vec{n},\theta}$ to show how to get from C to IC.

a) T



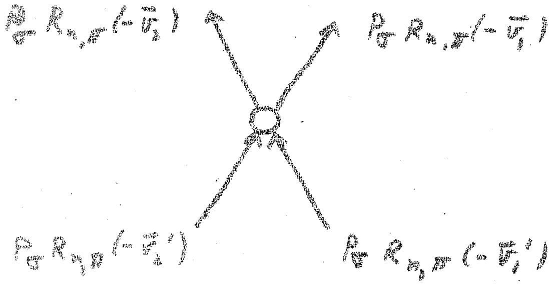
n is a unit vector normal to the C.M. velocity

b) $R_{\vec{n},\pi}$ – rotation by π about $\vec{n}$



The dashed line here denotes a plane σ normal to $\vec{n}$.

c) P_σ – reflection in σ



This is the inverse collision.

So if the forces have invariance under T, P, and R, then [illegible] are equal and detailed balance is valid in the form stated. This is the case for molecular forces which are electromagnetic in origin.

We now apply this in 1 dimension (in the case of 3 dimensions see K. Huang - Statistical Mechanics) and use a typical argument.

Assume there are 2 kinds of molecules: 1 and 2 with masses m_1 and m_2 respectively.

Let

$M_1(v)\ dv$ = number of 1 molecules in dv

$M_2(v)\ dv$ = number of 2 molecules in dv

At this stage we have made a continuum model of what is in fact a discrete distribution. The collision considered takes

$$v_1, v_2 \rightarrow v_1', v_2'$$

Using detailed balance and cancelling cross-sections we get

$$M_1(v)dv_1\ M_2(\quad_2)\ dv_2 = M_1(v_1')\ dv_1'\ M_2(v_2')\ dv_2' \qquad (2.1)$$

Conservation Laws for the collision give:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad (2.2)$$

$$\tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 v_1'^2 + \tfrac{1}{2} m_2 v_2'^2 \qquad (2.3)$$

Define:

$$V_{cm} = [m_1 + m_2]^{-1}\ [m_1 v_1 + m_2 v_2] \qquad (2.4)$$

$$V\ rel = v_1 - v_2 \qquad (2.5)$$

Then

$$v_1 = v_{cm} + \mu_1\ v\ rel \qquad (2.6)$$

$$v_2 = V_{cm} - \mu_2\ v\ rel \qquad (2.7)$$

with

$$\mu_1 = \frac{m_2}{m_1 + m_2} \qquad (2.8)$$

$$\mu 2 = \frac{m_1}{m_1 + m_2} \qquad (2.9)$$

[illegible]

$$v_{cm} = v'_{cm} \tag{2.10}$$

V^2 rel $-$ V'^2 rel and since we are interested in the case in which a collision actually occurs

$$V \text{ rel} = - V' \text{ rel} \tag{2.11}$$

Now write

$$M_j (v) = \exp L_j(v) \tag{2.12}$$

and assume $M_j(v)$ (or $L_j(v)$) has 3 derivatives. More general assumptions are also possible. We have in fact also assumed $M_j(v) > 0$.

(2.1) can now be rewritten to read:

$$\begin{aligned} &L_1 (v_{cm} + \mu_1 \, v \text{ rel}) + L_2 (v_{cm} - \mu_2 \, v \text{ rel}) \\ &= L_1 (v_{cm} - \mu_1 \, v \text{ rel}) + L_2 (v_{cm} + \mu_2 \, v \text{ rel}) \end{aligned} \tag{2.13}$$

We now show that the most general solution of (2.13) is a quadratic in V. To do this, differentiate (2.13) with respect to V rel 3 times and set V rel = 0.

1 differentiation:

$$\mu_1 L_1' (v_{cm}) = \mu_2 L_2' (v_{cm}) \tag{2.14}$$

3 differentiations:

$$\mu_1^3 L_1''' (v_{cm}) = \mu_2^3 L_2''' (v_{cm}) \tag{2.15}$$

2 differentiations yields an identity.

(2.14) and (2.15) combined imply that

$$L_j''' (v_{cm}) = 0 \qquad \text{for all } v_{cm}$$

$$L_j(v) = a_j + b_j v + c_j v^2 \tag{2.16}$$

Differentiating (2.16) and comparing with (2.14) we conclude that

$$b_j = m_j b \quad \text{and} \quad c_j = m_j c$$

Hence

$$M_j = d_j \exp [c m_j (v - v^{(0)})^2] \tag{2.17}$$

Comment:

In the derivation we didn't worry about the positions of the particles etc. and it is not clear what explicit microscopic assumptions are involved.

3. Boltzmann's Equation

Using the local conservation laws for particles we want to find an equation for the distribution function $f(\vec{x}, \vec{v}, t)$.

$f(\vec{x}, \vec{v}, t)$ = fraction of particles in $d\vec{x}\, d\vec{v}$ at time t. Traditionally the 6 dimensional $(\vec{x}, \vec{v})$ space is called μ-space, and the 6N dimensional $(x_1, \ldots \vec{x}_n, \vec{p}_1, \ldots, \vec{p}_n)$ space is called Γ space. The equation will be determined by the motion (drift and collisions) of particles in μ-space. Thus we write

$$\frac{\partial f}{\partial t}(\vec{x}, \vec{v}, t) = \left(\frac{\partial f}{\partial t}\right)_{drift} + \left(\frac{\partial f}{\partial t}\right)_{coll.} \tag{3.1}$$

where

$$\left(\frac{\partial f}{\partial t}\right)_{drift} = -\vec{v} \cdot \vec{\nabla}_x f - \left(\frac{\vec{F}}{m}\right) \cdot \vec{\nabla}_v f \tag{3.2}$$

Up to this point, the only statistical assumption involved is the introduction of the function f. Also notice that when the direction of $\vec{v}$ and increasing f agree, we lose particles from dv. This is the reason for the minus sign in the first term of (3.2). A similar reason holds for the minus sign in front of the second term.

To obtain an expression for ($\frac{\partial f}{\partial t}$) coll. we need more statistical assumptions.

One of these is the Stosszahlansatz. We also make the approximation of considering densities at a point $\vec{x}$ instead of a region about $\vec{x}$. Also only binary collisions are considered.

Now write:

$$\left(\frac{\partial f}{\partial t}\right)_{coll.} = \left(\frac{\partial f}{\partial t}\right)_{coll.\,out} + \left(\frac{\partial f}{\partial t}\right)_{coll.\,in} \tag{3.3}$$

Consider molecules at x, v and x_1, v_1. For a molecule at (x, v) there is a beam flux

$$f(\vec{x}, \vec{v}_1, t)\, d\vec{v}_1\, |\vec{v}_1 - \vec{v}|\, \sigma(\vec{v}, \vec{v}_1 : \vec{v}', \vec{v}_1')\, d\Omega' \tag{3.4}$$

To get the number of collisions / unit volume in μ-space we must multiply (3.4) by the density of particles at $(\vec{x}, \vec{v})$. Hence we get that the number of collisions/sec/unit volume of μ-space at $(\vec{x},\vec{v})$ which carry particles from $\vec{v}$, $\vec{v}_1$ to the neighborhood of $\vec{v}'$, $\vec{v}_1'$ is:

$$f(\vec{x},\vec{v},t)\, f(\vec{x},\vec{v}_1,t)\, d\vec{v}_1\, |\vec{v}_1-\vec{v}|\, \sigma(\vec{v},\vec{v}_1\,;\,\vec{v}',\vec{v}_1')\, d\Omega' \tag{3.5}$$

∴ No. of collisions/sec/unit volume of μ-space which lead out is:

$$\int d\vec{v}_1\, d\Omega' f\, f_1\, |\vec{v}_1-\vec{v}|\, \sigma \tag{3.6}$$

where we have introduced the shorthand notation:

$$\begin{aligned} f &= f(\vec{x},\vec{v},t) & f' &= f(\vec{x},\vec{v}',t) \\ f_1 &= f(\vec{x},\vec{v}_1,t) & f_1' &= f(\vec{x},\vec{v}_1',t) \end{aligned} \tag{3.7}$$

To get the collisions in, consider a particle at $(\vec{x},\vec{v})$ for which there is a beam of particles in the velocity range $d\vec{v}_1'$ around $\vec{v}_1'$ of flux

$$f_1'\ |\vec{v}_1' - \vec{v}'|\ d\vec{v}_1'$$

Thus there are

$f_1'\ |\vec{v}_1' - \vec{v}'|\ d\vec{v}_1'\ \sigma\ (\vec{v}', \vec{v}_1'; \vec{v}, \vec{v}_1)\ d\Omega$ collisions/sec which result in a pair of particles in $d\vec{v}_1\ d\Omega$ around $\vec{v}$ and $\vec{v}_1$. We now sum over all the particles to get:

Number of collisions/sec/unit volume of x space is

$$I = \int d\vec{v}'f_1'\ |\vec{v}_1' - \vec{v}'|\ \ \sigma'\ \ d\Omega\, d\vec{v}_1' \tag{3.8}$$

Since the collisions are elastic we can change variables to get number of collisions/sec/unit volume of μ-space. We use:

$$d\vec{v}\, d\vec{v}_1 = d\vec{v}'\ d\vec{v}_1' \tag{3.9}$$

which follows from either Liouville's theorem or a direct computation of the Jacobian.

$$|\vec{v}_1' - \vec{v}'| = |\vec{v}_1 - \vec{v}| \tag{3.10}$$

$$d\Omega = d\Omega' \tag{3.11}$$

$$\sigma = \sigma' \tag{3.12}$$

The last equation was essentially derived in lecture [illegible].

Hence

$$I = \int d\vec{v}\; f'\, f_1' \quad |\vec{v}_1 - \vec{v}|\; \sigma d\Omega'\; d\vec{v}_1 \tag{3.13}$$

Here $\vec{v}_1'$ is to be regarded as a function of $\vec{v}$. Combining (3.6) and (3.13) we obtain:

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \int d\Omega'\; d\vec{v}_1 \quad |\vec{v}_1 - \vec{v}|\, \sigma (f' f_1' - f\, f_1) \tag{3.14}$$

And so we can finally write the Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v}\cdot \vec{\nabla}_x f + \frac{\vec{F}}{m} \cdot \vec{\nabla}_v\, f = \int d\Omega'\; d\vec{v}_1 \quad |\vec{v}_1 - \vec{v}|\; \sigma \quad (f'\; f_1' - f\; f_1) \tag{3.15}$$

4. H Theorem

We shall now use (3.15) to derive Boltzmann's H Theorem. Notice, however, first that the Maxwell-Boltzmann distribution

$$f_o\; (x, \vec{v}) = C \exp \;\{-\beta(\tfrac{1}{2} m\, \vec{v}^2 + V(x)\,)\} \tag{3.16}$$

satisfies (3.15) with $F = -\vec{\nabla}_x \; V(x)$

We shall suppress the x dependence and define

$$H = \int f(\vec{v}, t)\; \log f(\vec{v}, t)\; d\vec{v} \tag{3.17}$$

The H Theorem states:

$$\frac{dH}{dt} \leq 0$$

$$\text{and} \qquad \frac{dH}{dt} = 0 \qquad \text{only if } f'\; f_1' = f\; f_1$$

The proof will depend on the fact that

$$\begin{aligned} &(x-y)\;(\log x - \log y) \;\geq 0 \\ \text{and} \qquad &\qquad = 0 \qquad \text{only for } x = y \end{aligned} \tag{3.18}$$

which is easily verified.

Proof of H Theorem

$$\frac{dH}{dt} = \int \frac{\partial f}{\partial t} [\log f + 1] \, d\vec{v} \qquad (3.19)$$

There are no drift terms.

$$\frac{dH}{dt} = \int d\vec{v}\, d\vec{v}_1\, d\Omega\, |\vec{v}_1 - \vec{v}|\, \sigma\, (f'f_1' - f f_1)[\log f + 1] \qquad (3.20)$$

Now interchange $\vec{v}$ and $\vec{v}_1$ in (3.20) and average the resultant equation with (3.20).

$$\frac{dH}{dt} = \tfrac{1}{2}\int d\vec{v}\, d\vec{v}_1\, d\Omega\, |\vec{v}_1 - \vec{v}|\, \sigma\, (f' f_1' - f f_1)\, [2 + \log f f_1] \qquad (3.21)$$

Now interchange the primed and unprimed coordinates and use (3.10), (3.11), and (3.12) and average the resultant equation with (3.21) to get:

$$\frac{dH}{dt} = \tfrac{1}{4}\int d\vec{v}\, d\vec{v}_1\, d\Omega\, |\vec{v}_1 - \vec{v}|\, \sigma\, (f'f_1' - f f_1)\, (\log f f_1 - \log f'f_1') \qquad (3.22)$$

Now $\qquad |\vec{v}_1 - \vec{v}| > 0$

$\qquad \sigma > 0$

And using (3.18) we get:

$$\frac{dH}{dt} < 0 \qquad (3.23)$$

and $\dfrac{dH}{dt} = 0$ only if $f' f_1' = f f_1$ (3.24)

One can deduce in 3 dimensions (see K. Huang) that $f'f_1' = f f_1$ implies the Maxwell-Boltzmann distribution. Thus any non-equilibrium distribution will tend to a Maxwell-Boltzmann distribution.

In the preceding lecture we have shown that on the basis of Boltzmann's equation the quantity H decreases monotonically with time and will keep on decreasing until it reaches the equilibrium value H . This result suggests that a system which is not in equilibrium will always approach equilibrium; we have here the first prong of Boltzmann's argument for arriving at thermodynamic properties from statistical mechanics. Another prong of Boltzmann's argument is the so called "Ergodic Hypothesis" introduced by Boltzmann in 1879. ("Ergodic" comes from the Greek: ΕΡΓΟΝ - work, energy; ΟΔΟΣ - path).

Ergodic Hypothesis: There is a unique orbit on a connected component of an energy surface in phase space; i.e., the orbit goes through every point. (See Truesdall's article in the Proceedings of the International School of Physics "Enrico Fermi", Course 14: Ergodic Theories, Academic Press)

The Ergodic Hypothesis aroused suspicion since it intuitively cannot be true. It was later re-interpreted to mean that the orbit is dense in the energy surface, but then one cannot so easily draw the tidy conclusions that are easily obtained from the stronger form of the hypothesis. The above re-interpretation is called the "Quasi-Ergodic Hypothesis".

Quasi-Ergodic Hypothesis: Each orbit is dense in a connected component of an energy surface in phase space.

In our treatment of the von Neumann and Birkhoff Ergodic theorems we shall not make use of the Ergodic and Quasi-Ergodic Hypotheses but rather work with the hypothesis of the irreducibility of the flow; it is instructive nevertheless to derive some of the consequences of the Ergodic Hypothesis assuming it to be true.

Consequences of the Ergodic Hypothesis:

We first show that the time average of a function defined on an energy surface is constant there. Let f be a function on the energy surface, define

$$\overline{f}^t(x_o) = \lim_{t\to\infty} \frac{1}{T} \int_0^T f(x_t)\, dt \qquad (4.1)$$

Let x_{T_o} be a point on the same orbit as x_o. Assuming the time average limit exists, we have:

$$\overline{f}^t(x_o) = \lim_{T\to\infty} \frac{1}{T} \left(\int_o^{T_o} + \int_{T_o}^{T}\right) f(x_t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{T_o}^{T} f(x_t)\, dt = \lim_{T\to\infty} \frac{1}{T-T_o} \int_{T_o}^{T} f(x_t)\, dt = \qquad (4.2)$$

$$= \overline{f}^t(x_{T_o}).$$

The second equality follows since $\int_o^{T_o} f(x_t)\, dt$ is a fixed number independent of T; the third follows from $\lim_{T\to\infty} \frac{T-T_o}{T} = 1$. By the Ergodic Hypothesis the orbit passes through every point, hence by picking T_o appropriately, x_{T_o} can be any point and we conclude that $\overline{f}^t$ is independent of position. We further conclude that

$$\overline{f}^t = \overline{f}^E = \text{position average}, \qquad (4.3)$$

since

$$\overline{f}^t = \overline{(\overline{f}^t)}^E = \overline{(\overline{f}^E)}^t = \overline{f}^E \qquad (4.4)$$

The first and third equalities follow since both [illegible] and [illegible] are [illegible] position independent numbers and hence equal to their respective position and time averages. The interchange of the order of integration leading to the second equality will be legitimate if the volume element on the energy surface is invariant under the flow and the total surface is finite. By Liouville's theorem the phase space volume element is invariant under the flow, and therefore if we use the volume element on the energy surface induced by the phase space volume element, the second equality is justified.

Said differently, $\overline{f}^t = \overline{f}^E$ shows that on the average (time average) an isolated system behaves like a microcanonical ensemble; this is how the Ergodic Hypothesis justifies the main contentions of Boltzmann's theory. If one admits that time averages exist (which is not <u>a priori</u> clear) then one must admit Boltzmann's argument if the Ergodic Hypothesis is true. The Ergodic Hypothesis is however false for systems of differential equations with smooth boundary conditions although it is not false in general since there do exist space-filling Peano curves. The existence of time averages is justified by the Birkhoff Ergodic Theorem, and at least for hard spheres in a box the equality of space and time averages was proved by Sinai. The Quasi-Ergodic Hypothesis does not in general imply equality of space and time averages, though for smooth enough systems it implies the irreducibility of the flow needed to prove the equality of the two averages.

The third prong of Boltzmann's analysis is a statistical one which was finally refined by Boltzmann and Ehrenfest in view of the criticism of Zermelo and Loeschmidt.

<u>Paradoxes and Boltzmann's resolution of them.</u>

<u>Reversibility paradox</u> (Loeschmidt, 1876):

If a mechanical system possesses time inversion invariance, i.e. if whenever x_t [illegible] is a solution of the equations of motion, then so is

$\hat{x}_t = x_{-t}$, then, if H decreases for a family of orbits $\{x_t\}$ it will increase for the corresponding family $\{\hat{x}_t\}$.

Recurrence paradox (Zermelo, 1896):

Zermelo used the Poincaré recurrence theorem to assert that no mechanical system satisfying the hypotheses of the Poincaré recurrence theorem can ever be irreversible. In particular the quantity H cannot decrease monotonically to its equilibrium value but must return in the course of time arbitrarily close to its initial value. The Poincare recurrence theorem states that under appropriate hypotheses, almost every orbit starting from any set of positive measure will after arbitrarily long times return to that set.

Boltzmann and the Ehrenfests introduced a statistical treatment of this situation. One associates with each system a set of macroscopic variables and then one makes a probabilistic assumption concerning the orbits corresponding to the values of the macroscopic variables; in effect, one associates a stochastic process with the system. A well defined set of macroscopic variables should be insensitive to the particular stochastic process attached to the system.

Let X be the phase space of the system, and x_t the flow given by a deterministic dynamics with no probabilities involved anywhere. Let X now be partitioned into a finite set of regions $\{S_i\}$. Take some finite set of functions $\{f_j\}_{j=1,\ldots,N}$ on X and form their "coarse-grained average" with respect to the partition:

$$\hat{f}_j(x) = \frac{1}{V(S_i)} \int_{S_i} f_j(x')dx', \text{ for } x \in S_i , \qquad (4.5)$$

where $V(S_i)$ is the volume of region S_i taken with respect to the canonical phase space volume element which we denote by dx.

[illegible] which those values are taken. One associates a stochastic process with the system by assigning a probability distribution on these x's. Looking at the flow of $\hat{f}_j$'s one can ask for the probability of a given distribution of $\hat{f}_j(x_\tau)$'s. This is obtained from the flow on X and the probability distribution on the x's which determine a set of given values of the $\hat{f}_j(x)$. We have the diagram.

Flow in X	$\longrightarrow$	Assignment of probability distribution on the x's which give a set of macro-variables
$\downarrow$		$\downarrow$
Flow in macroscopic variables		Stochastic process

and the formula:

$$\text{Probability for } (a_1 < \hat{f}_1(\tau_1) < b_1,\ a_2 < \hat{f}_2(\tau_2) < b_2, \ldots$$

$$\ldots, a_N < \hat{f}_N(\tau_N) < b_N) =$$

$$= \mu\,(\{x \mid a_1 < \hat{f}_1(x_{\tau_1}) < b_1, \ldots, a_N < \hat{f}_N(x_{\tau_N}) < b_N\})\ , \qquad (4.6)$$

where μ is the probability measure on the x's and picked by the physicist.

Boltzmann's statistical argument is that the assertion of the decrease of H will hold with overwhelming probability for the coarse-grained H. Previous statements about phase space now become probability statements about the above stochastic process.

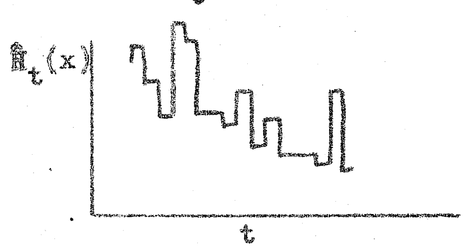
We recapitulate Boltzmann's statistical explanation by considering a more specific example. Consider N molecules in a box; we then have a 6N-dimensional phase or Γ-space and a (6N-1)-dimensional energy surface. Partition the μ-space into subsets S_i of physically significant size, i.e. each S_i contains many molecules but is sufficiently small to yield useful information about the distribution of quantities in the system. In the previous lecture we considered partitioning the phase space, the same set of ideas are applicable here. Let x denote the phase point and $n(S_i, x)$ denote the number of molecules in S_i when the system has phase x. Define the "course-grained" quantity $\hat{H}$ by

$$\hat{H}(x) = \Sigma\, n(S_i, x) \ln n(S_i, x) \,. \qquad (5.1)$$

The flow $x \to x_t$ induces a change in $\hat{H}$: $\hat{H}(x) \longrightarrow \hat{H}(x_t)$. It is convenient to introduce the functions of position $\hat{H}_t$ defined by

$$\hat{H}_t(x) = \hat{H}(x_t). \qquad (5.2)$$

We note that $\hat{H}_t(x)$, $-\infty < t < \infty$ for fixed x is piecewise constant since $\hat{H}_t(x)$ is constant as long as x_t remains in a single s_i.

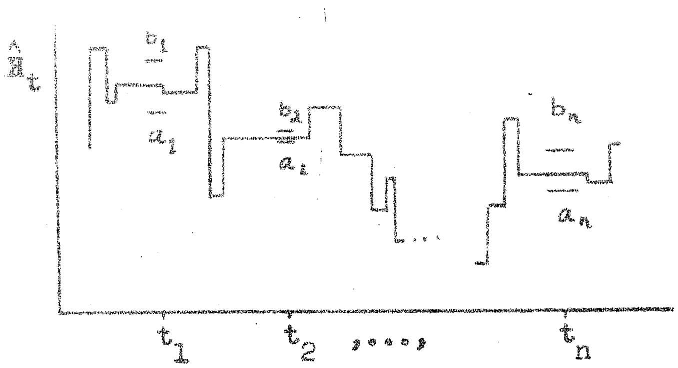


Boltzmann introduced a measure in the family of all $\hat{H}$ curves.

Probability of $(a_1 < \hat{H}_{t_1} < b_1, \ldots, a_n < \hat{H} < b_n)$

$$= \mu(\{x \mid a_1 < H_{t_1}(x) < b_1, \ldots, a_n < H_{t_n}(x) < b_n\}) \qquad (5.3)$$

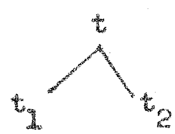
The left hand side is the probability that the observed $\hat{H}$ curve passes through the "gates" of the diagram below:



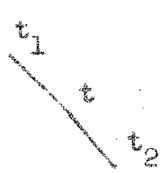
On the right hand side the assignment of the measure μ is a physical hypothesis depending on the wickedness or skill of the person giving you a set of macroscopic measurements; that is, assignment of μ is a "theory". The test of the theory consists in taking a set of copies of the system and making statistical frequency measurements on the H curves which appear. If for μ one takes the microcanonical measure (this is Boltzmann's choice), i.e. the phase volume on the energy surface induced by the canonical flow-invariant measure in phase space, then one gets a theory giving the results of the kinetic theory and which escapes the objections raised by Zermelo and Loeschmidt.

Quotes of typical Boltzmann statements as decoded by Ehrenfest:

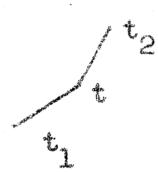
1. If $\hat{H}$ at time t has a value remote from its equilibrium value, then



is overwhelmingly more probably than



or



2. If one looks a $\hat{H}_t$ at a sequence of times $t_1,\ldots,t_n$ then there is a sequence of most probable values which is given approximately by the H curve of the Boltzmann equation; that is, consider a set of gates obtained from the

Boltzmann equation, then the H curve is overwhelmingly likely to go through them.

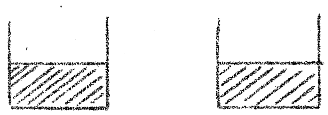
The ideas in Boltzmann's statistical explanation are applicable to any stochastic process, for which see:

> M. Kac, Probability and Related Topics in the Physical Sciences, Interscience; Chapter II.

Ehrenfest Model:

Consider 2R balls numbered consecutively which can be distributed in an arbitrary manner between two urns. The stochastic process consists of randomly picking a number from 1 to 2R and then transferring the ball having that number from the urn in which it is to the other urn; the procedure being repeated indefinitely. One can see that starting from a configuration

since one is more likely to pick a number of a ball in the more nearly full urn, the system will tend to a configuration



(R balls in each urn)

which is its equilibrium state. A system nearly in equilibrium will tend to fluctuate about the equilibrium state.

This model is explicitly soluble for the various quantities of interest such as H. These solutions are given in Kac's book.

Mixing

The mixing property forms a cornerstone of Gibb's work but was greatly underestimated by the Ehrenfests.

Let A and B be sets in phase space and $x \to x_t$ a measurable flow under which $A \to A_t$. The flow is said to have the mixing property with respect to the finite measure μ if

$$\mu(A_t \cap B) \xrightarrow[t \to \infty]{} \mu(A)\,\mu(B)\,/\mu(X)$$

Flows

We shall deal with flows on three levels of generality:

1) Measure spaces

2) Topological spaces

3) Differentiable Manifolds

Definition: A measure space is a triple (X, Σ, μ) consisting of a set X (called the space), a family of subsets Σ (called measurable sets) and a measure μ.

Σ is a non-empty σ-algebra of sets:

1) $S \in \Sigma \Rightarrow S^c \in \Sigma$ where $(\ .\ )^c$ denotes complementation.

2) $S_i \in \Sigma,\ i = 1, 2, \ldots \Rightarrow \bigcup_{i=1}^{\infty} S_i \in \Sigma$

3) $\bigcup_{\Sigma} S = X$

μ is a non-negative real- or $+\infty$- valued function on Σ which is countably additive:

$$S_i \in \Sigma,\ i = 1, 2, \ldots,\ S_i \cap S_j = \emptyset,\ i \neq j$$
$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} S_i\right) = \sum_{i=1}^{\infty} \mu(S_i).$$

We will assume that (X, Σ, μ) is σ-finite:

$$X = \bigcup_{i=1}^{\infty} X_i \quad \text{and} \quad \mu(X_i) < \infty.$$

As an example of a measure space one can take the Euclidean space of any number of dimensions in which Σ is generated by complementation and countable unions from all possible boxes and μ is taken to be the Euclidean volume.

<u>Definition.</u> An isomorphism of two measure spaces (X, Σ, μ) and (X', Σ', μ') is a one-to-one map ϕ of X onto X' which makes Σ correspond to Σ' and such that

$$\mu'(\phi(S)) = \mu(S) \quad S \in \Sigma$$

If the two spaces are the same, ϕ is called an automorphism.

<u>Definition.</u> A flow in a measure space (X, Σ, μ) is a representation of the additive group of real numbers in the automorphism group of (X, Σ, μ). That is, for all real t there exists an automorphism $x \to x_t$ having the property

$$(x_t)_s = x_{t+s} \tag{6.1}$$

Furthermore $(t,x) \to x_t$ should be a measurable correspondence.

A flow in a measure space will be called a <u>measurable flow.</u>

Suppose X has in addition a <u>topology</u>; that is, a distinguished class $\mathcal{T}$ of subsets, called <u>open sets</u>, satisfying

1) $X, \phi \in \mathcal{T}$
2) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$
3) $S_i \in \mathcal{T}, i \in I \Rightarrow \bigcup_{i \in I} S_i \in \mathcal{T}$

where I is an <u>arbitrary</u> index set.

<u>Definition.</u> A set S in a space with a topology is said to be <u>compact</u> in case every open covering of S contains a finite subcovering. In other words S is compact if and only if

$$S \subset \bigcup_{i \in I} G_i, \; G_i \in \mathcal{T} \Rightarrow \exists \; i_1, \ldots, i_n \in I \quad \text{such that}$$

$$S \subset G_{i_1} \cup \ldots \cup G_{i_n}$$

The algebra of sets generated by complementation and countable unions from compact sets is a σ-algebra called the σ-algebra of Borel sets. We require that the Borel sets be contained in Σ and that $\mu(S) < \infty$ if S is compact.

[illegible] that the topology is compatible with the measure space, and X in such a case will be called a topological measure space.

A continuous flow on a topological measure space is a flow on the underlying measure space which is required to preserve the additional structure imposed by the topology; specifically the correspondence $(t,x) \rightarrow x_t$ is required to be continuous.

If X has in addition the structure of a differentiable manifold, then by analogy with the above we have the notion of a differentiable flow.

EXAMPLES

1. Hamiltonian Mechanics for N Particles in a Box

Let V be a non-negative C^∞ (infinitely differentiable) function of $\vec{x}_1, \ldots, \vec{x}_N$ which approaches ∞ uniformly as $|\vec{x}_1{}^2 + \ldots + x_N{}^2| \rightarrow \infty$; for example,

$$V(\vec{x}_1, \ldots, \vec{x}_N) = \Sigma\left(\frac{|\vec{x}_i|^2}{L^2}\right)^{5,000,000} \tag{6.2}$$

V is the potential of the box.

Suppose $V_{jk}(x)$; $j,k = 1, \ldots, N$, $j \neq k$ are C^∞ functions which vanish for $|\vec{x}| \geq a > 0$. If E is a sufficiently large real number,

$$H(p,x) - E = 0$$

defines a connected C^∞ manifold. Here

$$H(p,x) = \sum_{j=1}^{n} \frac{p_j{}^2}{2m_j} + \sum_{j \neq k} V_{jk}(\vec{x}_j - \vec{x}_k) + V(\vec{x}_1, \ldots, \vec{x}_n) \tag{6.3}$$

and $m_j > 0$. The surface so obtained is of course the energy surface of energy E, and indeed by working hard one can show that $\Sigma |\nabla_{p_j} H|^2 + \Sigma |\nabla_{x_j} H|^2 \neq 0$ and hence the surface is smooth as was claimed. The flow on the surface is locally provided by Hamilton's equations

$$\frac{dx_j}{dt} = \nabla p_j H \qquad (6.4)$$

$$\frac{dp_j}{dt} = -\nabla x_j H$$

By the existence theorem for ordinary differential equations we know there will be solutions for all time provided the potentials are smooth enough, which is the case at hand. Moreover the solutions depend smoothly on the initial conditions.

Hamilton's equations in fact provide us with a C^∞ field of tangent vectors on the energy surface; this can be seen by explicitly computing

$$\Sigma \nabla p_j H \cdot dp_j + \Sigma \nabla x_j H \cdot dx_j$$

where dp_j and dx_j are given by Hamilton's equations, and showing that it indeed does vanish. By the fundamental existence theorem for flows one gets a differentiable flow generated by this field of tangents.

The invariant measure in phase space:

$$\mu(S) = \int_S \cdots \int d\vec{x}_1 \ldots d\vec{x}_N \, d\vec{p}_1 \ldots d\vec{p}_N \qquad (6.5)$$

induces on the energy surface H-E = 0 the invariant measure

$$\mu(S) = \int_S \cdots \int \delta(H-E) \, d\vec{x}_1 \ldots d\vec{x}_N \, d\vec{p}_1 \ldots d\vec{p}_N \qquad (6.6)$$

where $\delta(H-E)$ can be expressed locally by solving H-E=0 for some one of the 6N variables $\vec{x}_1, \ldots, \vec{x}_N; \vec{p}_1, \ldots \vec{p}_N)$ in terms of the others; e.g.,

$$(x_1)_1 = f\left((x_1)_2, (x_1)_3; \vec{x}_2, \ldots, \vec{x}_n; \vec{p}_1, \ldots, \vec{p}_n\right) \qquad (6.7)$$

where $((x_1)_1, (x_1)_2, (x_1)_3)$ are the components of $\vec{x}_1$ then,

$$\delta(H-E) = \frac{1}{\left|\frac{\partial H}{\partial (x_1)_1}\right|} \delta\left((x_1)_1 - f\right) \qquad (6.8)$$

This representation holds locally about any point for which $\partial H / \partial (x_1)_1 \neq 0$. There is always at least one of the 6N variables for which such a representation is possible since $\Sigma |\nabla x_j H|^2 + \Sigma |\nabla p_j H|^2 \neq 0$ for sufficiently large E.

For sufficiently small E the energy surface contains no points, and for intermediate values of E it may consist of disconnected pieces some of which may be singular.

2. Flows on the n-torus T

We begin with the Euclidean n-dimensional space R^n regarded as an additive group of n-tuples. Let Z^n be the subgroup with integer coordinates. The factor group $T^n = R^n / Z^n$ is an n-dimensional torus and constitutes the space X for this example. To get the flow take the one parameter subgroup of R^n

$$G_a = \{ta \mid -\infty < t < \infty, \ 0 \neq a \in R^n\}$$

(a is held fixed). This group, when projected into T^n, gives a subgroup of T^n isomorphic to the group $(G_a / G_a \cap Z_n)$. The image of G_a in T^n gives a single flow line, the flow of the other points is given by the correspondence

$$x \pmod{Z^n} \to (x + ta) \pmod{Z^n}$$

If the coordinates of a are relatively rational the flow lines are closed paths which wind several times around the torus and then come back upon themselves; in the other case the flow lines never close but wind densely in the torus.

For a detailed treatment of flows on groups see:

Auslander, Green, and Hahn, Flows on Homogeneous Spaces, Princeton University Press.

Lecture No. 7

Discussion of Integral Invariants

Suppose we are given a differential equation (more generally it will be some differentiable flow).

$$\frac{dx_i}{dt} = V_i(x) \qquad i = 1,\ldots,n \tag{7.1}$$

Find a differentiable positive function ρ such that

$$\mu(S) = \int_S \rho(x)\, dx_1 \ldots dx_n \tag{7.2}$$

is an invariant measure. Since we are working locally in E^n we may take S to be one of the family of Borel sets.

Thus we want the $\mu(s)$ defined by (7.2) to satisfy

$$\mu(S_t) = \mu(S) \tag{7.3}$$

where by definition

$$\mu(S_t) = \int_{S_t} \rho(x_t)\, dx_{t_1} \ldots dx_{t_n} \tag{7.4}$$

Method:

We shall find a differential equation satisfied by ρ.

We first express $\mu(S_t)$ as an integral over S:

$$\mu(S_t) = \int_S \rho(x_t(x))\, J\left(\frac{x_t}{x}\right) dx_1 \ldots dx_n \tag{7.5}$$

For the single-valuedness and differentiability of $x_t(x)$ we appeal to the existence theorem for (7.1).

Here

$$J\left(\frac{x_t}{x}\right) = \det\left(\frac{\partial x_{t_k}}{\partial x_i}\right) \tag{7.6}$$

is the Jacobian determinant.

Then

$$\frac{d}{dt}\mu(S_t) = \int[(\frac{\partial\rho}{\partial t} + \sum_{i=1}^{n} \frac{\partial\rho}{\partial x_{t_i}} \frac{dx_{t_i}}{dt}) J + \rho \frac{dJ}{dt}]dx_1\ldots dx_n \tag{7.7}$$

We now prove that

$$\frac{dJ}{dt} = (\text{div } V)\, J \tag{7.8}$$

where

$$\text{div } V = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V_i(x) \tag{7.9}$$

<u>Proof:</u>

$$\frac{d}{dt}\{\det (\frac{\partial x_{t_1}}{\partial x_i} \ldots\ldots \frac{\partial x_{t_n}}{\partial x_i})\} = \sum_{k=1}^{n} \left|\frac{\partial x_{t_1}}{\partial x_i} \ldots \frac{d}{dt}\frac{\partial x_{t_k}}{\partial x_i} \ldots \frac{\partial x_{t_n}}{\partial x_i}\right| \tag{7.10}$$

Here we have explicitly written out the columns of the determinant.

But

$$\frac{d}{dt}(\frac{\partial x_{t_k}}{\partial x_i}) = \frac{\partial}{\partial x_i}(\frac{dx_{t_k}}{dt}) = \frac{\partial}{\partial x_i}(V_k(x_t)) = \sum_{\ell=1}^{n} \frac{\partial V_k(x_t)}{\partial x_{t_\ell}} \cdot \frac{\partial x_{t_\ell}}{\partial x_i} \tag{7.11}$$

Now if we substitute (7.11) in (7.10) the only term that will contribute from the sum will be the one with $\ell = i$ since by appropriate multiplication of the other terms we can make these columns equal to columns already present in the determinant.

Thus

$$\frac{d}{dt} J = \sum_{k=1}^{n} \left|\frac{\partial x_{t_1}}{\partial x_i} \ldots\ldots \frac{\partial x_{t_k}}{\partial x_i} \ldots \frac{\partial x_{t_n}}{\partial x_i}\right| \frac{\partial V_k(x_t)}{\partial x_{t_k}} = (\text{d iv } V)\cdot J$$

which is equation 7.8.

Since S in (7.7) is an arbitrary set we therefore get that if $\frac{d}{dt}\mu(S_t) = 0$ then

$$J\frac{\partial\rho}{\partial t} + \text{div } (\rho V)J = 0$$

or

$$\frac{\partial\rho}{\partial t} + \text{div } (\rho V) = 0 \tag{7.12}$$

Conversely the equation of continuity implies that

$$\int_S \rho\, dx_1\ldots dx_n$$

is an integral invariant.

Definition

If $\operatorname{div} V = 0$ then the flow defined by (7.1) and the measure (7.2) is incompressible. In this case ρ = constant defines an integral invariant.

Theorem (Liouville)

A differentiable flow arising from a Hamiltonian system of differential equations is incompressible.

Proof:

$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q_i} \qquad \frac{\partial q_i}{\partial t} = \frac{\partial H}{\partial p_i}$$

$$\operatorname{div} V = \sum_i -\frac{\partial^2 H}{\partial p_i \partial q_i} + \sum_i \frac{\partial^2 H}{\partial q_i \partial p_i} = 0$$

There are more general theorems about these integral invariants as formulated by Krylov and Bogoliubov. These will not be discussed here. See, however, D. C. Oxtoby, Bulletin of the American Mathematical Society (about 1951 or 1952) - Ergodic Sets. He discusses the existence of invariant measures in more general cases.

We shall now give a discussion of Koopman's Idea for Classical Flows.

Given a measurable flow, there is a family of mappings of measurable function on X (phase space) defined by:

$$f_t(x) = f(x_{-t}) \tag{7.13}$$

where x_{-t} is to be regarded as a function of x. If among these measurable functions we now select certain specific ones, we get the various function spaces. Thus the integrable functions give $L^1(X,\mu)$, and the square integrable functions give $L^2(X,\mu)$. We shall be especially interested in $L^2(X,\mu)$ which is well-known to be a Hilbert space, and hence the notion of unitary operators on $L^2(X,\mu)$ is defined. We shall now prove the theorem of Koopman.

Theorem (Koopman)

Each measurable flow in its action on $L^2(X,\mu)$ defines a continuous unitary one-parameter group by the formula:

$$(U_t f)\ (x) = f_t(x) \tag{7.14}$$

Proof:

We first note that U_t not only carries measurable, square integrable functions into themselves, but that the range is every such function. This is clear since any $g(x)$ can be written as some $f_t(x)$.

Explicitly,

$$g(x) = (g_{-t})_t(x) \tag{7.15}$$

Therefore U_t is a bijection of $L^2(X,\mu)$.

Also U_t is clearly linear, since

$$\begin{aligned}(U_t(\alpha f + \beta g))\ (x) &= (\alpha f + \beta g)\ (x_{-t}) \\ &= \alpha f(x_{-t}) + \beta g(x_{-t}) = \alpha f(x) + \beta g_t(x) \\ &= \alpha(U_t f)\ (x) + \beta(U_t g)\ (x)\end{aligned}$$

Furthermore U_t is isometric:

$$\begin{aligned}\int |(U_t f)\ (x)|^2\ d\mu(x) &= \int |f_t(x)|^2\ d\mu(x) \\ &= \int |f(x)|^2\ d\mu(x_{-t}) \\ &= \int |f(x)|^2\ d\mu(x)\end{aligned}$$

where the last equality follows from the fact that the measure μ is invariant under the flow.

$$\therefore\quad ||\ U_t\ f\ ||^2 = ||f||^2 \tag{7.16}$$

Combining this with the fact that U_t is linear and a bijection, we get that U_t is unitary.

Thus $\qquad U_t^* U_t = 1 \qquad U_t U_t^* = 1 \qquad (7.17)$

It can also be verified directly that U_t preserves the scalar product.

$$(U_t f, U_t g) = \int \overline{(U_t f)(x)}\,(U_t g)(x)\,d\mu(x)$$
$$= \int \overline{f(x_{-t})}\,g(x_{-t})\,d\mu(x)$$
$$= \int \overline{f(x)}\,g(x)\,d\mu(x_t)$$

$$\therefore \quad (U_t f, U_t g) = (f,g) \qquad (7.18)$$

Also

$$(U_s U_t f)(x) = (U_t f)(x_{-s})$$
$$= f((x_{-s})_{-t})$$
$$= f(x_{-s-t})$$

$$U_s U_t = U_{s+t} \qquad (7.19)$$

$$\text{with } U_o = 1 \qquad (7.20)$$

Continuity Properties.

Since the U_t form a one-parameter group, the continuity of U_t can be interpreted in at least two ways which are equivalent, namely:

1. Weak continuity:

$(f, U_t g)$ is continuous for all f, $g \in L^2(X,\mu)$

2. Strong continuity:

$U_t f$ is a continuous function for all $f \in L^2(X,\mu)$. As usual the strong continuity implies weak continuity, the converse follows from:

$$\|U_t f - U_{t'} f\|^2 = \|U_{t-t'} f - f\|^2 = 2\|f\|^2 - 2\mathrm{Re}(U_{t-t'} f, f) \qquad (7.21)$$

Then clearly as $t \to t'$ with weak continuity assumed

$$2\|f\|^2 - 2\mathrm{Re}(U_{t-t'} f, f) \to 2\|f\|^2 - 2\mathrm{Re}(U_o f, f)$$
$$= 2\|f\|^2 - 2\|f\|^2 = 0$$

Hence $\lim_{t \to t'} \|U_t f - U_{t'} f\|^2 = 0$

By the way, by reversing the argument we also get weak continuity from strong. Thus 1. and 2. above are equivalent. So to complete the proof of the theorem of Koopman we need only show that U_t is (strongly) continuous.

To do this we sketch the proof from Dunford and Schwartz - Linear Operators Part I, page 616. But first we need a definition.

Definition:

An operator valued function U_t such that

$$U_{t+s} = U_t U_s \ , \ U_o = 1 \quad t,s \ge 0 \ .$$

is called a semi-group of operators.

Now for the theorem.

Theorem:

Let $\{U_t\}$ be a semi-group of bounded linear operators in a Hilbert space $\mathcal{H}$ defined for $0 \le t < \infty$ so that $U_t h$ is measurable on $[0,\infty)$ for each $h \in \mathcal{H}$, then $U_t h$ is continuous at every point in $[0,\infty)$.

Proof:

We assume $\|U_t\|$ is bounded on each open interval $(\delta, 1/\delta)$, $\delta > 0$.

Let $0 < \alpha < \beta < t_o$ and pick a δ such that

$$2\delta < 1 \ , \ \alpha \ , \ t_o - \beta \ , \ (t_o + 1)^{-1}$$

since

$U_{t_o} h = U_t(U_{t_o - t} h)$ and since the right side in independent of t, it is integrable over the finite interval (α,β). Now if $|\epsilon| < \delta$ we have:

$$(\beta-\alpha) [(U_{t_o+\epsilon} - U_{t_o})h] - \int_\alpha^\beta U_t \, [U_{t_o+\epsilon-t}]h \, dt$$

Now by hypothesis we have an $M > 0$ such that

$$\|U_t\| < M \quad \text{for} \quad t \in (\alpha,\beta)$$

Hence

$$\begin{aligned}(\beta-\alpha)\ \|(U_{t_o+\epsilon} - U_{t_o})h\| &< M \int_\alpha^\beta \|(U_{t_o+\epsilon-t} - U_{t_o-t})h\|\, dt \\ &= M \int_{t_o-\beta}^{t_o-\alpha} \|(U_{s+\epsilon} - U_s)h\|\, ds\end{aligned} \tag{7.22}$$

where we have set $s = t_o - t$.

Now it can be shown that (although it is a rather technical argument) the right side of (7.22) tends to zero as $\epsilon \to 0$ and hence

$$\|(U_{t_o+\epsilon} - U_{t_o})h\| \xrightarrow[\epsilon \to 0]{} 0$$

Thus U_t is continuous for every t_o in any finite interval $0 \le \alpha \le t_o \le \beta < \infty$

To apply this theorem in our case we need only notice that $\|U_t\| = 1$ i.e., all U_t are uniformly bounded, and all $U_t f$ are measurable for $f \in L^2(X,\mu)$, and furthermore the $U_t (t > 0)$ form a semi-group. We now extend the theorem to $t \le 0$ by noticing that all the above conditions also apply to $U_t^* = U_t^{-1}$ for $t \ge 0$

But

$$U_t^{-1} = U_{-t}$$

$\therefore$ The theorem also holds in our case for $t \le 0$

Comments on Koopmanism:

We consider the unitary group of one-parameter operators. Then it is false to say that the properties of these groups up to equivalence determine the flows. However, it is true that one of these one-parameter groups in concrete form determines all properties of the flow.

Thus given U_t, the properties of the flow are determined. i.e., U_t contains all the information. However, if we go to $V_t = S\, U_t\, S^{-1}$ (arbitrary S) then V_t contains only information about the spectral invariants: the spectrum and the multiplicity of the spectrum.

Koopman, therefore, gives us the following program:

1. Study the one-parameter groups.
2. Determine what the properties of the unitary operators are in terms of the dynamics of the system.

Special Properties of the Groups Obtained

1. The U_t's which occur have a very special multiplicative property:

If $f, g \in L^2(X,\mu)$ and the point wise product $(fg)(x) - f(x)\, g(x) \in L^2(X,\mu)$, then

$$U_t(fg) = (U_t f)\ (U_t g) \tag{8.1}$$

Although this is not a meaningful equation in terms of functions on Hilbert space, it is a meaningful equation in terms of functions on measure space,

Furthermore, the validity of (8.1) has a large effect on the eigenvalue problem of U, for if f and g are eigenfunctions, so is fg if $fg \in L^2(X,\mu)$. We now turn to a more explicit examination of this statement.

2. Let $e^{i\lambda_1 r} \ldots e^{i\lambda_k t}$ be proper values of U_t (the unitarity of U_t guarantees the reality of the λ_k) with the corresponding proper functions $\phi_1 \ldots \phi_k$. Then, if $\phi_1^{m_1} \ldots \phi_k^{m_k}$ ($m_1 \ldots m_k$ a set of non-negative integers) is square integrable

we have:

$$U_t\ \phi_1^{m_1}\ \phi_2^{m_2}\ldots\phi_k^{m_k} = \left[\exp\left(i\ \left(\sum_{j=1}^{k} m_j\lambda_j\right)\ t\right)\right]\phi_1^{m}\ldots\phi_k^{m_k}. \qquad (8.2)$$

So we see that the eigenvalues form a semi-group.

For simplicity assume U_t has a purely discrete spectrum. Thus, we expect to get a complete set of ϕ_k. Then taking powers as before, we expect to get all eigenvalues as linear combinations with integral coefficients of a basic set of eigenvalues.

For completeness we state at this stage without proof a theorem regarding groups of unitary operators.

<u>Stone's Theorem:</u>

Every continuous one-parameter group of unitary transformation U_t is generated by an infitesimal transformation iA, where A is a self-adjoint transformation which is, in general, unbounded:

$$U_t = e^{itA} \qquad (8.3)$$

$$iA = \lim_{h\to o} \frac{1}{h}\ (U_h - 1) \qquad (8.4)$$

If the space is separable, continuity follows from measurability.

For the proof of this important theorem see Riesz and Sz.-Nagy - Functional Analysis - chapter X.

From the above considerations there arises a very natural mathematical problem, the so-called "Conjugacy Problem," which was first considered by von Neumann: Find a set of "invariants" to characterize non-isomorphic measurable flows.

Although this problem is natural from the mathematicians' view point it it not natural from the physicists' view point. To clarify this let us recall the definition of isomorphism.

If we have two flows x_t and x'_t defined on 2 measure spaces (X,Σ,μ) and (X', Σ', μ') then the 2 flows are isomorphic if we have a bijection ϕ of X onto X' which makes Σ correspond to Σ' and is such that

$$\mu'(\phi(s)) = \mu(s) \quad \forall s \in \Sigma$$

And furthermore satisfies:

$$\phi(x_t) = x'_t = (\phi(x))_t \tag{8.5}$$

So it is clear that interesting structure, from the physicists viewpoint can be destroyed by such mappings. For instance flows on a plane and line are in certain cases isomorphic, but the mapping (Peano curve) although it is continuous, destroys differentiability.

We get the following answers to von Neumann's question.

1. If the spectrum of the one-parameter group of operators is discrete, then the spectral invariants are all the invariants needed. Thus all the information is contained in

$$U_t = \int_{-\infty}^{\infty} \exp(it\lambda)\, dE_\lambda \tag{8.6}$$

(von Neumann - 1932)

2. If the spectrum is not discrete, more invariants are needed (Kolmogorov - 1958).

There are many more results in Koopmanism. More later.

We shall now introduce some more terminology for classifying flows.

1. The old "Quasi-Ergodic" Hypotheses in one possible interpretation): A continuous flow is <u>topologically transitive</u> if it possesses a dense orbit. That is, if there is a single point such that it comes arbitrarily close to every point.

2. It is <u>minimal</u> if every orbit is dense. Thus, minimal is equivalent to the quasi-ergodic hypothesis in the strictest possible interpretation.

3. A measurable flow is <u>ergodic</u> if measurable subsets of X invariant under the flow are either of measure zero or differ from X by a set of measure zero. This means that if the flow is ergodic, the underlying measure space cannot be cut up into parts of non-zero measure which are invariant under the flow. This condition was also called metrically transitive by G. D. Birkhoff.

4. A measurable flow in a finite measure space is called <u>weakly mixing</u> if:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left| \mu (A_t \cap B) - \frac{\mu(A)\mu(B)}{\mu(X)} \right| dt = 0$$

for all measurable sets A, B.

5. A measurable flow in a finite measure space is <u>strongly mixing</u> or <u>mixing</u> if

$$\lim_{t\to\infty} \{\mu(A_t \cap B) - \frac{\mu(A)\mu(B)}{\mu(X)}\} = 0$$

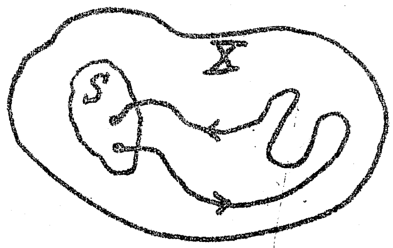
for all measurable A, B.

The notion of mixing may be viewed as a spreading out of dye in a fluid (the measure space). Then the flow of the fluid is strongly mixing if starting with the dye concentrated in some region A_o at $t = 0$, the dye is spread uniformly throughout the fluid as $t \to \infty$.

We may also think of mixing as stating that eventually there will be no correlation between measurable sets.

Lecture No. 9

We shall now do some ergodic theory. The oldest theorem of this type is the Recurrence Theorem of Poincaré (1890). In the context of flows, Poincaré's theorem pertains to the question about what happens if we take a region S and see if a given particle returns to S. It is neater to discuss



instead of U_t, to see if a particle returns in a fixed time say (1 second) and thus discuss U_1. Therefore, we will prove a discrete case of Poincaré's theorem.

In statistical mechanics it is, of course, of interest to compare the relaxation time with the recurrence times. Birkhoff (1932) Proc. Nat'l. Acad. has shown that for an ergodic transformation the average recurrence time $\bar{\tau}$ is given by

$$\bar{\tau} = \frac{\mu(X)}{\mu(S)} \tag{9.1}$$

which is precisely what phtsicists intuition would have suggested. Now for the

Recurrence Theorem

Let S be a measurable subset of a finite mesure space X, and let T be a measure-preserving transformation of X into itself. Then almost all points of S are infinitely recurrent. That is, if $x \in S$ then $T^n x \in S$ for infinitely many positive n, with the possible exception of a set of x's of measure zero.

Digression:

We recall that a transformation T is measure-preserving or measurable if

$$\mu(T^{-1}A) = \mu(A) \tag{9.2}$$

where $T^{-1}A$ is the set of antecedents of A under T and the notation does not imply the existence of an inverse transformation T^{-1}. For comparison recall that a mapping is continuous if the antecedent of an open set is open.

Now for the

Proof:

Let N be the set of points of S which are not recurrent. That is, Tx, Tx^2,... etc. $\notin$ S. We show that N is measurable and has measure zero, by actually displaying N.

$$N = S \cap T^{-1}(X-S) \cap T^{-2}(X-S) \cap \ldots\ldots\ldots\ldots$$

Here X-S is the set theoretic difference of X and S, that is

$$X - S = \{x \in X \mid x \notin S\} \tag{9.3}$$

Here we have displayed N as a countable intersection of measurable sets and since any intersection of measurable sets is measurable, N is measurable.

We now study the sets $T^{-n}N$ n = 1, 2,..., that is, the sets that are mapped into N by T^n. We will show that these are disjoint (their mutual intersections are empty).

Now the statement that

$$\forall x \in N;\ Tx,\ T^2x, \ldots \notin N$$

is equivalent to the statement

$N \cap T^{-n}N = \emptyset$ for n = 1, 2,... because the set $N \cap T^{-n}N$ is precisely the set of those x which are in N, and which are such that $T^n x$ is in N.

But $N \cap T^{-n} N = \emptyset$ implies that N, $T^{-1}N$, $T^{-2}N$,... are **disjoint,** because $T^{-n} N \cap T^{-n-k} N = T^{-n}(N \cap T^{-k}N) = \emptyset$

Because we have assumed the invariance of μ under T, the measure of all these sets N, T^{-1} N,... is the same. But this leads to a contradiction unless they all have measure zero,

since $\mu(X) < \infty$ and unless $\mu(T^{-n}N) = 0\ \forall\ n$

$$\mu\left(\bigcup_{k=1}^{\infty} T^{-k}N\right) = \sum_{k=1}^{\infty} \mu(T^{-k}N) = \infty.$$

Now, this would mean that the measure of a measurable subset of X, exceeds the (finite) measure of X. This, however, cannot be since for measurable sets $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

[illegible] infinitely with the [illegible] exception of a set of measure zero, we apply the preceding argument to T^n. Let N_n be the set of those $x \in S$ which do not recur under T^n. Then we have shown

$$\mu\,(N_n) = 0 \qquad (9.4)$$

Hence

$$\mu\,(\bigcup_n N_n) = 0 \qquad (9.5)$$

because it is a countable union.

So if we can now show that the points of $(S - \bigcup_{k=1}^{\infty} N_k)$ are infinitely recurrent we are finished.

Now if $x \in (S - \bigcup_{k=1}^{\infty} N_k)$

Then surely $x \in (S-N_1)$ so that there is a positive integer $n_1 \ni T^{n_1} x \in S$. But for the same reason

$$x \in S - N_{n_1} \text{ and hence } \exists \text{ a positive integer } n_2 \ni (T^{n_1})^{n} \, x \in S.$$

We could now use formal induction and thus conclude that the terms

$$T^{n_1}x,\ T^{n_1 n_2}x,\ T^{n_1 n_2 n_3}x,\ \ldots \in S.$$

The second part of this proof can be considerably shortened as follows.

The points of S whose last recurrence in S is at time $k = S \cap T^{-k}N$

This is clearly measurable.

$\therefore$ The set of points recurring only finitely many times $= A = \bigcup_{k=1}^{\infty}(S \cap T^{-k}N) \subset \bigcup_{k=1}^{\infty} T^{-k}N$

$\therefore \quad \mu(A) = 0$ since $\mu(\bigcup T^{-k}N) = 0.$

It is also possible to study the behaviour of a system on a time average. An example is von Neumann's Mean Ergodic Theorem. The proof given is due to F. Riesz.

<u>Mean Ergodic Theorem:</u>

Let U be an isometry of the Hilbert space $\mathcal{H}$, then the sequence

$$A_n = \frac{1}{n} \sum_{j=0}^{n-1} U^j$$

has a strong limit P which is the projection onto the subspace invariant under U.

<u>Remarks:</u>

Let $\mathcal{H}$ be 1 dimensional, then U is multiplication by u, a complex number of modulus 1.

If $u = 1 \quad A_n = 1$

If $u \neq 1 \quad A_n = \frac{1}{n} \left(\frac{1 - u^n}{1 - u} \right)$

And

$$\|A_n\| \leq \frac{2}{n} \frac{1}{\|1-u\|} \xrightarrow[n \to \infty]{} 0$$

So it is clear that P is a projection since its eigenvalues are 1, 0. In this case the subspace is all of $\mathcal{H}$.

We continue with some remarks concerning the Mean Ergodic Theorem. 2) Let H be n-dimensional. The isometry of U implies

$$U^* U = 1 \tag{10.1}$$

$$\therefore \quad \det U^* \det U = |\det U|^2 = 1 \Rightarrow \det U \neq 0 .$$

Hence U^* is the inverse of U and we also have

$$U U^* = 1 \tag{10.2}$$

Hence for a finite dimensional Hilbert space every isometry is a unitary transformation, and U can be brought into a diagonal form by an appropriate choice of basis:

$$U = \begin{pmatrix} e^{i\lambda_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & e^{i\lambda_n} \end{pmatrix} \tag{10.3}$$

By remark D we see that

$$A_n \to \begin{pmatrix} 1 & & & & & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1 & 0 & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & 0 \end{pmatrix} \tag{10.4}$$

where 1 occurs in case $e^{i\lambda\nu} = 1$ and 0 in case $e^{i\lambda\nu} \neq 1$. 3) When H is infinite dimensional an isometry need not be unitary:

$$\|Uf\| = \|f\| \qquad U^* U = 1 \tag{10.5}$$

$$U^* U = 1 \quad \not\Rightarrow \quad U U^* = 1 \tag{10.6}$$

In any case $U U^*$ will be an orthogonal projection. For an operator F to be an orthogonal projection it is necessary and sufficient that $F = F^*$ and $F = F^2$.

We have:

$$(UU^*)^* = U^{**} U^* = U U^* \tag{10.7}$$

$$(UU^*)^2 = UU^* UU^* = U1U^* = UU^* . \tag{10.8}$$

UU^* can however be a proper projection since in the infinite dimensional case U may map onto a proper subspace and still preserve norm.

4) There is a special property of isometry needed for the proof:

$$Uf = f \iff U^* f = f \tag{10.9}$$

The necessity is immediate:

$$Uf = f \Rightarrow U^* Uf = U^* f \Rightarrow f = U^* f$$

For the sufficiency we proceed as follows:

$$\|Uf - f\|^2 = (Uf, Uf) + (f,f) - (Uf,f) - (f,Uf) =$$
$$= 2\|f\|^2 - (f, U^* f) - (U^* f,f) = 0 \Rightarrow f = U^* f$$

Proof: (See F. Riesz Comm. Math. Helv. 17, 221-239 (1945)).

Consider first a vector f satisfying $Uf = f$. For these

$$A_n f = \frac{1}{n} \Sigma_{j=o}^{n-1} U^j f = f \; . \tag{10.10}$$

If now f is vector of the form $g - Ug$, $g \in \mathcal{H}$ we have:

$$A_n f = \frac{1}{n} \Sigma_{j=o}^{n-1} U^j (g - Ug)$$
$$= \frac{1}{n} (\Sigma_{j=o}^{n-1} U^j - \Sigma_{j=o}^{n-1} U^{j+1})g$$
$$= \frac{1}{n} (g - U^n g) \tag{10.11}$$

$$\therefore \quad \|A_n f\| \le \frac{1}{n} 2\|g\| \to 0 \tag{10.12}$$

$$\therefore \quad A_n f \to 0 \quad \text{strongly} \tag{10.13}$$

Now the f of the form $g - Ug$ form a linear manifold M ; the idea of the rest of the proof is to show that the closure $\bar{M}$ of M and the set of f satisfying $f - Uf$ are orthogonal complements. A general f then decomposes

$$f = f_1 + f_2 \; ; \; Uf_1 = f_1 \; , \; f_z \in \bar{M} \tag{10.14}$$

One then has to show that $A_n f_2 \to 0$ as $n \to \infty$. Clearly $Pf = f_1$ and $f_2 = (1 - P)f$, so if the above statements are established the theorem is proved.

We compute $\bar{M}^{\perp} = M^{\perp}$ where $(\cdot)^{\perp}$ denotes the orthogonal complement.

$$\forall g \in \mathcal{H}, \ (h, g - Ug) = 0 \iff$$

$$\iff \forall g \in \mathcal{H}, \ (h - U^* h, g) = 0 \iff$$

$$\iff h - U^* h = 0 \iff h = Uh \tag{10.15}$$

$$\therefore \quad M^{\perp} = \{f | \ Uf = f\} \tag{10.16}$$

Note that $\forall f \in \mathcal{H}$

$$\|A_n f\| \le \frac{1}{n} \Sigma_{j=0}^{n-1} \|U^j f\| = \|f\| \tag{10.17}$$

Consider now a general element of $\bar{M}$. Such an element is a strong limit of a sequence $\{f_k\}_{k=1,2,\ldots}$ where f_k are of the form:

$$\left.\begin{aligned} f_k &= g_k - Ug_k \\ f_k &\to f \ \text{strongly} \end{aligned}\right\} \tag{10.18}$$

$$\begin{aligned} \|A_n f\| &= \|A_n(f - f_k) + A_n f_k\| \le \\ &\le \|A_n(f - f_k)\| + \|A_n f_k\| \le \|f - f_k\| + \|A_n f_k\| \end{aligned} \tag{10.19}$$

Let $n \to \infty$, then

$$\lim_{n \to \infty} \|A_n f\| \le \|f - f_k\| \tag{10.20}$$

Since this is true for all k and since $f_k \to f$ strongly we have

$$\lim_{n \to \infty} \|A_n f\| = 0 \qquad \text{Q.E.D.} \tag{10.21}$$

Corollary. If U is an isometry of the Hilbert space $\mathcal{H}$, then the double sequence of operators

$$A_{n,m} = \frac{1}{n-m}\sum_{j=m}^{n-1} U^j \,, \quad n > m \geq 0 \tag{11.1}$$

converges strongly to P, the projection onto the subspace of $\mathcal{H}$ consisting of those vectors left invariant by U, whenever n and m vary over a sequence of pairs such that $n-m\to\infty$.

If U is unitary the same statement holds with $n>m\geq 0$ replaced by $n>m$.

The proof of the corollary is the same as the proof of the Mean Ergodic Theorem with the substitution

$$A_n \to A_{n,m} \tag{11.2}$$

and whenever $f = g - Ug$ we have instead:

$$A_{n,m} f = \frac{U^m g - U^n g}{n-m} \tag{11.3}$$

$$\| A_{n,m} f \| \leq \frac{2\|g\|}{n-m} \tag{11.4}$$

Theorem (Mean Ergodic Theorem, continuous case)

Let U_t $0\leq t<\infty$ be a continuous semigroup (i.e. $U_t U_s = U_{t+s}$, $U_0 = 1$) of isometries of the Hilbert space $\mathcal{H}$, then there exists a strong limit

$$\lim_{T-S\to\infty} \frac{1}{T-S}\int_S^T U_t \, dt = P \,, \quad T > S \geq 0 \tag{11.5}$$

where P is the orthogonal projection onto the subspace of $\mathcal{H}$ consisting of all vectors invariant under U_t, $t\geq 0$.

If U_t is a one-parameter group, then the same statement holds with $T > S \geq 0$ replaced by $T > S$.

Proof

We first note that just as in the discrete case if f satisfies $U_t f = f$, $\forall t$ then if the strong limit exists, $Pf = f$.

Second; if the theorem can be proved for T and S integers, then it is true for arbitrary real T and S. This can be seen as follows: ([x] = the greatest integer in x). If T>S+1,

$$\frac{1}{T-S}\int_S^T U_t\,dt = \frac{[T]-[S]}{T-S}\,\frac{1}{[T]-[S]}\int_{[S]}^{[T]} U_t\,dt + $$

$$+ \frac{1}{T-S}\left(\int_S^{[S]} U_t\,dt + \int_{[T]}^{T} U_t\,dt\right). \qquad (11.6)$$

The last two terms → 0 in the limit since

$$\left\| \int_S^{[S]} U_t\,dt\, f \right\|^2 = \int_S^{[S]}\int_S^{[S]} (U_r f, U_t f)\,dr\,dt \le \|f\|^2 \qquad (11.7)$$

hence $\left\| \int_S^{[S]} U_t\,dt \right\| \le 1$ and similarly for $\left\| \int_{[T]}^{T} U_t\,dt \right\|$.

Also $([T]-[S])/(T-S) \to 1$, hence

$$\lim_{T-S\to\infty} \frac{1}{T-S}\int_S^T U_t\,dt = \lim_{[T]-[S]\to\infty} \frac{1}{[T]-[S]}\int_{[S]}^{[T]} U_t\,dt \qquad (11.8)$$

provided either limit exists.

For the rest of the proof we consider T and S to be integers, we then have by the semigroup property

$$\frac{1}{T-S}\int_S^T U_t\,dt = \left(\frac{1}{T-S}\sum_{j=S}^{T-1} U_1^j\right)\int_0^1 U_t\,dt \qquad (11.9)$$

By the discrete form of the theorem this has a strong limit which we call P. We show that

$$\left.\begin{array}{l} \text{(i)}\ \forall f \in \mathcal{H},\quad \|Pf\| \leq \|f\| \\ \text{(ii)}\ \forall t,\ U_t P = P = P U_t \end{array}\right\} \tag{11.10}$$

(i) is immediate by the Schwartz inequality applied to (11.9). To show (ii) consider:

$$U_t \frac{1}{T}\int_0^T U_\tau \, d\tau = \frac{1}{T}\int_0^T U_\tau \, d\tau \, U_t$$
$$= \frac{t+T}{T} \frac{1}{t+T}\left(\int_0^{t+T} U_\tau \, d\tau - \int_0^t U_\tau \, d\tau \right) \tag{11.11}$$

This identity shows that if f is in the range of P, *i.e.* $f = Pg$, then $U_t f = f$; this proves (ii). It is also true that P annihilates every vector that is orthogonal to all vectors left invariant by U_t. This is shown by repeating the Riesz argument: by (ii)

$$\forall t \text{ and } \forall g \in \mathcal{H},\quad P(g - U_t g) = 0. \tag{11.12}$$

moreover

$$\forall t \text{ and } \forall g \in \mathcal{H},\quad (f, g - U_t g) = 0 \iff$$
$$\iff \forall t \text{ and } \forall g \in \mathcal{H},\quad (f - U_t^* f, g) = 0 \iff$$
$$\iff \forall t,\ f = U_t^* f \iff \forall t,\ f = U_t f. \tag{11.13}$$

Hence **annihilates** the orthocomplement of the set of vectors f satisfying $\forall t,\ f = U_t f$. Q.E.D.

Physical Interpretation of the mean convergence of time averages

Von Neumann argued that the above ergodic theorem and not Birkhoff's, which we shall state and prove later in these notes, is the physically relevant one. The argument being that if f is a physical quantity the physical question is: is there a c such that the quantity

$$\int \left| \frac{1}{T-S} \int_S^T f(x_t)\,dt - c \right|^2 \mu(dx) \qquad (11.14)$$

is arbitrarily small provided T-S is sufficiently large? However, there are typical physical situations which cannot be formulated in the above manner. A typical Boltzmann statement is:

For a system in which $\mu(\mathfrak{X}) < \infty$

$$\text{Probability}\left(\lim_{T-S\to\infty} \frac{1}{T-S}\int_S^T f(x_t)\,dt = \frac{1}{\mu(\mathfrak{X})}\int f(x)\mu(dx)\right) \qquad (11.15)$$

$= 1$,

By the statistical interpretation this is translated to:

$$\mu\left(\left\{x \,\middle|\, \lim_{T-S\to\infty} \frac{1}{T-S}\int_S^T f(x_t)\,dt = \frac{1}{\mu(\mathfrak{X})}\int f(x)\mu(dx)\right\}\right) = \mu(\mathfrak{X}). \qquad (11.16)$$

This result can be proved by Birkhoff's theorem assuming the flow is ergodic, but is impossible to get from von Neumarn's argument. A result in such a direction is contained in the following theorem:

If $f_k \to f$ in the mean, then $(\forall \varepsilon > 0)(\forall \delta > 0)$ $(\exists N)$ such that $(\forall k > N)$ $\mu(\{x \mid |f_k(x) - f(x)| > \delta\}) < \varepsilon$.

This result however cannot lead to Birkhoff's statement.

Conclusion: There is a physical point to Birkhoff's theorem.

Integrals of Motion and Ergodicity

In classical mechanics one usually learns of the existence of a large number of integrals of motion. In fact, in Hamilton-Jacobi theory one usually meets with a transformation to a new set of canonical variables in which one increases linearly with time and all the rest are constant. Such coordinates however exist only locally and in general due to ergodicity cannot be patched together to give global invariants. This situation is illustrated in the following theorem:

Let (X, Σ, μ) be a measure space and T a measure preserving transformation of X into X, then T is ergodic $\Leftrightarrow$ every measurable function which is invariant (i.e. f(Tx) = f (x) a.e.) is constant almost everywhere.

If in place of T we have a measurable flow, the same statement holds. (a measurable function is invariant under a flow if for almost all x, $f(x_t) = f(x) \quad \forall t$)

Lecture 12

Proof of assertion

If S and $\mathbb{X} - S$ are measurable sets of measure greater than zero which are invariant under T or $x \to x_t$ respectively, then the characteristic functions of S,

$$\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases} \tag{12.1}$$

will be measurable and invariant, but not constant almost everywhere.

Conversely, suppose f is a measurable function invariant under T or the flow. We assume ergodicity and propose to prove f is consta nt almost everywhere. Consider the set

$$\mathbb{X}(k,n) = \{x \mid \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\} \tag{12.2}$$

Then, because f is invariant, the set $\mathbb{X}(k,n)$ will differ from an invariant set by at most a set of measure zero. By ergodicity either $\mu(\mathbb{X}(k,n)) = 0$ or $\mu(\mathbb{X} - \mathbb{X}(k,n)) = 0$. For a fixed n, there can be at most one k such that $\mu(\mathbb{X}(k,n)) > 0$, we construct the intersection

$$\bigcap_{\substack{n \\ \mu(\mathbb{X}(k,n)) > 0}} \mathbb{X}(k,n) \tag{12.3}$$

f is constant on this intersection, and its complement is of measure zero. Q.E.D.

The above theorem shows that if one has ergodicity that there are no non-trivial measurable integrals of motion. In general one cannot eliminate the sets of measure zero appearing in the theorem.

The following theorem relates this notion of ergodicity to one of the interpretations of the "Quasi-Ergodic hypothesis."

Theorem

Let (X, Σ, μ) be a measure space and T a measure preserving ergodic transformation of X into itself. Suppose in addition that X is a topological measure space with a countable base (i.e. $\exists \{G_i\}_{i=1,2,\ldots} \subset \mathcal{T}$ such that whenever $x \in G \in \mathcal{T}$, $\exists n$ such that $x \in G_n \subset G$) such that each non-empty open set has a non-zero measure, then almost every orbit is dense; that is, $\{T^n x, n = 0, 1, 2, \ldots\}$ is a dense set for almost all x.

Proof

If x has an orbit which is not dense, then there must be some open set G of the countable base such that

$$\forall n > 0, \ T^n x \notin G \iff$$

$$\iff \forall n > 0, \ x \in X - T^{-n} G = T^{-n}(X - G) \qquad (12.4)$$

Consider the intersection

$$\bigcap_n T^{-n}(X - G) \ ; \qquad (12.5)$$

this is a measurable invariant set, it is contained in the complement of G, and $\mu(G) > 0$. By ergodicity this intersection has zero measure, but

$$\{x \mid \text{orbit}(x) \text{ not dense}\} \subset \bigcup_G \bigcap_n T^{-n}(X - G) \qquad (12.6)$$

where the G run over the countable base. Since the right hand side is a countable union of sets of measure zero, it has measure zero, therefore almost all orbits are dense, Q.E.D.

Maximal Ergodic Theorem

Let (X, Σ, μ) be a measure space and T a measure preserving transformation of X into itself. Let f be an integrable function on X . Let

$$E = \{x \mid \text{at least one of the sums } f(x) + f(Tx) + \cdots + f(T^n x) \text{ is positive}\} \qquad (12.7)$$

Then

$$\int_E f(x) \mu(dx) \geq 0 \qquad (12.8)$$

Lecture 13

We shall now prove the maximal ergodic theorem, but first we need a definition and a lemma.

Definition:

Let $a_1\ a_2 \ldots a_n$ be a sequence of real numbers and m an integer ($1 \leq m \leq n$); then an element a_k of the sequence is said to be an m-leader if there exists an integer p ($1 \leq p \leq m$) such that

$$a_k + a_{k+1} + \ldots + a_{k+p-1} \geq 0.$$

Then, a_k is said to lead this sum.

Lemma

The sum of all m-leaders of $a_1 \ldots a_n$ is non-negative.

Proof

If $a_1 \ldots a_n$ has no m-leaders, the lemma is vacuously true. Therefore, assume a_k to be the first m leader and $a_k + \ldots + a_{k+p-1}$, the shortest sum it leads. Then I assert that each a_h occurring in this sum is also an m-leader and in fact

$$a_h + a_{h+1} + \ldots + a_{h+p-1} \geq 0.$$

For suppose $a_h + \ldots + a_{k+p-1} < 0$. Then $a_k + a_{k+1} + \ldots a_{h-1} > 0$ which violates the assumption that $a_k + \ldots + a_{k+p-1}$ is the shortest sum that a_k leads, so the assertion is proved. To complete the lemma it suffices to break up the sequence into blocks of shortest non-negative sums as shown

$$a_1 \ldots + (a_k + \ldots + a_{k+p-1}) + \ldots + (\qquad) + \ldots$$

Then a_k is an m-leader and by the assertion just proved each term in the block is an m-leader and their sum is non-negative. This is true for each block so the lemma is proved. We state this argument more explicitly. Apply the process used in the assertion to the sequence $a_{k+p} + \ldots + a_n$; that is, look for the first m-leader and the shortest sum it leads. The above argument shows that every term of that sum is an m-leader. Proceed in this manner until the set of m-leaders is exhausted. The result is a break-up of $a_1 \ldots a_n$ into blocks $a_1 \ldots (a_k \ldots) \ldots (a_\ell \ldots) \ldots (\quad) \ldots a_n$. Each block consists entirely of m-leaders and each m-leader is in a block and the sum over each block is non-negative. So the lemma is true.

Proof of the Maximal Ergodic Theorem

Let $E_m = \{x \mid f(x) + f(Tx) + \cdots + f(T^p x) \geq 0$ for at least one integer $p \leq m\}$ (13.1)

Since f is assumed to be measurable and T is a measurable transformation, E_m is a measurable set. Also

$$E_m \subset E_{m'}, \quad \text{if } m < m' \tag{13.2}$$

and $\bigcup_m E_m = E \equiv$ set of all χ such that at least one such sum as occurs in the definition of E_m is non-negative. Thus, to prove the theorem, it is clearly sufficient to prove that

$$\int_{E_m} f(x)\,d\mu(x) \geq 0 \tag{13.3}$$

We now do some skulduggery that allows us to make good use of the lemma just proved. Let n be a positive integer and consider for each point x the m-leaders of the sequence $f(x)$, $f(Tx)$, $f(T^{n+m-1}x)$, We let n be the running index and we are looking for m leaders, hence the final term is $f(T^{n+m-1}x)$. Let $\mathcal{S}(x)$ be the sum of these m-leaders. Let S_k be the union of all x such that $f(T^k x)$ is an m-leader of this sequence. Finally let $\chi_{S_k}(x)$ be the characteristic function of S_k.

Then

$$\mathcal{S}(x) = \sum_{k=0}^{n+m-1} f(T^k x)\,\chi_{S_k}(x) \tag{13.4}$$

It is clear that $s(x)$ is measurable and integrable. So $\int s(x)\,d\mu(x)$ exists and is non-negative and hence applying this to (13.4)

$$\sum^{n+m-1} \int f(T^k x)\,d\mu(x) \geq 0 \tag{13.5}$$

We make the following assertion:

The first n terms of this sum are equal and each of the last m is bounded by $\int |f(x)| \, d\mu(x)$.

If we admit this, the proof is easy to complete because then

$$n \int_{S_o} f(x)\, d\mu(x) + m \int |f(x)|\, d\mu(x) \geq 0 \quad , \tag{13.6}$$

and since $S_o = E_m$

$$\int_{E_m} f(x)\, d\mu(x) + \frac{m}{n} \int |f(x)|\, d\mu(x) \geq 0 \tag{13.7}$$

for all n.

Letting $n \to \infty$, we get

$$\int_{E_m} f(x)\, d\mu(x) \geq 0$$

which is the desired equation (13.3).

We now justify the assertion. To do this, notice that for $k = 1, \ldots, n$, the statement $x \in S_k$ is equivalent to $f(T^k x) + f(T^{k+1} x) + \ldots + f(T^{k+p-1} x) \geq 0$ for some $p \leq m$. But this is equivalent to $T^k x \in S_o$. That is,

$$S_k = T^{-k} S_o \qquad k = 1, \ldots, n \tag{13.8}$$

Hence

$$\int_{S_k} f(T^k x)\, d\mu(x) = \int_{T^{-k} S_o} f(T^k x)\, d\mu(x)$$

$$= \int_{S_o} f(g)\, d\mu(g)$$

where we have used the invariance of the measure. This establishes the equality of the first m terms. The bound for the last terms follows from

$$\left| \int_{S_k} f(T^k x)\, d\mu(x) \right| \leq \int_X |f(T^k x)|\, d\mu(x) = \int_X |f(x)|\, d\mu(x)$$

So, the assertion and hence the theorem is proved.

We now introduce some further terminology for sequences of real numbers.

$\inf a_n$ = g. l. b. a_n = greatest lower bound of a_n

$\sup a_n$ = l.u. b. a_n = lowest upper bound a_n

$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf_{n\geq N} a_n$$

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup_{n\geq N} a_n$$

It is notorious in the theory of real analysis that lim inf and lim sup always exist and that when they are equal the limit exists and is equal to them. See for example Apostol - Mathematical Analysis - Chapter 12.

Theorem (Individual Ergodic Theorem - Discrete Case)

Let T be a measure - preserving transformation of a measure space (X, Σ, μ) into itself, and let f be a real integrable function, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$$

exists for almost all x and defines an integrable invariant function f^*. Furthermore if $\mu(X) < \infty$ then

$$\int f^*(x)\, d\mu(x) = \int f(x)\, d\mu(x)$$

and we can identify it with the function in the mean ergodic theorem.

Proof

Consider the sequence of averages $\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$. Let a, b be two real numbers a < b and let Y (a,b) be the set of all x such that

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) < a < b < \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$$

We can without loss of generality take $b > 0$ since we can always consider $-f$ instead of f if necessary. The next step is to prove that

$\mu(Y(a,b)) < \infty$ and then to prove that

$\mu(Y(a,b)) = 0$. For simplicity we suppress the a, b on $Y(a,b)$.

Y is certainly a measurable set. Furthermore Y is an invariant set since

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^{j+1}x) = \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$$

That is, the equality holds with x replaced by Tx. Let C be any measurable subset of Y of finite measure and consider $f - b\chi_C$.

Lecture 14.

The function $f - b\chi_c$ is certainly integrable and so we can apply the maximal ergodic theorem to it. Let F be the analogue for this function of the set E occurring in the statement of the maximal ergodic theorem. Thus we have

$$\int_F [f(x) - b\chi_c(x)]\, d\mu(x) \geq 0 \tag{14.1}$$

Now $Y \subseteq F$

for if $x \in Y \Rightarrow b < \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$

and this implies that at least one of the sums

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) > b$$

This in turn implies that

$$\sum_{j=0}^{n-1} [f(T^j x) - b\chi_c(T^j x)] > 0$$

and hence $x \in F$. Thus $Y \subseteq F$.

Hence

$$\int_X |f(x)|\, d\mu(x) \geq \int_F |f(x)|\, d\mu(x) \geq b\mu(c) \tag{14.2}$$

$$\therefore \mu(c) \leq \frac{1}{b}\int_X |f(x)|\, d\mu(x)$$

This implies $\mu(Y) < \infty$. To prove this statement recall that μ is σ-finite. That is,

$$\exists\, S_i \ni X = \cup S_i \text{ and } \mu(S_i) < \infty.$$

Without loss of generality we can take the S_i as non-overlapping. Then

$$Y = \bigcup_i (Y \cap S_i) \tag{14.3}$$

Hence

$$\mu(Y) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} (Y \cap S_i)\right)$$

$$\leq \frac{1}{b}\int |f(x)|\, d\mu(x) \tag{14.4}$$

To see now that $\mu(Y)$ is in fact zero, apply the maximal ergodic theorem to the functions $f - b$ and $a - f$ on the set $Y(a,b)$ itself. This is legitimate because $Y(a,b)$ is invariant under T (already shown) and both $f - b$ and $a - f$ are integrable there (Y has finite measure).

So we get

$$\int_Y (f(x) - b)\, d\mu(x) \geq 0 \qquad (14.5a)$$

$$\int_Y (a - f(x))\, d\mu(x) \geq 0 \qquad (14.5b)$$

Adding yields:

$$(a-b) \int_Y d\mu(x) \geq 0 \qquad (14.6)$$

But b > a

$$\therefore \mu(Y) = 0 \qquad (14.7)$$

Applying this result for all rational a,b ∋ a < b we see that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \qquad (14.8)$$

for all x with the exception of a set of measure zero. The limit function f^* is also, in fact, integrable. To see this we will need a lemma which we state without proof. For the proof see Halmos - Measure page 113.

<u>Fa tou's Lemma</u>

If f_n is a sequence of integrable functions for which $\liminf_{n\to\infty} \int f_n(x) d\mu(x) < \infty$. Then the limit function $f(x)$ defined by $f(x) = \liminf_{n\to\infty} f_n(x)$ is integrable and

$$\int f(x)\, d\mu(x) \leq \liminf_{n\to\infty} \int f_n(x)\, d\mu(x)$$

Now back to our assertion that f^* is integrable.

Consider

$$\int \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \right| d\mu(x) \leq \frac{1}{n} \sum_{j=0}^{n-1} \int |f(T^j x)|\, d\mu(x) \leq \int |f(x)|\, d\mu(x)$$

and apply Fatou's lemma, then it follows that f^* is integrable.

The invariance of f^* under T is simply a consequence of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^{j+1} x)$$

whenever the limit exists. But this states that

$$f^*(Tx) = f^*(x) \qquad (14.9)$$

It remains to prove that if $\mu(X) < \infty$ then

$$\int f^*(x)\, d\mu(x) = \int f(x)\, d\mu(x)$$

Note that this is the only place where we use $\mu(X) < \infty$.

Apply the maximal ergodic theorem again.

If $f^*(x) \geqslant a$

everywhere on an invariant set $X' \subseteq X$ (obviously measurable), then at least one of the sums

$$\sum_{j=0}^{n-1} \left(f(T^j x) - a + \varepsilon \right)$$

must be non-negative for each $\varepsilon > 0$ and $x \in X'$

and hence

$$\int_{X'} f(x)\, d\mu(x) \geqslant a\, \mu(X') \tag{14.10}$$

similarly $f^*(x) \leqslant b$ guarantees that

$$\int_{X'} f(x)\, d\mu(x) \leqslant b\mu(X') \tag{14.11}$$

Now let

$$X(k,n) = \{ x \mid k2^{-n} \leqslant f^*(x) < (k+1)2^{-n};\ n > 0,\ k \text{ and } n \text{ integers} \} \tag{14.12}$$

and apply (14.10) and (14.11) to $X(k,n)$. This is again legitimate since $X(k,n)$ is invariant.

Hence we get

$$k2^{-n}\mu(X(k,n)) \leqslant \int_{X(k,n)} f(x)\, d\mu(x) < (k+1)2^{-n}\mu(X(k,n)) \tag{14.13}$$

Butit also follows from integrating

$$k2^{-n} \leqslant f^*(x) < (k+1)2^{-n} \tag{14.14}$$

that

$$k2^{-n}\mu(X(k,n)) \leqslant \int_{X(k,n)} f^*(x)\, d\mu(x) < (k+1)2^{-n}\mu(X(k,n)) \tag{14.15}$$

subtracting (14.15) from (14.13)

$$-2^{-n}\mu(X(k,n)) \leqslant \int_{X(k,n)} (f(x) - f^*(x))\, d\mu(x) \leqslant 2^{-n}\mu(X(k,n)) \tag{14.16}$$

Now sum over k for fixed n to get

$$\left| \int_X (f - f^*)\, d\mu(x) \right| \leq \frac{\mu(X)}{2^m} \qquad (14.17)$$

Since this is true for all positive integers, the left hand side vanishes.

This completes the discrete form of the theorem. For applications to statistical mechanics we need the continuous case of the theorem. This, however, follows readily if we observe that the original proof did not really depend on being discrete. We first, however, state a corollary whose proof is left as an exercise.

<u>Corollary</u>

Let T be a measure-preserving transformation of a measure space (X,Σ,μ) into itself. Then every subsequence of the family $\frac{1}{n-m}\sum_{j=m}^{n-1} f(T^j x)$ $n > m \geq 0$

for which $n-m \to \infty$, converges almost everywhere to the same function f^* which by the above theorem is integrable. Suppose further that there exists a set S of measure zero in X so that T is a bijection of X-S, then the same conclusion holds with $n>m\geq 0$ replaced by $n>m$.

We know that if T is ergodic then f^* is a constant almost everywhere and f^* is integrable.

Hence it follows that if

$$\mu(X) = \infty \quad \text{then } f^* = 0 \quad \text{a.e.}$$

and if $\mu(X) < \infty$ then

$$\int f(x)\, d\mu(x) = \int f^*(x)\, d\mu(x) = f^*(x)\mu(X)$$

So finally, for ergodic T we get

$$\lim_{n-m\to\infty} \frac{1}{n-m}\sum_{j=m}^{n-1} f(T^j x) = \frac{\int f(x)\, d\mu(x)}{\mu(X)} \quad \text{a.e.} \qquad (14.18)$$

That is, the time average is almost everywhere equal to the space average. Notice, however, that this does not prove ergodicity. That will come later.

Lecture 14

The function $f - b\chi_C$ is certainly integrable and so we can apply the maximal ergodic theorem to it. Let F be the analogue for this function of the set E occurring in the statement of the maximal ergodic theorem. Thus we have

$$\int_F [f(x) - b\chi_C(x)]\, d\mu(x) \geq 0 \tag{14.1}$$

Now $Y \subseteq F$

for if $x \in Y \Rightarrow b < \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$

and this implies that at least one of the sums

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) > b$$

This in turn implies that

$$\sum_{j=0}^{n-1} [f(T^j x) - b\chi_C(T^j x)] > 0$$

and hence $x \in F$. Thus $Y \subseteq F$.

Hence

$$\int_X |f(x)| d\mu(x) \geq \int_F |f(x)| d\mu(x) \geq b\mu(C) \tag{14.2}$$

$$\therefore \mu(C) \leq \frac{1}{b}\int_X |f(x)|\, d\mu(x)$$

This implies $\mu(Y) < \infty$. To prove this statement recall that μ is σ-finite. That is,

$$\exists\ S_i \ni X = \bigcup S_i \text{ and } \mu(S_i) < \infty.$$

Without loss of generality we can take the S_i as non-overlapping. Then

$$Y = \bigcup_i (Y \cap S_i) \tag{14.3}$$

Hence

$$\mu(Y) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} (Y \cap S_i)\right)$$

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To see now that $\mu(Y)$ is in fact zero, apply the maximal ergodic theorem to the functions $f - b$ and $a - f$ on the set $Y(a,b)$ itself. This is legitimate because $Y(a,b)$ is invariant under T (already shown) and both $f - b$ and $a - f$ are integrable there (Y has finite measure).

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$$\int f(x)\, d\mu(x) \leq \liminf_{n\to\infty} \int f_n(x)\, d\mu(x)$$

Now back to our assertion that f^* is integrable.

Consider

$$\int \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \right| d\mu(x) \leq \frac{1}{n} \sum_{j=0}^{n-1} \int |f(T^j x)|\, d\mu(x) \leq \int |f(x)|\, d\mu(x)$$

and apply Fatou's lemma, then it follows that f^* is integrable.

The invariance of f^* under T is simply a consequence of

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It remains to prove that if $\mu(X) < \infty$ then

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Note that this is the only place where we use $\mu(X) < \infty$.

Apply the maximal ergodic theorem again.

If $f^*(x) \geq a$

everywhere on an invariant set $X' \subseteq X$ (obviously measurable), then at least one of the sums

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and apply (14.10) and (14.11) to X(k,n). This is again legitimate since X(k,n) is invariant.

Hence we get

$$k2^{-n}\mu(X(k,n)) \leq \int_{X(k,n)} f(x)\, d\mu(x) < (k+1)2^{-n}\mu(X(k,n)) \tag{14.13}$$

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subtracting (14.15) from (14.13)

$$-2^{-n}\mu(X(k,n)) \leq \int_{X(k,n)} (f(x) - f^*(x))\, d\mu(x) \leq 2^{-n}\mu(X(k,n)) \tag{14.16}$$

Now sum over k for fixed n to get

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Corollary

Let T be a measure-preserving transformation of a measure space (X,Σ,μ) into itself. Then every subsequence of the family $\frac{1}{n-m}\sum_{j=m}^{n-1} f(T^j x)$ $n > m \geq 0$ for which $n-m \to \infty$, converges almost everywhere to the same function f^* which by the above theorem is integrable. Suppose further that there exists a set S of measure zero in X so that T is a bijection of X-S, then the same conclusion holds with $n>m\geq 0$ replaced by $n>m$.

We know that if T is ergodic then f^* is a constant almost everywhere and f^* is integrable.

Hence it follows that if

$$\mu(X) = \infty \quad \text{then } f^* = 0 \quad \text{a.e.}$$

and if $\mu(X) < \infty$ then

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So finally, for ergodic T we get

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That is, the time average is almost everywhere equal to the space average. Notice, however, that this does not prove ergodicity. That will come later.

Lecture 15

Theorem

Individual Ergodic Theorem, Continuous Case. Let $x \to x_t$ $-\infty < t < \infty$ be a measurable flow on a measure space (X, Σ, μ), then

$$\lim_{T-S\to\infty} \frac{1}{T-S} \int_S^T f(x_t)\, dt \quad T > S$$

exists for almost all x independent of the individual behaviour of T and S provided only that $T - S \to \infty$. It defines an invariant integrable function f^*. And, furthermore, if $\mu(X) < \infty$ then

$$\int f^*(x)\, d\mu(x) = \int f(x)\, d\mu(x)$$

Proof:

Before the proof begins we have to ascertain that the indicated integral makes sense, i.e., that $f(x_t)$ is locally integrable in t for almost all x.

This, however, follows from 3 things.

1. Invariance of the measure under $x \to x_t$
2. Measurability of $(t,x) \to x_t$
3. Fubini's Theorem:

If $|f(x,t)|$ is an integrable function on $X \times \mathbb{R}$, then $\int d(x) \int |f(x,t)| dt$ or any other order exists and hence $\int |f(x,t)| dt$ exists for almost all x.

In our case we have from 1. & 2. that

$\int |f(x_t)| d\mu(x)$ is independent of t and finite and hence that

$\int_S^T dt |f(x_t)|$ exists for almost all x.

Furthermore, we can restrict our attention to integral S and T because

$$\frac{1}{T-S}\int_S^T f(x_t)\,dt = \frac{[T]-[S]}{T-S}\cdot\frac{1}{[T]-[S]}\int_{[S]}^{[T]} f(x_t)\,dt + \frac{1}{T-S}\left[\int_{[T]}^{T} f(x_t)\,dt + \int_S^{[S]} f(x_t)\,dt\right] \quad T > S+1 \tag{15.1}$$

We are interested in $T - S \to \infty$ and so $T > S + 1$ always.

Furthermore, the remainder term

$$R = \frac{1}{T-S} \left[\int_{[T]}^{T} f(x_t)dt + \int_{S}^{[S]} f(x_t)dt\right] \to 0 \text{ because}$$

$$|R| \leq \frac{1}{T-S} \left[\int_{[T]}^{[T]+1} |f(x_t)|\, dt + \int_{[S]-1}^{[S]} |f(x_t)|dt\right]$$

Now let A be the flow through 1 second

Then

$$|R| \leq \frac{1}{T-S} \left[A^{[T]} \int_0^1 |f(x_t)|dt + A^{[S]+1} \int_0^1 |f(x_t)|dt\right]$$

Now use the corollary of the discrete form of the theorem in the form that

$$a_{n,m} = \frac{1}{n-m} \sum_{j=m}^{n-1} f(T^j x) \text{ converges a.e.}$$

$$|a_{n,m} - a_{n-1,m+1}| \to 0 \text{ a.e.}$$

But

$$|a_{n,m} - a_{n-1,m+1}| = \left|\frac{1}{n-m}\sum_{j=m}^{n-1} f(T^j x) - \frac{1}{n-m-2}\sum_{j=m+1}^{n-2} f(T^j x)\right|$$

$$= \left|\frac{1}{n-m}\sum_{j=m}^{n-1} f(T^j x) - \frac{1}{n-m-2}\sum_{j=m+1}^{n-2} f(T^j x) + \frac{1}{n-m}\sum_{j=m+1}^{n-2} f(T^j x) - \frac{1}{n-m}\sum_{j=m+1}^{n-2} f(T^j x)\right|$$

$$= \left|\frac{1}{n-m}\left(f(T^m x) + f(T^{n-1} x)\right) - \frac{2}{n-m} \cdot \frac{1}{n-m-2}\sum_{j=m+1}^{n-2} f(T^j x)\right|$$

Now

$$\frac{1}{n-m-2}\sum_{j=m+1}^{n-2} f(T^j x) \to f^* \text{ a.e}$$

And hence

$$\therefore \frac{2}{n-m} \cdot \frac{1}{n-m-2}\sum_{j=m+1}^{n-2} f(T^j x) \to 0 \quad \text{a.e.}$$

$$\frac{1}{n-m} |f(T^m_x) + f(T^{n-1}_x)| \to 0$$

We now replace T by A and f(x) by

$$h(x) = \int_0^1 |f(x_t)|dt$$

Hence

$$\frac{[T]-[S]}{T-S} \cdot \frac{1}{[T]-[S]} \left| A^{[T]}\int_0^1 |f(x_t)|dt + A^{[S]+1}\int_0^1 |f(x_t)|\, dt\right| \to 0$$

Since $\frac{[T]-[S]}{T-S} \to 1.$

$\therefore \quad |R| \to 0$ as needed.

Now notice that when S and T are integers

$$\frac{1}{T-S}\int_S^T f(x_t)\,dt = \frac{1}{T-S}\sum_{j=S}^{T-1} g(A^j x) \tag{15.2a}$$

where $A: x \to x_1$ Ie. $Ax = x_1$ (15.2b)

and

$$g(x) = \int_o^1 f(x_t)\,dt \tag{15.2c}$$

Now using the corollary of the discrete form of the individual ergodic theorem we know that the right side of (15.2a) has a limit almost everywhere. So the first part of the theorem is proved. That is, f^* exists and is independent of the individual behaviour of T and S as long as $T - S \to \infty$. We must verify that f^* is invariant and integrable. But from the discrete form of the theorem we get integrability and invariance under A. To get invariance under the whole group consider

$$f^*(x_t) = \lim_{T-S\to\infty} \frac{1}{T-S}\int_S^T f(x_{t+\tau})dt$$

$$= \lim_{T-S\to\infty} \frac{1}{T-S}\int_{S+\tau}^{T+\tau} f(x_t)\,dt$$

$$= f^*(x) \quad \text{a.e.} \tag{15.3}$$

We now prove the last part of the theorem.

If $\mu(X) < \infty$, the discrete form of the theorem applies.

$$\int f^*(x)\,d\mu(x) = \int g(x)\,d\mu(x) \tag{15.4}$$

But the right side is in fact $\int(\int_o^1 f(x_t)dt)\,d\mu(x)$

so we get

$$\int f^*(x)\,d\mu(x) = \int_o^1 dt(\int f(x_t)\,d\mu(x))$$

$$= \int f(x_t)\,d\mu(x) \tag{15.5}$$

The last equality follows from the fact that

$\int f(x_t)\,d\mu(x)$ is independent of t and $\int_o^1 dt = 1$.

So the theorem is proved.

We now state another consequence of this convergence that will be needed later.

Theorem

If T is a measure-preserving transformation of a measure space (X, Σ, μ) into itself, if $\mu(X) < \infty$, and if f is an integrable function, then

$$\lim_{n-m\to\infty} \int \left| \frac{1}{n-m} \sum_{j=m}^{n-1} f(T^j_x) - f^*(x) \right| d\mu(x) = 0 \qquad (15.6)$$

That is, the convergence is in L^1.

Proof:

1. If f is a bounded function, that is $|f(x)| < M$ then

$$\left| \frac{1}{n-m} \sum_{j=m}^{n-1} f(T^j_x) \right| \leq \frac{1}{n-m} \sum |f(T^j_x)| \leq M$$

and the result follows by the following standard theorem of Lebesgue theory.

The Lebesgue Dominated Convergence Theorem – Halmos page 110

If f_n $n = 1,2,\ldots$ is a sequence of integrable functions which converges a. e. to f and g is an integrable function such that

$$|f_n(x)| \leq |g(x)| \qquad \text{a.e.}$$

then f is integrable and

$$\lim_{n\to\infty} \int |f_n(x) - f(x)| \, d\mu(x) = 0$$

2. 2. If f is not bounded it can be approximated arbitrarily well in norm by a bounded function. That is, for each $\varepsilon > o$ there exists a g which is bounded and such that

$$\int d\mu(x) \; |f(x) - g(x)| < \varepsilon$$

Now

$$\left\| \frac{1}{n-m} \sum_{j=m}^{n-1} f(T^j_x) - f^*(x) \right\|_1 \leq \left\| \frac{1}{n-m} \sum_{j=m}^{n-1} (f(T^j_x) - g(T^j_x)) \right\|_1 + \left\| \frac{1}{n-m} \sum_{j=m}^{n-1} g(T^j_x) - g^*(x) \right\|_1 + \| f^*(x) - g^*(x) \|_1 \qquad (15.7)$$

where $\| \cdot \|_1$ means the L^1 norm and the existence of g^* is guaranteed by our previous theorem.

We consider the terms on the right side individually.

$$\| \text{first term} \|_1 \leq \frac{1}{n-m} \sum_{j=m}^{n-1} \| f(T^j x) - g(T^j x) \|_1 = \| f-g \|_1 < \varepsilon$$

where we have used the invariance of the measure under T.

For the second term the first part of the proof applies. To control the third term, apply Fatou's Lemma to $h = f-g$ and get that

$$\| f^* - g^* \|_1 \leq \| f-g \|_1 < \varepsilon$$

Recall that Fatou says.

If g_n is a sequence of positive functions and if $\liminf \int g_n(x)\, d\mu(x) < \infty$ then

$$\int \liminf g_n(x)\, d\mu(x) \leq \liminf \int g_n(x)\, d\mu(x))$$

In our case take

$$g_{n-m}(x) = \frac{1}{n-m} \left| \sum_{j=m}^{n-1} h(T^j x) \right|$$

Then by the same argument as we used above,

$$\liminf \int g_{n-m}(x)\, d\mu(x) \leq \int |h(x)|\, d\mu(x)$$

and hence

$$\int \lim_{n-m\to\infty} \frac{1}{n-m} \left| \sum_{j=m}^{n-1} h(T^j x) \right| d\mu(x) \leq \int |h(x)|\, d\mu(x) < \varepsilon$$

So the proof is complete.

Example

Flow on T_n (n -Torus)

We shall use a "Fundamental Theorem of Fourier analysis" namely that the functions

$\exp(2\pi i \sum_{j=1}^{n} k_j x_j)$ are a complete orthonormal set in L^2 $(T_n, dx_1, dx_2 \dots dx_n)$ where $k_1 \dots k_n$ are integers. So we have

$$f(x_1 \cdots x_n) \sim \sum_{k_1 \cdots k_n} b_{k_1 \cdots k_n} \exp(2\pi i \sum_{j=1}^{n} k_j x_j) \qquad (15.8)$$

where $\sim$ means that the right side converges to the left side in the mean, and

$$b_{k_1 \cdots k_n} = \int_{T_n} dx_1 \cdots dx_n \exp(-2\pi i \sum_{j=1}^{n} k_j x_j) f(x_1 \cdots x_n) \qquad (15.9)$$

x_i always modulo 1.

The flow we had before was

$$(U_t f)^{\prime}(x) = f(x + a\,t) \tag{15.10}$$

where a is a vector and there are no integers l_j such that $\Sigma l_j a_j = 0$ except $l_j = 0$ if this flow is to be ergodic. Note that the exponentials are eigenfunctions of this flow, since

$$U_t \exp(2\pi i \sum k_j x_j) = \exp(2\pi i \sum k_j (x_j + a_j t))$$

$$= \exp(2\pi i \sum k_j a_j\, t) \cdot \exp(2\pi i \sum k_j x_j) \tag{15.11}$$

So every exponential is a proper function with the proper value

$$\exp(2\pi i\, \Sigma\, k_j a_j\, t).$$

From this it is clear that the linear independence of the components of a over the integers is a necessary condition for ergodicity. It is also sufficient because $U_t f$ has the Fourier expansion whose coefficients are

$$b_{k_1 \ldots k_n} \exp(2\pi i \sum_{j=1}^{n} k_j a_j t)$$

These coefficients can not be $b_{k_1 \ldots k_n}$ if the linear independence holds. This disposes of L^2 invariant functions. If there is an invariant function f(x) not in L^2, one can form arctan f(x). This function will be non constant if and only if f is non constant. Also arctan f is bounded and hence in L^2. So the preceding argument shows that arctan f(x) is constant almost everywhere. Hence f is constant almost everywhere.

This flow was studied by H. Weyl prior to the ergodic theorems. (Math. Ann. 77 (1916) 313-352). He was able to show

Theorem

If f is a Riemann integrable function on T_n, then under any of the above ergodic flows

$$\lim_{T-S\to\infty} \frac{1}{T-S} \int_S^T f(x_t)\, dt = \int f(x)\, dx \text{ for all } x.$$

As we shall see, this flow is not mixing. The flows that one usually wants to consider in statistical mechanics, are mixing. We will now characterize mixing and other properties of flows in terms of the spectrum of the associated one-parameter group.

Lecture 16

The flows on T^n are closely related to what are called integrable cases in Hamiltonian mechanics. Consider a system whose phase space is T^n cross (subset of R^n) where the coordinates lie in the T^n, and the momenta in the R^n, and such that H is a function of p's alone. Hamilton's equations now are

$$\dot{q}_j = \omega_j(p), \quad j = 1,\ldots,n \tag{16.1}$$

$$\dot{p}_j = 0 \quad , \quad j = 1,\ldots,n$$

where

$$\omega_j(p) = \frac{\partial H}{\partial p_j}(p) \quad . \tag{16.2}$$

Each torus p = constant is left invariant by the flow. If the $\omega_i(p)$ are linearly independent over the integers, then the flow is said to be conditionally periodic. Variables q, p for which the equations of motion take the form (16.1) are called angle and action variables. Hamiltonians which can be brought into this form by a canonical transformation are called integrable.

There is a well known sufficient condition that H be integrable; it is that there exist n integrals of motion in involution, meaning that if $f_1,\ldots,f_n$ are these integrals of motion that

$$\{f_j, f_k\} = 0 \quad j \neq k \tag{16.3}$$

where { , } is the Poisson bracket:

$$\{f,h\} = \sum_{\ell=1}^{n}\left(\frac{\partial f}{\partial p_\ell}\frac{\partial h}{\partial q_\ell} - \frac{\partial f}{\partial q_\ell}\frac{\partial h}{\partial p_\ell}\right) \tag{16.4}$$

Almost all the problems interesting for statistical mechanics are not integrable.

The relation between spectrum and ergodicity.

Theorem

An invertible measure-preserving transformation T of a measure space (X, Σ, μ) of finite measure is ergodic if and only if 1 is a simple (i.e., non-degenerate) proper value of the induced unitary operator U.

If T is ergodic then the absolute value of every proper function is constant almost everywhere, every proper value is simple, and the proper values form a subgroup of the multiplicative group of the unit circle.

Proof

If $\mu(X) < \infty$ then the constant functions are integrable and $U1 = 1$; so 1 is a proper function of proper value 1. An integrable function is invariant if and only if it is a proper function of proper value 1, this is so since

$$(Uf)(x) = f(Tx) \tag{16.5}$$

and, therefore,

$$Uf = f \text{ as vector of } L^2(\mu) \Leftrightarrow f(x) = f(Tx) \text{ a.e.}. \tag{16.6}$$

The theorem stating the equivalence of ergodicity and the constancy almost everywhere of invariant integrable functions implies that the dimension of the subspace of $L^2(X, \mu)$ consisting of functions invariant under U is one, and conversely if the dimension is one the same theorem implies T is ergodic.

We now look at other proper functions. If g is one such

$$Ug = \lambda g \Leftrightarrow g(Tx) = \lambda g(x) \quad \text{a.e.} \tag{16.7}$$

where since U is unitary $|\lambda| = 1$.

Consequently,

$$|g(Tx)| = |\lambda| \, |g(x)| = |g(x)|, \tag{16.8}$$

$$\therefore \quad U|g| = |g|. \tag{16.9}$$

Conclusion: if T is ergodic $|g|$ is constant almost everywhere.

If there were two linearly independent proper functions f, g associated with the proper value λ, then f/g would be a proper function associated with the proper value 1. By ergodicity of T, f/g = const. a.e. which contradicts the linear independence; hence, λ is a simple proper value. Note that the above division is legitimate since by what we have already shown $|g|$ = const. a.e., , hence $|g| \neq 0$ a.e., hence $|f|/|g|$ = const. a.e., and f/g is integrable.

Finally, if f is a proper function of proper value λ, and g is a proper function of proper value μ, then f/g is a proper function of proper value λ/μ. We see, therefore, that the proper values form a subgroup of the multiplicative group of the unit circle. Q.E.D.

We now state a continuous version of the above theorem.

Theorem

A measurable flow on a finite measure space is ergodic if and only if 1 is a simple proper value; that is, the subspace of L^2 of vectors left invariant by the induced one-parameter group is one dimensional.

If M_λ is the subspace of all square integrable functions satisfying

$$U_t f = \lambda^t f \quad \forall t, \ |\lambda| = 1 \tag{16.10}$$

then M_λ is one dimensional and $|f|$ is constant almost everywhere.

The proof of this theorem is the same as for the discrete case with U_t replacing U, x_t replacing Tx, and λ^t replacing λ.

We now want to return to the question of mixing and study some examples and properties. As a first example we show that the flow on T^n discussed before is not mixing though it is ergodic. To see this directly from the definition of mixing take

$$\int (\exp 2\pi i \ \Sigma k_j(x_j + a_j t)) \ \chi_S(x) dx_1 \ldots dx_n \tag{16.11}$$

where χ_S is a characteristic function such as

$$\chi_S(x) = \begin{cases} 1, & 0 \leqslant x_j \leqslant \alpha_j < 1 \\ 0 & \text{otherwise} \end{cases} \tag{16.12}$$

The integral (16.11) has no limit as $t \to \infty$ since it always oscillates with a constant amplitude; this cannot happen for a mixing flow for which the following is true:

$$\lim_{t \to \infty} \int f(x_t)\, \chi_S(x)\, \mu(dx) = \frac{\mu(S) \int f(x)\mu(dx)}{\mu(X)} . \tag{16.13}$$

Lecture 17

Unilateral and Bilateral Shifts; the Baker's Transformation

This section provides an example of a mixing transformation (see Halmos Measure Theory pp. 154 - 158).

Consider the set X_o of sequences $x_1, x_2, x_3, \ldots$, where $x_j \in \{0,1\}$. Such sequences can be mapped onto real numbers by the formula

$$x = \sum_{j=1}^{\infty} 2^{-j} x_j \tag{17.1}$$

then $0 \leq x \leq 1$ and x just that real number whose binary decimal is $.x_1x_2x_3\ldots$. Conversely each real number in the interval $[0,1)$ has a binary expansion unique except for the equivalence

$$.x_1x_2\ldots x_n100\ldots0\ldots = .x_1x_2\ldots x_n011\ldots1\ldots. \tag{17.2}$$

By convention we exclude the sequences ending in $1, 1, \ldots, 1, \ldots$. If we denote the map $\{x_i\}_{i=1,2,\ldots} \to x$ by S, then S is a bijection of $X = X_o$ - {sequences ending in $1, 1, \ldots, 1, \ldots$} onto $[0, 1)$. To avoid annoying complications we shall from now on work solely with X.

Lebesgue measure assigns a measure L(A) to each measurable subset of $[0,1)$; this measure being ordinary length when A is an interval. This induces a measure on X via the formula

$$\mu(Y) = L(S(Y)) \tag{17.3}$$

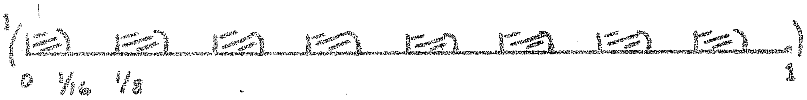
where evidently the measurable subsets Y of X are by definition the antecedents under S of the measurable subsets of $[0,1)$

Consider in particular the cylinder set

$$\Delta^{\alpha_{k_1}\cdots\alpha_{k_n}} = \{x_1, x_2, \ldots \mid x_{k_1} = \alpha_{k_1}, \ldots, x_{k_n} = \alpha_{k_n}\} \tag{17.4}$$

where $k_1, \ldots, k_n$ are distinct positive integers and $\alpha_{k_1}, \ldots, \alpha_{k_n}$ are 0 or 1. An example of such a set is:

$$\Delta^{0} = S^{-1}\Big(\quad\Big) \qquad (17.5)$$



We see that α_{k_i} being 0 or 1 determines whether the points we are considering are in the left or respectively the right half of the intervals formed by the $(k_i - 1)$-st binomial subdivision of the interval $[0,1)$.

One easily finds that

$$\mu(\Delta^{\alpha_{k_1} \cdots \alpha_{k_n}}) = 2^{-n} \qquad (17.6)$$

We define the transformation T by

$$(Tx)_j = x_{j+1}, \qquad (17.7)$$

hence

$$T^{-1}(\Delta^{\alpha_{k_1} \cdots \alpha_{k_n}}) = \Delta^{\alpha_{k_1+1} \cdots \alpha_{k_n+1}}, \qquad (17.8)$$

and $\mu(T^{-1}(\Delta)) = \mu(\Delta)$ whenever Δ is a cylinder.

To show that T is measure-preserving in general, one must show that the cylinders generate the σ-algebra of measurable sets. For this it is sufficient to show that by means of σ-algebra operations the cylinders can generate every set of the form $S^{-1}([a,b))$ where $0 \leq a < b \leq 1$, since the intervals generate the Borel sets of $[0,1)$ and the operation of taking inverse images commutes with the σ-algebra operations. However,

$$S^{-1}([a,b)) = \bigcap_{n} \bigcup_{\substack{\{\alpha_1, \ldots, \alpha_n\} \\ \sum_{j=1}^{n}(S^{-1}a)_j 2^{-j} \leq \sum_{i=1}^{n} \alpha_i 2^{-i} < b}} \Delta^{\alpha_1 \cdots \alpha_n} \qquad (17.9)$$

The right hand side is a countable intersection of finite unions of cylinders and hence is a set in the σ-algebra generated by the cylinders.

T induces a transformation $T^1 = STS^{-1}$ on the interval $[0,1)$

$$T^1 x = \Sigma_{j=1}^{\infty} 2^{-j}(Tx)_j = \Sigma_{j=1}^{\infty} 2^{-j} x_{j+1}$$

$$= \begin{cases} 2x, & x_1 = 0 \\ 2x-1, & x_1 = 1 \end{cases} = 2x \pmod 1. \qquad (17.10)$$

Consider now the space X_1 of all bilateral sequences

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots \quad , \qquad x_j = 0, 1 \tag{17.11}$$

where we leave out all sequences that are eventually constantly 1 in either the positive or the negative direction. We introduce the same cylinder sets as before but with the restriction to positive integers of the k_i removed. We introduce a map S_1 of X_1 onto $X \times X$ via

$$\begin{aligned} &\ldots, x_{-2}, x_{-1}, x_o, x_1, x_2, \ldots \\ &\rightarrow (x_1, x_2, \ldots; x_o, x_{-1}, x_{-2} \ldots) \quad , \end{aligned} \tag{17.12}$$

and then map this pair into two numbers from $[0,1)$ via the maps S and S' defined as:

$$\begin{aligned} x &= \Sigma_{j=1}^{\infty} \quad 2^{-j} x_j \quad ; \quad S : x_1, x_2, \ldots \rightarrow x \in [0,1) \\ y &= \Sigma_{j=-\infty}^{o} \quad 2^{j-1} x_j \quad ; \quad S' : x_o, x_{-1}, x_{-2}, \ldots \rightarrow y \in [0,1). \end{aligned} \tag{17.13}$$

The map $S_2 = (S \times S') \circ S_1 : \{x_i\} \rightarrow (x,y)$

maps X_1 onto $[0,1) \times [0,1)$ and we define

$$\mu\ (Y) = L_2\ (S_2\ (Y)) \tag{17.14}$$

where L_2 is the Lebesgue measure on the square and Y is an antecedent under S_2 of a Lebesgue measurable set.

It is still true that $\mu(\Delta^{\alpha_{k_1} \ldots \alpha_{k_n}}) = 2^{-n}$ and if $(Tx)_j = x_{j+1}$ then $\mu(T^{-1}(\Delta)) = \mu(\Delta)$ for all cylinders and again in fact for all measurable sets. The induced transformation T_1' on the square is

$$T'_1(x,y) = (x',y') \tag{17.15}$$

where

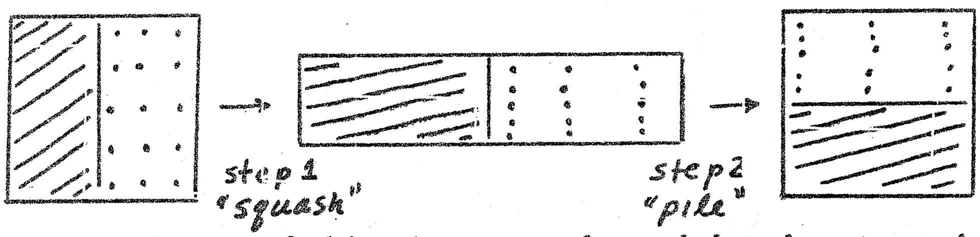
$$x' = \Sigma_{j=1}^{\infty} \quad 2^{-j} \; x_{j+1} \; = \; 2x \pmod 1 \tag{17.16}$$

$$y' = \Sigma_{j=-\infty}^{0} \quad 2^{j-1} \; x_{j+1}$$

$$= \tfrac{1}{2} \left(x_1 + \Sigma_{j=-\infty}^{0} \quad 2^{j-1} \; x_j\right)$$

$$= \begin{cases} \tfrac{1}{2} (y) \;, & x_1 = 0 \Leftrightarrow 0 \leq x < \tfrac{1}{2} \\ \tfrac{1}{2} (y+1), & x_1 = 1 \Leftrightarrow \tfrac{1}{2} \leq x < 1 \end{cases} \tag{17.17}$$

The transformation may be pictorially conceived of as taking place in two steps:



(17.18)

The similarity of this picture to what a baker does to a piece of dough is responsible for the name "Baker's transformation."

The two shift operations T introduced above are <u>mixing</u>; that is

$$\lim_{n \to \infty} \mu \left(T^{-n}(A) \cap B\right) = \mu(A) \; \mu(B). \tag{17.19}$$

To prove this we will first show that (17.19) holds for cylinders and then show that any measurable set can be approximated in measure by a finite union of cylinders thereby proving mixing in general.

Lecture 18

Proof of the mixing property of T

Let
$$A_1 = \Delta^{\alpha_{k_1} \cdots \alpha_{k_p}}$$
$$A_2 = \Delta^{\beta_{h_1} \cdots \beta_{h_q}} \qquad (18.1)$$

be two cylinders, then it is immediate that

$$A_1 \cap A_2 = \begin{cases} \phi \text{ if } \exists\, k_i = h_j \text{ for some } i, j, \text{ such that } \alpha_{k_i} \neq \beta_{h_j}. \\ \Delta^{\gamma_{\ell_1} \cdots \gamma_{\ell_r}} \text{ if } \alpha_{k_i} = \beta_{h_j} \text{ whenever } k_i = h_j \\ \text{and where } \ell_s \in \{k_1, \ldots, k_p\} \cup \{h_1, \ldots, h_q\} \\ \text{and } \gamma_{\ell_s} = \alpha_{k_i} \text{ if } \ell_s = k_i \text{ and } \gamma_{\ell_s} = \beta_{h_j} \\ \text{if } \ell_s = h_j. \end{cases} \qquad (18.2)$$

Now

$$T^{-n}(A_1) = \Delta^{\alpha_{k_1+n} \cdots \alpha_{k_p+n}} \qquad (18.3)$$

hence for sufficiently large n

$$\{k_1 + n, \ldots, k_p + n\} \cap \{h_1, \ldots, h_q\} = \phi \;; \qquad (18.4)$$

and in this case

$$T^{-n}(A_1) \cap A_2 = \Delta^{\alpha_{k_1+n} \cdots \alpha_{k_p+n} \beta_{h_1} \cdots \beta_{h_q}} \qquad (18.5)$$

moreover we then have

$$\mu(T^{-n}(A_1) \cap A_2) = 2^{-(p+q)} = \mu(A_1)\ \mu(A_2) \qquad (18.6)$$

and thus the mixing property holds for cylinders.

Let now

$$A = \bigcup_{i=1}^{r} A_i$$
$$B = \bigcup_{j=1}^{s} B_j \qquad (18.7)$$

...e finite unions of cylinders. We have shown in case r=s=1 that for sufficiently large n, $\mu(T^{-n}(A) \cap B) = \mu(A)\,\mu(B)$. By making use of (18.2) and the identity $\mu(S \cup T) = \mu(S) + \mu(T) - \mu(S \cap T)$ for any measurable sets S, T we can proceed by induction on either r or s. We illustrate the procedure by induction on r, the argument on s is entirely similar. Assuming by induction mixing holds for r and smaller unions we have for sufficiently large n

$$\mu(T^{-n}(\bigcup_{i=1}^{r} A_i \cup A_{r+1}) \cap B)$$

$$= \mu((T^{-n}(\bigcup_{i=1}^{r} A_i) \cap B) \cup (T^{-n}(A_{r+1}) \cap B)$$

$$= \mu(T^{-n}(\bigcup_{i=1}^{r} A_i) \cap B) + \mu(T^{-n}(A_{r+1}) \cap B)$$

$$- \mu(T^{-n}(\bigcup_{i=1}^{r} (A_i \cap A_{r+1})) \cap B)$$

$$= \mu(\bigcup_{i=1}^{r} A_i)\ \mu(B) + \mu(A_{r+1})\ \mu(B)$$

$$- \mu(\bigcup_{i=1}^{r} (A_i \cap A_{r+1}))\ \mu(B) = \mu(\bigcup_{i=1}^{r} A_i \cup A_{r+1})\ \mu(B)$$

We have used (18.2) to conclude that each of $A_i \cap A_{r+1}$ is a cylinder.

To prove the theorem in general we first note that the complement of a cylinder is a finite union of cylinders, in fact:

$$\left(\Delta^{\alpha_{k_1}\cdots\alpha_{k_n}}\right)^c = \bigcup_{i=1}^{n} \Delta^{(\alpha_{k_i}+1)(\mathrm{mod}\ 2)} \tag{18.9}$$

This follows directly from the definition of a cylinder. We introduce the symmetric difference $A \Delta B$ of two sets A, B:

$$A \Delta B = (A-B) \cup (B-A) \tag{18.10}$$

The symmetric difference has the following properties which we shall need

(I) $A^c \Delta B^c = A \Delta B$

(II) $(\bigcup_{i\in I} A_i) \Delta (\bigcup_{i\in I} B_i) \subset \bigcup_{i\in I}(A_i \Delta B_i)$ (18.11)

(III) $A \Delta B \subset (A \Delta D) \cup (D \Delta B)$

(I) follows from:

$$A - B = A \cap B^c = A^{cc} \cap B^c = B^c - A^c \; ; \tag{18.12}$$

(II) follows from:

$$x \in (\bigcup_{i\in I} A_i) - (\bigcup_{i\in I} B_i) \Leftrightarrow \exists i,\ x \in A_i \text{ and } \forall j,\ x \notin B_j \tag{18.13}$$

$$\Rightarrow \exists i,\ x \in A_i \text{ and } x \notin B_i \Rightarrow \exists i,\ x \in A_i - B_i$$

(III) follows from

$$A - B \subset (A - D) \cup (D - B). \tag{18.14}$$

Let Σ_o be the set of those measurable sets that can be approximated in measure arbitrarily well by finite unions of cylinders; that is,

$$A \varepsilon \Sigma_o \Leftrightarrow \forall \varepsilon > 0 \ \exists \text{ cylinders } D_1, \ldots, D_n$$

$$\text{such that } \mu(A \Delta (\bigcup_{k=1}^{n} D_k)) < \varepsilon \qquad (18.15)$$

If $A \varepsilon \Sigma_o$ and $\mu(A\Delta(\bigcup_{k=1}^{n} D_k)) < \varepsilon$, then by (18.11(I))

$$\mu(A^c \Delta (\bigcup_{k=1}^{n} D_k)^c) = \mu(A \Delta (\bigcup_{k=1}^{n} D_k)) < \varepsilon, \qquad (18.16)$$

but $(\bigcup_{k=1}^{n} D_k)^c = \bigcap_{k=1}^{n} D_k{}^c$ and (18.2) and (18. 9) show that this too is a finite union of cylinders, therefore

$$A \varepsilon \Sigma_o \Rightarrow A^c \varepsilon \Sigma_o. \qquad (18.17)$$

Let now $A_i \varepsilon \Sigma_o$, $i=1,2,\ldots$ and set $A = \bigcup_{i=1}^{\infty} A_i$. For a given $\varepsilon >$ pick an N such that

$$\mu(A \Delta (\bigcup_{i=1}^{N} A_i)) < \varepsilon/2, \qquad (18.18)$$

For each i, $i = 1,2,\ldots, N$ pick a finite union of cylinders $\bigcup_{k=1}^{n_i} D_k^i$ such that

$$\mu(A_i \Delta (\bigcup_{k=1}^{n_i} D_k^i)) < \varepsilon/2N. \qquad (18.19)$$

We now have using (18.11(II)) and (18.11(III))

$$\mu(A\Delta(\bigcup_{i=1}^{N} \bigcup_{k=1}^{n_i} D_k^i)) \leq$$

$$\leq \mu(A\Delta(\bigcup_{i=1}^{N} A_i)) + \mu((\bigcup_{i=1}^{N} A_i) \Delta (\bigcup_{i=1}^{N} \bigcup_{k=1}^{n_i} D_k^i)) \leq$$

$$\leq \varepsilon/2 + \mu(\bigcup_{i=1}^{N} (A_i \Delta (\bigcup_{k=1}^{n_i} D_k^i))) \leq$$

$$\leq \varepsilon/2 + \sum_{i=1}^{N} \mu(A_i \Delta (\bigcup_{k=1}^{n_i} D_k^i)) <$$

$$< \varepsilon/2 + N\varepsilon/2N = \varepsilon \qquad (18.20)$$

Therefore

$$A_i \varepsilon \Sigma_o,\ i=1,2,\ldots \Rightarrow \bigcup_{i=1}^{\infty} A_i \varepsilon \Sigma_o. \qquad (18.21)$$

By (18.17) and (18.21) we see that Σ_o is a σ-algebra and since it contains the cylinders it must coincide with Σ the σ-algebra of measurable sets since the latter is the smallest σ-algebra containing the cylinders.

We conclude therefore that any measurable set can be approximated arbitrarily well in measure by finite unions of cylinders.

Let F, G, now be any two measurable sets and A, B be finite unions of cylinders which approximate them in measure.

Now

$$
\begin{aligned}
&(T^{-n}(F) \cap G) - (T^{-n}(A) \cap B) \\
&= (T^{-n}(F) \cap G) \cap (T^{-n}(A) \cap B)^c \\
&= ((T^{-n}(F) \cap G) \cap (T^{-n}(A))^c) \cup ((T^{-n}(F) \cap G) \cap B^c) \\
&= (T^{-n}(F-A) \cap G) \cup (T^{-n}(F) \cap (G-B)) \,. \qquad (18.22)
\end{aligned}
$$

A similar equation holds with (A,B) and (F,G) interchanged; combining this with (18.27) we conclude:

$$
\begin{aligned}
&(T^{-n}(F) \cap G) \,\Delta\, (T^{-n}(A) \cap B) \\
&= (T^{-n}(F-A) \cap G) \cup (T^{-n}(A-F) \cap B) \cup \\
&\quad (T^{-n}(F) \cap (G-B)) \cup (T^{-n}(A) \cap (B-G)) \\
&\subset T^{-n}(A\Delta F) \cup (G\Delta B). \qquad (18.23)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mu((T^{-n}(F) \cap G) \,\Delta\, (T^{-n}(A) \cap B)) \le \\
&\le \mu(T^{-n}(A\Delta F)) + \mu(G\Delta B) = \\
&= \mu(A\Delta F) + \mu(G\Delta B), \qquad (18.24)
\end{aligned}
$$

and the right hand side can be made arbitrarily small. By picking A and B appropriately we see that we can make

$$
\left.
\begin{aligned}
&|\mu(T^{-n}(F) \cap G) - \mu(T^{-n}(A) \cap B)| < \epsilon/2 \\
&|\mu(A)\,\mu(B) - \mu(F)\,\mu(G)| < \epsilon/2
\end{aligned}
\right\} \qquad (18.25)
$$

For sufficiently large n we have already shown that $\mu(T^{-n}(A)\cap B) = \mu(A)\mu(B)$ and, therefore, in view of (18.25) we have

$$\lim_{n\to\infty} \mu(T^{-n}(F)\cap G) = \mu(F)\mu(G) \tag{18.26}$$

for all measurable F and G and hence the transformation T is mixing. Q.E.D.

Statistical Regularity

We now come to a series of results which aim at explaining the "roulette wheel problem"; that is, how can a mechanical system such as a roulette wheel which is presumably governed by a deterministic mechanics give rise to random outcomes? The relevance of this question to statistical mechanics is evident. The theorems of statistical regularity give a partial answer to this problem, however, as they are idealized to infinite relaxation times and thereby lead to the physically too strong result that if mixing is present every measurable set is statistically regular, whereas an actual physical system with finite relaxation times will have a smaller number of such sets. Nevertheless the idealization is useful in developing an insight into the situation and we shall pursue it.

We consider a roulette wheel idealized as a measurable flow; it is started at a given point $x\epsilon X$ (position-angular velocity space) and allowed to run continuously. At a time t one then observes whether x_t is in a measurable set or not (say whether the ball is on red or not); this procedure being a substitution for the actual process of playing roulette in which the ball is allowed to come to rest. Suppose the wheel is spun many times so that the probability distribution of initial x is given by a positive integrable function f satisfying $\int f d\mu = 1$ (The existence of this f is in fact a strong macroscopic assumption). Then the probability that the outcome A is found at time t is:

$$\int f(x)\ \chi_A(x_t)\ \ \mu(dx) \tag{18.27}$$

where χ_A is the characteristic function of the outcome A. It is the dependence of this number on f and t that we wish to study.

Definition

An event A (i.e. a measurable subset of X) is statistically regular relative to a measurable flow if for every positive integrable function f,

$$\lim_{t\to\infty} \frac{(f_{-t}, \chi_A)}{(f, 1)} = L(A) \qquad (18.28)$$

exists and is independent of f.

Lecture No. 19

If we take the f in the definition of statistical regularity of a set A relative to a measurable flow to be the characteristic function of a measurable set B of non-zero finite measure, then the defining statement becomes

$$\lim_{t \to \infty} \frac{\mu(B_{-t} \cap A)}{\mu(B)} = L(A) \tag{19.1a}$$

$$= \frac{\mu(B \cap A_t)}{\mu(B)} \tag{19.1b}$$

where the second equality follows from the invariance of the measure under the flow.

Theorem

If $\lim_{t \to \infty} \frac{\mu(B_{-t} \cap A)}{\mu(B)} = L(A)$ holds for every measurable set B of non-zero finite measure, then A is statistically regular with respect to the flow.

Before we state the proof of the theorem we need a definition.

Definition

A simple function is one of the form $f = \Sigma c_i \chi_{S_i}$ where the sum is finite and $\mu(S_i) < \infty$. Ie. f is an integrable step function.

Proof

If (19.1a) holds then

$$\lim_{t \to \infty} (f_{-t} \cap \chi_A) = L(A) \ (f,1) \tag{19.2}$$

for simple functions. It is a well-known theorem in analysis (see Riesz and Nagy - Functional Analysis page 33) that integrable functions can be arbitrarily well approximated in norm by simple functions. That is, given an integrable f and $\varepsilon > 0$, there exists a simple function g such that

$$\int |f(x) - g(x)| \, d\mu(x) < \varepsilon \tag{19.3}$$

So

$$|(f_{-t}, \chi_A) - L(A)(f,1)| = |(f_{-t}, \chi_A) - (g_{-t}, \chi_A) + (g_{-t}, \chi_A)$$
$$- L(A)(g,1) + L(A)(g-f,1)|$$
$$\leq |(|f-g|, 1)| + |(g, \chi_{A_t}) - L(a)(g,1)|$$
$$+ |L(A)| (|g-f|, 1)$$

which is as small as one likes for a suitably chosen g and all sufficiently large t. So the proof is complete.

Now apply the criterion of the theorem to the case in which B is an invariant measurable set of non-zero finite measure.

Then

$$L(A) = \frac{\mu(B \cap A)}{\mu(B)} \tag{19.4}$$

If in particular $\mu(X) < \infty$, then $B = X$ yields

$$L(A) = \frac{\mu(A)}{\mu(X)} \tag{19.5}$$

So in that case the criterion for statistical regularity reads

$$\lim_{t\to\infty} \mu(B_{-t} \cap A) = \frac{\mu(B)\mu(A)}{\mu(X)} \qquad (19.6)$$

which holds for all measurable B since it is trivially true if B has measure zero. But this is precisely the equation that occurs in the definition of the mixing property except that there it must hold for all measurable A. In this connection we have the following theorem whose proof we have just given.

Theorem

A measurable flow in a finite measure space is mixing if and only if every measurable subset is statistically regular with respect to the flow.

In this sense, it is the mixing property that explains why an ideal roulette wheel works regardless of the particular ensemble that the house uses to spin the wheel. That is, on the average, black or red will occur a fixed fraction of the time regardless of the force with which the ball is started spinning.

Added Historical Remarks

See N. S. Krylov - Works on the Foundations of Statistical Mechanics - in Russian. This book consists of Krylov's Thesis (about 1943) and an unfinished manuscript on the mathematical foundations of Statistical Mechanics. Krylov realized the importance of the mixing property and emphasized its necessity for statistical mechanics. He is quite dogmatic in his approach and states that there are essentially only two problems in statistical mechanics.

1. Under what circumstances will mechanical systems give rise to laws that can be formulated statistically? The theorem just stated above is one partial answer to this question.

2. The second question at present isstill wide open. How does one relate a deterministic system such as Hamilton's equations to the stock models of the theory such as Boltzmann's equation?

Krylov furthermore made calculations on a hard sphere gas (which he called an ideal gas). He considered a sphere and followed it through a single collision (short time). Now he made a small change in the direction of one of the spheres and asked how big the resultant solid angle would be. This indicated that for a low density gas there would be a rapid spread of uncertainty. He concludes that the reason that stochastic models work is because the flow is mixing.

I shall now make some comments on various characterizations of ergodicity, mixing, weak mixing, and their physical significance.

Physical Interpretation of the Individual Ergodic Theorem

We know that

$$\lim_{T-S\to\infty} \frac{1}{T-S} \int_S^T f(x_t)\, dt = \frac{\int_X f(x)\, d\mu(x)}{\mu(X)}$$

Applying this to χ_A where A is a finite measurable set we get

$$\lim_{T-S\to\infty} \frac{\text{Time spent in A during time interval } (S,T)}{T-S} = \frac{\mu(A)}{\mu(X)} \qquad (19.7)$$

A natural question to ask now is, "Suppose we know that (19.7) holds, is the flow ergodic?" In this connection we get a host of theorems answering questions of this sort. All of these give alternate characterizations of ergodicity. We will state the theorems for cascades. It is a relatively simple matter to modify the proofs so that the statements hold for flows.

Theorem

If T is a measure-preserving transformation of a finite measure space, and for every integrable function

$$\frac{1}{n} \sum_{j=0}^{n} f\,(T^j x) \rightarrow \text{constant a.e.} \tag{19.8}$$

then T is ergodic. Furthermore the same statement holds if (19.8) is required only for characteristic functions.

Proof

If A is an invariant measurable set, the quantity

$$\frac{1}{n} \sum_{j=0}^{n-1} \chi_A\,(T^j x) = \chi_A(x) \qquad \text{a.e.} \tag{19.9}$$

If, as is assumed in the theorem, this is a constant almost everywhere, then A has either measure zero or $\mu\,(X)$. That is, T is ergodic. So we can state that, if the average time of sojourn in a set is equal to the measure of a set, we have ergodicity. In this case we have an immediate physical interpretation.

Theorem

Let T be a measure-preserving transformation of a finite measure space (X, Σ, μ), then T is ergodic if and only if for every pair of measurable sets F, G the quantity

$$\mu\,(T^{-n}F \cap G) \rightarrow \frac{\mu(F)\;(G)}{\mu\,(X)} \tag{19.10}$$

in the sense of Cesàro. That is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu\,(T^{-n}F \cap G) = \frac{\mu(F)\ \mu(G)}{\mu\ (X)} \qquad (19.11)$$

Before giving the proof we give an unforgettable physical interpretation of the theorem due to Halmos.

Let F denote the quantity of vermouth in a glass

at time $t = 0$

Similarly let $X-F$ denote the amount of gin. Then $T^{-n}F \cap G$ is the amount of vermouth in G after n steps (n stirrings). The interpretation of the statement

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{\mu(T^{-n}\ F \cap G)}{\mu\ (G)} = \frac{\mu\ (F)}{\mu\ (G)} \qquad (19.12)$$

is: <u>On the average</u>, the fraction of vermouth in G equals the fraction of vermouth there originally. On the other hand, the interpretation of mixing

$$\lim_{n\to\infty} \frac{\mu\mu(T^{-n}F \cap G)}{\mu\ (G)} = \frac{\mu\ (F)}{\mu\ (X)}$$

is: the limit of the fraction of the volume of G which is vermouth is the fraction of the original volume which is vermouth.

We shall see later what is the martini drinker's interpretation of weak mixing.

Lecture No. 20

We now give the proof of the theorem stated last time.

Proof

From the individual ergodic theorem, the characteristic function χ_F has a time average χ^*_F which is almost everywhere equal to the constant $\frac{\mu(F)}{\mu(X)}$ because

$$\int \chi^*_F(x)\, d\mu(x)$$

$$= \int \chi_F(x)\, d\mu(x)$$

$$= \mu(F)$$

Since for almost all x we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_F(T^j x)\, \chi_G(x) = \chi^*_F(x)\, \chi_G(x) \tag{20.1}$$

Integrating over x and using the Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(T^{-j}F \cap G) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \chi_F(T^j x)\, \chi_G(x)\, d\mu(x)$$

$$= \int \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_F(T^j x)\, \chi_G(x)\, d\mu(x)$$

$$= \int \chi^*_F(x)\, \chi_G(x)\, d\mu(x)$$

$$= \frac{\mu(F)\ \mu(G)}{\mu(x)}$$

So if T is ergodic we have Cesàro convergence. Conversely if $\mu\,(T^{-n}F \cap G)$ converges in the sense of Cesàro (or in any other sense) to $\frac{\mu\,(F)\,\mu(G)}{\mu\,(X)}$ then take F = G = S where S is any invariant measurable set. The resulting identity says

$$\mu(S) \;=\; \frac{\mu(S)\,\mu(S)}{\mu\,(X)} \tag{20.2}$$

so $\mu\,(S) = 0$ or $\mu\,(S) = \mu\,(X)$

which is precisely the condition of ergodicity. Here is still another version of this criterion of ergodicity.

Theorem

If T is a measure preserving transformation of a finite measure space (X, Σ, μ), then T is ergodic if and only if for each pair f,g of square integrable functions

$$\int f(T^n_{\ x})\; g\;(x)\;d\mu(x) \to \frac{\int f(x)d\mu(x).\int g(x)\;d\mu(x)}{\mu\;(X)} \tag{20.3}$$

in the sense of Cesàro.

Proof

Take f and g to be χ_F and χ_G, the characteristic functions of measurable sets. Then the criterion of the theorem of ergodicity reduces to that of the preceding theorem and so ergodicity indeed follows. On the other hand, to get the converse we must proceed from characteristic functions to square integrable functions. However, if f is square integrable, the mean ergodic theorem implies that

$\frac{1}{n}\sum_{j=0}^{n-1} f\;(T^j_{\ x})$ converges in the mean of order 2 to $f^*(x)$. Consequently

$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j_{\ x})\;g\;(x)$ converges in the mean of order 1 to $f^*(x)\;g(x)$. That is,

$$\lim_{n\to\infty} \int \left| \frac{1}{n} \sum_{j=0} f(T^j{}_x)\, g(x) - f^*(x)\, g(x) \right| d\mu(x) = 0 \tag{20.4}$$

as follows from Schwartz's inequality since

$$\int \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j{}_x)\, g(x) - f^*(x)\, g(x) \right| d\mu(x) \leq \left[\int d\mu(x) \left| \frac{1}{n}\sum_{j=0}^{n-1} f(T^j{}_x) - f^*(x) \right|^2\right]^{\frac{1}{2}} \cdot \left[\int |g(x)|^2 d\mu(x)\right]^{\frac{1}{2}}$$

But L^1 convergence implies that

$$\lim_{n\to\infty} \int \frac{1}{n} \sum_{j=0}^{n-1} f(T^j{}_x)\, g(x)\, d\mu(x) = \int f^*(x)\, g(x)\, d\mu(x) \tag{20.5}$$

And from ergodicity

$$f^*(x) = \frac{\int f(x)\, d\mu(x)}{\mu(X)}$$

So we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int f(T^j{}_x)\, g(x)\, d\mu(x) = \frac{\int f(x)\, d\mu(x) \cdot \int g(x)\, d\mu(x)}{\mu(X)} \tag{20.6}$$

We now proceed to the analogous situation for mixing, that is, we shall give several criteria for mixing.

<u>Theorem</u>

A measurable cascade (or alternatively a flow) on a finite measure space is mixing if and only if

a) $\lim_{n\to\infty} (U^n f, g) = \frac{(f,1)\ (1,g)}{\mu(X)}$ in the case of a cascade, or

$\lim_{t\to\infty} (U_t f, g) = \frac{(f,1)\ 1,g)}{\mu(X)}$ in the case of a flow, for every pair of square integrable functions f, g.

Or

b) $\lim_{n\to\infty} U^n = P$ (cascade)

$\lim_{t\to\infty} U_t = P$ (flow)

where P is the projection

$$Pf = 1\ \frac{(1,f)}{\mu(X)} \tag{20.7}$$

That is, P projects onto the subspace spanned by the function 1, and the convergence is to be understood as weak operator convergence.

Proof

As usual, if a) is satisfied, take f and g equal to χ_F and χ_G, where F and G are arbitrary measurable sets, and thus get the definition of mixing. Conversely, having the mixing property for any pair F, G of measurable sets, hold G fixed in the inner product

$(U^n \chi_F, \chi_G)$ and take linear combinations of χ_F's (different F's) to conclude that

$$\lim_{n\to\infty} (U^n f, \chi_G) = \frac{(f,1)\ (1,\chi_G)}{\mu\ (X)}$$

for any simple function f. Now pass to an arbitrary square-integrable function f by using

$$|(U^n f,\chi_G) - (U^n h, \chi_G)| < \|f-h\|_2\ (\mu(X))^{\frac{1}{2}} \tag{20.8}$$

and the fact that for a suitable h, $\|f-h\|_2$ is arbitrarily small. Now keep f fixed and repeat the procedure for χ_G to get a square integrable g.

Part b) of the theorem follows because

$$(Pf,g) = \frac{((1,f)\ 1,g)}{\mu\ (X)} = \frac{(f,1)\ (1,g)}{\mu\ (X)}$$

and from the definition of weak convergence.

Some elementary statements on Cesàro Convergence. We are leading up to a theorem on weak mixing, namely.

Theorem

A measurable cascade (flow) on a finite measure space is weakly mixing if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left|(U^n f,g) - \frac{(f,1)\ (1,g)}{\mu(X)}\right| = 0 \tag{20.9}$$

for every pair of square integrable functions.

Recall that T is weakly mixing if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left|\ \mu\ (T^{-j}F \cap G) - \frac{\mu(F)\mu(G)}{\mu(X)}\right| = 0 \tag{20.10}$$

This is strong Cesàro convergence of $\mu(T^{-j}F \cap C)$ rather than Cesàro convergence or even plain convergence. Furthermore, we get the following sequence of implications.

mixing $\Rightarrow$ weak mixing $\Rightarrow$ ergodicity

since for sequences

convergence $\Rightarrow$ strong Cesàro convergence $\Rightarrow$ Cesaro convergence as we shall now show.

Suppose $$\lim_{n\to\infty} a_n = a$$

This means that there is an N such that for $\varepsilon > 0$

$$|a_n - a| < \varepsilon \text{ for } n > N$$

But

$$\frac{1}{n}\sum_{j=0}^{n-1} |a_j - a| \leq \sum_{j=0}^{n-1} |a_j - a|$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} |a_j - a| + \frac{1}{n}\sum_{j=n}^{n-1} |a_j - a|$$

$$< \frac{N}{n} \sup_{j\leq N-1} |a_j - a| + \frac{n-1-N}{n}\varepsilon$$

$$< \varepsilon$$

if we take $$n > \left(\frac{N}{\varepsilon}\right) \sup_{j\leq N-1} |a_j - a|$$

so convergence implies strong Cesàro convergence.

Now assume strong Cesàro convergence. I.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} |a_j - a| = 0$$

Then

$$\frac{1}{n}\sum_{j=0}^{n-1} |a_j - a| < \varepsilon \text{ for } n>N$$

But

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}(a_j - a)\right| \leqslant \frac{1}{n}\sum_{j=0}^{n-1} |a_j - a|$$

$$= \left|\ \frac{1}{n}\sum_{j=0}^{n-1} |a_j - a|\ \right|$$

$$< \varepsilon \qquad \text{for } n > N$$

So our assertion is proved.

Definition

A set J of integers has zero density if

$$\lim_{N\to\infty} \frac{[\text{number of integers } k \in J \text{ with } k \leqslant N]}{N} = 0 \tag{20.11}$$

Similarly a set of real numbers has zero density if

$$\lim_{N\to\infty} \frac{L\,(J \cap [0,N])}{N} = 0 \tag{20.12}$$

Here L is the Lebesgue measure (length) of the interval indicated. Also [0,N] is the closed interval from 0 to N.

Lemma

If $\{a_n\}$ is a bounded sequence of complex numbers it converges strong Cesaro to a if and only if there is a subset J of the positive integers having zero density such at $a_n \to a$ for $n \in J^c$.

Before giving the proof we need a few preliminaries. If P is any set of positive integers we define for $m > n$; m, n integers.

$$\nu^m_n (P) = \text{number of } p \in P \text{ such that } n < p \leq m$$

$$\rho^m_n(P) = \frac{1}{m-n} \nu^m_n (P) \tag{21.1}$$

Thus if P has zero density $\rho^n_o (P) \to 0$ as $n \to \infty$. If P_1 and P_2 are two sets of positive integers we define P as the splice of P_2 onto P_1 at position n by

$$P = (P_1 \cap \{1,\ldots n\}) \cup (P_2 \cap \{n+1, n+2, \ldots\}) \tag{21.2}$$

In the particular case that both P_1 and P_2 have zero density we have

$$\rho^N_o(P) = \frac{\nu^n(P_1) + \nu^N_n (P_2)}{N} \leq$$

$$\leq \rho^n_o (P_1) + \rho^N_o (P_2) \quad \forall N > n. \tag{21.3}$$

If ε_1, ε_2 are positive numbers then for all sufficiently large n:

$$\rho^n_o (P_1) < \varepsilon_1 \tag{21.4}$$

$$\rho^N_o (P_2) < \varepsilon_2 \quad \forall N > n.$$

A splice satisfying (12.4) will be called an $(\varepsilon_1, \varepsilon_2)$ - splice.

Proof of Lemma

Let $\{a_n\}$ converge strong Cesaro to a.

Define for integer p

$$K_p = \{n \mid |a_n - a| > \frac{1}{p} \}. \tag{21.5}$$

We assume not all K_p are empty since otherwise $a_n = a \; \forall n$ and there is nothing to prove. Note that

$$p > q \Rightarrow K_p \subset K_q. \tag{21.6}$$

Each K_p is of zero density since

$$\rho^m_o (K_p) < p \frac{1}{m} \Sigma^m_{j=1} |a_j - a| \to 0 \tag{21.7}$$

[illegible] $J_1 = K_1$ and define J_p, $p > 1$ by induction as any $(\frac{1}{2p}, \frac{1}{2p})$ - splice of K_p onto J_{p-1}. This is possible since by (21.3) splicing any two zero density sets yields again a zero density set and hence by induction J_{p-1} has zero density. The sequence of sets J_p has a well defined limit J since eventually all our splices are beyond an arbitrarily large position. J has zero density by virtue of (21.3) and (21.4). By virtue of (21.6) we see that J contains all but a finite number of elements of each K_p, but this implies that $a_n \to a$ on J^c. We have thus proved necessity.

To prove sufficiency assume $a_n \to a$ for $n \in J^c$ where J is some set of zero density. Then

$$\frac{1}{n} \Sigma^n_{j=1} |a_j - a| = \frac{1}{n} \Big(\underset{j \in J}{\Sigma^n_{j=1}} + \underset{j \in J^c}{\Sigma^n_{j=1}}\Big) |a_j - a| \leq$$

$$\leq \rho^n{}_o(J) \sup_{1 \leq j \leq k} |a_j - a| + \frac{1}{n} \underset{j \in J^c}{\Sigma^n_{j=1}} |a_j - a| \qquad (21.8)$$

The first term becomes arbitrarily small as $n \to \infty$ since $\{a_n\}$ is bounded and J has zero density; the second since $a_j \to a$ on J^c. Q.E.D.

Lemma

If $\{a_n\}$, $\{b_n\}$ are bounded sequences such that $\{a_n\} \to a$. $\{b_n\} \to b$ both strong Cesaro, then $\{a_n b_n\} \to ab$ strong Cesaro.

Proof

By the previous lemma $a_n \to a$ on $J_1{}^c$ and $b_n \to b$ on $J_2{}^c$, where J_1 and J_2 have zero density. Therefore $a_n b_n \to ab$ on $J_1{}^c \cap J_2{}^c = (J_1 \cup J_2)^c$. But $J_1 \cup J_2$ is again of zero density and applying the lemma once again $\{a_n b_n\} \to ab$ strong Cesaro.

Theorem (Mixing Theorem)

Consider a measurable flow (cascade) on a finite measure space, then the following three statements are equivalent.

1. The flow (cascade) is weakly mixing

2. 1 is a simple proper value of the associated unitary operators of the flow (cascade), and there are no other proper values.

3. The square of the flow (cascade) is ergodic.

Remarks

Given (X, Σ, μ) we define a new measure space $(\hat{X}, \hat{\Sigma}, \hat{\mu})$ where $\hat{X} = X \times X$; $\hat{\Sigma}$ is the σ - algebra generated by the rectangles $A \times B$, $A, B \in \Sigma$ and $\hat{\mu}$ is the so-called product measure which is uniquely defined by its value on the rectangles: $\hat{\mu}(A \times B) = \mu(A)\mu(B)$.

If T is a measure preserving transformation of (X, Σ, μ) then we define $\hat{T}$, the square of T on $\hat{X}$ as:

$$\hat{T}(x, y) = (Tx, Ty), \ (x,y) \in \hat{X} \tag{21.9}$$

In a similar manner any flow or cascade induces a square on the product space $(\hat{X}, \hat{\varepsilon}, \hat{\mu})$

Proof

We shall prove the theorem according to the implication diagram (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1); furthermore, we shall prove it only for the case of a cascade since for the flow the proof is entirely similar.

Suppose T is a weakly mixing transformation; i.e., $\mu(T^{-n}(A) \cap B)$ converges strong Cesaro to $\frac{\mu(A)\mu(B)}{\mu(X)}$. To prove that its square is ergodic it suffices to show that $\hat{\mu}(\hat{T}^{-n}(\hat{A}) \cap \hat{B})$ converges Cesaro to $\frac{\mu(a)\mu(B)}{(\mu(X))^2}$, where $\hat{A}, \hat{B} \in \hat{\Sigma}$. Moreover, by an already familiar argument, since any set of $\hat{\Sigma}$ can be approximated in measure arbitrarily well by a rectangle it suffices to prove this convergence whenever $\hat{A}$ and $\hat{B}$ are of the form:

$$\left.\begin{aligned} \hat{A} &= C \times D \\ \hat{B} &= F \times G \end{aligned}\right\} C, D, F, G \in \Sigma \tag{21.10}$$

In this case

$$\hat{\mu}(\hat{T}^{-n}(\hat{A}) \cap \hat{B}) = \mu(T^{-n}(C) \cap F)\ \mu(T^{-n}(D) \cap G). \tag{21.11}$$

By assumption $\mu(T^{-n}(C) \cap F)$ converges strong Cesaro to $\frac{\mu(C)\mu(F)}{\mu(X)}$ and $\mu(T^{-n}(D) \cap G)$ converges the same way to $\frac{\mu(D)\mu(G)}{\mu(X)}$, so by the second of the preceding lemmas

$$\ddot{\mu}(\hat{T}^{-n}(\hat{A}) \cap B) \xrightarrow{\text{Cesaro}} \frac{\mu(C)\mu(D)\mu(F)\mu(C)}{(\mu(X))^2} =$$

$$= \frac{\hat{\mu}(\hat{A})\hat{\mu}(\hat{B})}{(\mu(X))^2} \tag{21.12}$$

We have thus proved (1) $\Rightarrow$ (3).

Assume now $\hat{T}$ is ergodic. If f is a proper function of U, the isometry corresponding to T, then

$$Uf = cf, \ |c| = 1 \tag{21.13}$$

Write

$$\hat{f}\,(x,y) = f\,(x)\,\overline{f}\,(y), \tag{21.14}$$

then

$$\hat{U}\,\hat{f} = \hat{f}.$$

By ergodicity of $\hat{T}$, $\hat{f}$ = constant a.e. $\Rightarrow$ f = constant a.e., but then c = 1. So U has no point spectrum except 1 and since f is necessarily constant a.e. the multiplicity is also 1. We have shown (3) $\Rightarrow$ (2).

Suppose finally that U has no point spectrum except 1 and that this proper value is simple. To prove T is weakly mixing it suffices to prove

$$\frac{1}{n} \Sigma_{j=0}^{n-1} \left| (U^j f,g) - \frac{(f,1)\ (1,g)}{\mu(X)} \right| \to 0 \tag{21.15}$$

for all square integrable f, g. We note that if f is constant the above convergence holds trivially since each term in the sum vanishes; we can therefore assume (f, 1) = 0 and it is enough to show that

$$\frac{1}{n} \Sigma_{j=0}^{n-1} \left| (U^j f,g) \right| \to 0 \tag{21.16}$$

for all $f \perp 1$ and for all g. Equation (21.16) is implied by

$$\frac{1}{n} \Sigma_{j=0}^{n-1} \left| (U^j f,g) \right|^2 \to 0 \tag{21.17}$$

since (21.17) is equivalent to the convergence off a set of **density** zero of $|(U^j f,g)|^2$ to zero which implies the convergence off the same zero density set of

$|(U^j f,g)|$ to zero which in turn implies (21.16). We now use the spectral theorem to write

$$(U^j f,g) = \int x^j (E(dx)\ f,\ g) \tag{21.18}$$

where E is the projection-valued spectral measure of U with the support of E contained in the unit circle. We have:

$$\frac{1}{n} \Sigma_{j=0}^{n-1} \ |(U^j f,g)|^2 = \frac{1}{n} \Sigma_{j=0}^{n-1} \ \iint x^j y^j (E(dx)\ f,g)\ \overline{(E(dy)f,g)}$$

$$= \iint \ \frac{1-(x\bar{y})^n}{n(1-x\bar{y})} \ (E(dx)f,g)\ \overline{(E(dy)f,g)} \tag{21.19}$$

Let M be the measure of the set $\{x = y\}$ in the product of unit circles taken with respect to the product measure $(E(dx)f,\ g)\ \overline{(E(dy)f,\ g)}$.

We have

$$M = \iint_{\{x=y\}} (E(dx)f,\ g)\ \overline{(E(dy)f,g)} \tag{21.20}$$

By the Fubini theorem we evaluate this by iterated integration by integrating over x first:

$$M = \int_{\{|y|=1\}} (E(\{y\})f,\ g)\ \overline{(E(dy)f,g)} \tag{21.21}$$

But U has no proper value other than 1, therefore $E(\{y\}) = 0$ for $y \neq 1$, therefore

$$M = (E(\{1\}\ )f,\ g)\ \overline{(E(\{1\}\ )f,g)} = 0 \tag{21.22}$$

since $f \perp 1$ and $E(\{1\})$ is the projection onto the constants. The integrand of (21.19) is bounded in modulus by 1 and for $x \neq y$ converges to 0 as $n \to \infty$. Using Lebesgue's dominated convergence theorem and (21.22) the integral must also converge to zero.

We have thus shown (2) $\Rightarrow$ (1) and the theorem is proved. Q.E.D.

<u>Application of Some Ideas of Information Theory</u>

<u>Entropy</u>

Given a set of mutually exclusive possibilities $A_1,\ldots,A_n$ with respective probabilities $p_1,\ldots,p_n$, information theory attributes an information $-\log p_j$ to the absorvation of A_j in an experiment. The average information associated with

$$H(A) = -\sum_{j=1}^{n} p_j \log p_j \qquad (21.23)$$

and is called the <u>entropy</u>. The base of the logarithm is usually taken to be 2 in information theory since then information is given in bits, i.e. one unit of information is given to the occurence of either one of two equally probably mutually exclusive events. For our purposes the base is irrelevant and its choice amounts to a choice of an overall constant factor. In (21.23) if 0 log 0 occurs it is by convention taken to be 0; this is consistent with the continuity of H.

Let now $B = (B_1, \ldots, B_m)$ be another set of mutually exclusive possibilities having the corresponding probabilities $q_1, \ldots, q_m$. We denote by $A \times B$ the experiment in which the possibilities are $A_j B_k$, that is the joint occurence of both A_j and B_k. If A and B are independent experiments the probability of $A_j B_k$ is $p_j q_k$ and

$$H(A \times B) = - \sum_{j=1}^{n} \sum_{k=1}^{m} p_j q_k \log p_j q_k =$$

$$= - \sum_{j=1}^{n} \sum_{k=1}^{m} (p_j q_k \log p_j + p_j q_k \log q_k) =$$

$$= - \sum_{j=1}^{n} p_j \log p_j - \sum_{k=1}^{m} q_k \log q_k =$$

$$= H(A) + H(B) \tag{22.1}$$

An analogous expression holds when the experiments are not independent. Let

$$p_{jk} = P(A_j | B_k) = \frac{P(A_j B_k)}{P(B_k)} \tag{22.2}$$

be the conditional probability that A_j occur given that B_k has occured. We have

$$\Sigma_{j=1}^{n} \; p_{jk} = 1$$

$$\Sigma_{k=1}^{m} \; p_{jk} q_k = p_j \tag{22.3}$$

In this case

$$H(A \times B) = - \sum_{j=1}^{n} \sum_{k=1}^{m} P(A_j B_k) \log P(A_j B_k) =$$

$$= - \sum_{j=1}^{n} \sum_{k=1}^{m} p_{jk}\, q_k \log p_{jk}\, q_k =$$

$$= - \sum_{k=1}^{m} q_k \sum_{j=1}^{n} p_{jk} \log p_{jk} - \sum_{k=1}^{m} q_k \log q_k$$

$$= H(A|B) + H(B). \tag{22.4}$$

This equation defines the <u>conditional entropy</u> $H(A|B)$. We also define the conditional entropy $H(A|B_k)$ by

$$H(A|B_k) = - \sum_{j=1}^{n} p_{jk} \log p_{jk} . \tag{22.5}$$

By symmetry we also have

$$H(A \times B) = H(B \times A) = H(B|A) + H(A). \tag{22.6}$$

When A and B are independent we have $H(A|B) = H(A)$.

Since all terms in (22.4) are positive we deduce some useful inequalities:

$$\begin{aligned} H(A|B) &\leqslant H(A \times B) \\ H(A) &\leqslant H(A \times B). \end{aligned} \tag{22.7}$$

The second of these has the intuitive meaning that a refined experiment produces more information. Again intuitively it should be true in general that

$$H(A|B) \leqslant H(A) \tag{22.8}$$

on the ground that knowing B reduces what can be learned from A. The proof uses the convexity of $x \log x$.

Digression on convexity (see Hardy, Littlewood and Polya, Inequalities, Cambridge University Press).

A real valued function on an interval I of the real numbers is called convex if

$$f\left(\frac{x_1 + x_2}{2}\right) \leqslant \frac{f(x_1) + f(x_2)}{2} \qquad x_1, x_2 \varepsilon I \tag{22.9}$$

Applying this twice:

$$f\left(\frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}}{2}\right) \leqslant \frac{f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right)}{2} \leqslant$$

$$\leqslant \frac{\Sigma_{j=1}^{4} f(x_j)}{4} \tag{22.10}$$

This is a special case of

$$f(2^{-n} \Sigma_{j=1}^{2^n} x_j) \leqslant 2^{-n} \Sigma_{j=1}^{2^n} f(x_j), \qquad x_j \varepsilon I \tag{22.11}$$

which is proved by induction:

$$f\left(\frac{2^{-n}\,\Sigma_{j=1}^{2^n} x_j + 2^{-n}\, x_{2^{n+1}}}{2}\right) \leqslant$$

$$\tfrac{1}{2}(f(2^{-n}\,\Sigma_{j=1}^{2^n}\, x_j) + f(x_{2^{n+1}})) \leqslant$$

$$\leqslant 2^{-(n+1)}\,\Sigma_{j=1}^{2^{n+1}}\, f(x_j)\ . \tag{22.12}$$

Inequality (22.11) is itself a special case of

$$f\,(\tfrac{1}{n}\,\Sigma_{j=1}^{n}\, x_j) \leqslant \tfrac{1}{n}\,\Sigma_{j=1}^{n}\, f(x_j), \qquad x_j \epsilon I \tag{22.13}$$

To show that this holds for all n it is sufficient to show that if it holds for n, it holds for n-1 since by (22.11) we can work down from a number of the form 2^k. We have:

$$f(\frac{1}{n-1}\;\Sigma_{j=1}^{n-1}\; x_j) = f(\frac{1}{n}\;[\Sigma_{j=1}^{n-1}\, x_j + \frac{1}{n-1}\;\Sigma_{j=1}^{n-1}\, x_j]\;); \tag{22.14}$$

letting $x_n = \frac{1}{n-1}\,\Sigma_{j=1}^{n-1}\; x_j$ and applying the assumption we see that the left-hand side of (22.14) is then less than or equal to

$$\frac{1}{n}\;[\Sigma_{j=1}^{n}\, f(x_j) + f\,(\frac{1}{n-1}\;\Sigma_{j=1}^{n-1}\; x_j)] \tag{22.15}$$

Taking the second term to the left-hand side to combine with the left-hand side of (22.14) we obtain the required inequality.

Inequality (22.13) in turn implies that for all non-negative rationals α_j, $j = 1,\ldots,n$, $\Sigma\, \alpha_j \neq 0$:

$$f\left(\frac{\Sigma\,\alpha_j\, x_j}{\Sigma\,\alpha_j}\right) \leqslant \frac{1}{\Sigma\,\alpha_j}\;\Sigma\,\alpha_j\; f\,(x_j), \qquad x_j \epsilon I \tag{22.16}$$

To prove this write all α_j as a ratio of integers each ratio having the same denominator; after cancelling this denominator in the ratios appearing in (22.16) the inequality reduces to (22.13) if we interpret a variable with integer coefficient as a sum of identical variables with unit coefficient.

Theorem

If f is a continuous convex function then for all sets of n non-negative real α_j with $\Sigma\,\alpha_j \neq 0$.

$$f\left(\frac{\Sigma\,\alpha_j x_j}{\Sigma\,\alpha_j}\right) \leqslant \frac{1}{\Sigma\,\alpha_j}\,\Sigma\,\alpha_j\,f\,(x_j). \qquad (22.17)$$

Proof

If (22.17) was violated for some set α_j then by continuity it would be violated by some set of rationals approximating α_j which would contradict (22.16)

Theorem

f is a convex function with two continuous derivatives defined on (a, b) if and only if

$$f''\,(x) \geqslant 0, \quad a < x < b. \qquad (22.18)$$

Proof

Assume f is convex and rewrite the defining inequality of convexity as

$$f(t) \leqslant \frac{f(t+h) + f(t-h)}{2}\ , \qquad t = \frac{x_1 + x_2}{2}\ , \quad h = \frac{x_1 - x_2}{2} \qquad (22.19)$$

Suppose $f''(t) < 0$, $t \in (a,b)$, then there is a positive δ and h such that

$$f'(t+u) - f'(t-u) < -\delta u,\ 0 < u \leqslant h. \qquad (22.20)$$

Integrate this inequality with respect to u from 0 to h:

$$f(t+h) - f(t) + f(t-h) - f(t) < -\frac{\delta h^2}{2} \qquad (22.21)$$

Solving this inequality for $f(t)$ we obtain a contradiction.

Conversely assume $f'' \geqslant 0$ on (a,b) and let q_1, q_2 be real non-negative numbers with $q_1 + q_2 = 1$. Define $x = q_1 x_1 + q_2 x_2$ where $x_1, x_2 \in (a,b)$. We have by Taylor's formula

$$f(x_j) = f(x) + (x_j - x)\, f'(x) + \frac{(x_j - x)^2}{2!} \quad f''(\xi_j) \tag{22.22}$$

where $j = 1, 2$ and ξ_j is in the interval between x_j and x. Multiply (22.22) by q_j and sum to get

$$\Sigma\, q_j\, f(x_j) = f(x) + \Sigma\, q_j \;\frac{(x_j - x)^2}{2!} \quad f''\,(\xi_j). \tag{22.23}$$

The second term is non-negative since $f'' \geqslant 0$ and therefore

$$f(x) = f(\Sigma q_j x_j) \leqslant \Sigma\, q_j\; f(x_j) \tag{22.24}$$

thereby proving convexity. Q.E.D.

References

P. Billingsley, Ergodic Theory and Information, John Wiley & Sons, 1965.

According to the experts the best discussion of the convergence of entropy is found in A. Ionescu Tulcea. Arkiv för Math. (1963)

We are aiming at the following two papers:

Ya. G. Sinai, On the Concept of Entropy of Dynamical Systems, Doklady Acad. Nauk, 124, 768-771 (1959)

Kushnirenko, An Estimate from Above for the Entropy of a Classical Dynamical System, Doklady Acad. Nauk, 161, 37-38 (1965) Sovmat. Dok. 6 360-362 (1965).

Applying our convexity criterion to the function x log x:

$$\frac{d^2}{dx^2} \; x \log x = \frac{1}{x} > 0 \text{ for } x > 0, \tag{23.1}$$

we find it to be convex for x > 0. We return to our two experiments $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_m)$ with probabilities $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_m)$ and conditional probabilities p_{jk}.

We have:

$$p_j = \Sigma_{k=1}^{m} \; p_{jk} \, q_k \tag{23.2}$$

By convexity we find

$$p_j \log p_j = \Sigma_{k=1}^{m} \; p_{jk} \, q_k \log (\Sigma_{\ell=1}^{m} \; p_{j\ell} q_\ell) \leq$$

$$\leq \Sigma_{k=1}^{m} \; q_k \, p_{jk} \log p_{jk}, \tag{23.3}$$

which when summed over j implies

$$H(A|B) \leq H(A) \tag{23.4}$$

thereby proving our intuitive conjecture (22.8) of the previous lecture.

H(A) is a continuous function of n real variables $p_1, \ldots, p_n$ defined on the set $p_j > 0$, $\Sigma \, p_j = 1$. This function vanishes whenever any $p_j = 1$. By convexity we have

$$\frac{1}{n} \log \frac{1}{n} = \frac{1}{n} \; \Sigma \, p_j \log \frac{1}{n} \; \Sigma \, p_j \leq$$

$$\leq \frac{1}{n} \; \Sigma \, p_j \log p_j \quad \Rightarrow$$

$$\Rightarrow H(A) \leq \log n \tag{23.5}$$

The value log n is realised when $p_j = \frac{1}{n}$, $\forall_j$, this being in fact the only place where the maximum is attained. The maximum information therefore results from an experiment in which all the possibilities are equally likely.

The uniqueness of entropy is the content of

Theorem

Let H_n be a sequence of real continuous functions; each H_n being defined and symmetric in the arguments $p_j \geqslant 0$, $\Sigma p_j = 1$, $j = 1,..,n$, and satisfying

(1) $H_n(p_1, \ldots, p_n)$ takes its largest value at

$$p_j = \frac{1}{n} \quad \forall_j$$

(2) $H_{n+1}(p_1,\ldots,p_n, 0) = H_n(p_1,\ldots,p_n)$. In other words if we add a possibility with no probability of occurrence we do not change the experiment.

(3) If $p_j = \Sigma_{k=1}^m p_{jk} q_k$; $p_j \geqslant 0$, $_k \geqslant 0$,

$p_{jk} \geqslant 0$; $\Sigma p_j = 1 = \Sigma q_k$, $\Sigma_{j=1}^n p_{jk} = 1$,

then

$$H_{nm}(p_{11}q_1,\ldots,p_{jk}q_k,\ldots,p_{nm}q_m) =$$
$$= H_n(q_1,\ldots,q_m) + \Sigma_{k=1}^m q_k H_n \quad (p_{1k},\ldots,p_{nk}).$$

then

$$H_n(p_1\ldots,p_n) = -\lambda \Sigma p_j \log p_j,$$

where λ is some real positive number independent of n.

Proof (Khinchin.)

Let

$$L(n) = H_n\left(\frac{1}{n},\ldots,\frac{1}{n}\right). \tag{23.6}$$

By (1) and (2)

$$L(n) = H_{n+1}\left(\frac{1}{n},\ldots,\frac{1}{n}, 0\right) \leqslant H_{n+1}\left(\frac{1}{n+1},\ldots, \frac{1}{n+1}\right) = L(n+1) \tag{23.7}$$

showing that L(n) is non-decreasing. Let now $p_j = \frac{1}{n}$, $q_k = \frac{1}{m}$, $p_{jk} = \frac{1}{n}$

and using (3) deduce

$$L(nm) = L(n) + L(m). \tag{23.8}$$

$$\frac{r}{s} \leqslant \frac{\log m}{\log n} \leqslant \frac{r+1}{s} \Rightarrow$$

$$\Rightarrow \quad n^r \leqslant m^s \leqslant n^{r+1} \tag{23.9}$$

Using (23.8) and (23.7) we deduce

$$L(n^r) = rL(n) \leqslant L(m^s) = sL(m) \leqslant L(n^{r+1}) = (r+1) L(n) \Rightarrow$$

$$\Rightarrow \quad \frac{r}{s} < \frac{L(m)}{L(n)} < \frac{r+1}{s} \tag{23.10}$$

Combining (23.9) and (23.10):

$$\left| \frac{L(m)}{L(n)} - \frac{\log m}{\log n} \right| < \frac{1}{s} \tag{23.11}$$

and since s can be arbitrarily large we conclude

$$L(m) = \lambda \log m \tag{23.12}$$

where $\lambda > 0$ by (23.7). For $m = 1$ (23.8) implies $L(m) = 0$ which is consistent with (23.12).

Let now $q_1, \ldots, q_m$ be any set of rational numbers $\Sigma q_k = 1$. Express q_k in the form

$$q_k = \frac{d_k}{d} \tag{23.13}$$

where d_k are integers and $\Sigma\, d_k = d$. Define

$$p_{jk} = \begin{cases} 1/d_k \ , & d_1 + \ldots + d_{k-1} \leqslant j < d_1 + \ldots + d_k \\ 0 \ , & \text{otherwise} \end{cases} \tag{23.14}$$

Each $p_{jk} q_k$ is either 0 or $1/d$, and using (2) and (3) we conclude

$$L(d) = H_m(q_1, \ldots, q_m) + \Sigma \frac{d_k}{d} L(d_k) \tag{23.15}$$

whereupon using (23.12) we get

$$H_m(q_1, \ldots, q_m) = -\lambda \, \Sigma \frac{d_k}{d} \log \frac{d_k}{d} =$$

$$= -\lambda \, \Sigma q_k \log q_k \tag{23.16}$$

Since H_m was assumed continuous (23.16) must in fact hold for all $q_k \geqslant 0$, $\Sigma\, q_k = 1$.

Q.E.D.

Entropy of a Partition of a Measure Space and Entropy of a Measure Preserving Transformation.

If (X, Σ, μ) is a finite measure space a measurable partition mod 0 is a family of measurable subsets $\mathcal{A} = \{A_i, i \in I\}$ such that

(1) $\mu(\cup A_i) = \mu(X)$

(2) $\mu(A_i \cap A_j) = 0, i \neq j$

(3) $0 < \mu(A_i) < \infty$

We allow the trivial partition {X - a set of measure zero}. If the set I of indices is finite the partition is called a finite partition. We associate with each $A_i \in \mathcal{A}$ a probability $\mu(A_i) / \mu(X)$. Normalizing the measure $\mu(X) = 1$ we define the entropy of $\mathcal{A}$ as

$$H(\mathcal{A}) = -\Sigma\, \mu(A_i) \log \mu(A_i) \tag{23.17}$$

Given two finite partitions $\mathcal{A} = \{A_i, i \in I\}$ and $\mathcal{B} = \{B_j, j \in J\}$ we can define a conditional entropy

$$H(\mathcal{A} | \mathcal{B}) = -\Sigma_{j\in J}\, \mu(B_j)\, \Sigma_{i\in I} \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \log \frac{\mu(A_i \cap B_j)}{\mu(B_j)} =$$

$$= -\Sigma_{i\in I,\, j\in J}\, \mu(A_i \cap B_j) \log \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \tag{23.18}$$

The family of finite partitions admitts a partial order $\subseteq$ called refinement, and an operation V:

$\mathcal{A} \vee \mathcal{B} = \{A_i \cap B_j \mid \mu(A_i \cap B_j) \neq 0, i \in I, j \in J\}$,

$\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow$ each set of $\mathcal{A}$ is a union of sets of $\mathcal{B}$ with a possible exception of a set of measure zero.

There is a partition $\mathcal{N}$ such that $\mathcal{N} \subseteq \mathcal{A}, \forall \mathcal{A}$; $\mathcal{N}$ being in fact the trivial partition {X} .

It is immediate from the definitions that

$$\left.\begin{array}{c} (\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C} = \mathcal{A} \vee (\mathcal{B} \vee \mathcal{C}) \\ \mathcal{A} \vee \mathcal{B} = \mathcal{B} \vee \mathcal{A} \\ \mathcal{A} \leq \mathcal{A} \vee \mathcal{B}, \quad \mathcal{B} \leq \mathcal{A} \vee \mathcal{B} \\ \mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{A} \vee \mathcal{B} = \mathcal{B} \\ \mathcal{A} \leq \mathcal{C}, \mathcal{B} \leq \mathcal{C} \Rightarrow \mathcal{A} \vee \mathcal{B} \subset \mathcal{C} \end{array}\right\} \qquad (23.19)$$

We have already shown but in a different notation that

$$H(\mathcal{A} \vee \mathcal{B}) = H(\mathcal{B}) + H(\mathcal{A} \mid \mathcal{B}) \qquad (23.20)$$

from which immediately follows

$$H(\mathcal{B}) \leq H(\mathcal{A} \vee \mathcal{B}), \qquad (23.21)$$

and as in eqn. 23.4 we also have

$$H(\mathcal{A} \mid \mathcal{B}) \leq H(\mathcal{A}) \qquad (23.22)$$

hence

$$H(\mathcal{A}) \leq H(\mathcal{A} \vee \mathcal{B}) \leq H(\mathcal{A}) + H(\mathcal{B}). \qquad (23.23)$$

By (23.19)

$$H((\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C}) = H(\mathcal{B} \vee (\mathcal{A} \vee \mathcal{C})). \qquad (23.24)$$

Expanding both sides according to (23.20) we get

$$H(\mathcal{A} \vee \mathcal{B} \mid \mathcal{C}) = H(\mathcal{B} \mid \mathcal{A} \vee \mathcal{C}) + H(\mathcal{A} \mid \mathcal{C}), \qquad (23.25)$$

and using (23.19) we deduce

$$\mathcal{A} \leq \mathcal{B} \Rightarrow H(\mathcal{A} \mid \mathcal{C}) \leq H(\mathcal{B} \mid \mathcal{C}). \qquad (23.26)$$

The conditional entropy $H(\mathcal{A} \mid \mathcal{B})$ is thus an increasing function of the first variable $\mathcal{A}$. To show that it is a decreasing function in the second variable, i.e.,

$$\mathcal{B} \supseteq \mathcal{C} \Rightarrow H(\mathcal{A} \mid \mathcal{B}) \leq H(\mathcal{A} \mid \mathcal{C}) \qquad (23.27)$$

one must again make use of the convexity of x log x and proceed as in the proof of (23.22). Details are to be supplied in the next lecture.

We shall now prove some of the relations stated last time

$$H(\mathcal{A}|\mathcal{C}) \leq H(\mathcal{A}|\mathcal{B}) \quad \text{if} \quad \mathcal{B} \subseteq \mathcal{C} \tag{24.1}$$

Proof:

Since $\mathcal{B} \subseteq \mathcal{C}$

$$\therefore \quad \mu(B_j \cap C_k) = \begin{cases} 0 & \text{if } C_k \nsubseteq B_j \\ \mu(C_k) & \text{if } C_k \subseteq B_j \end{cases}$$

Hence

$$\frac{\mu(A_i \cap B_j)}{\mu(B_j)} = \sum_{k\in K} \frac{\mu(A_i \cap C_k)}{\mu(C_k)} \cdot \frac{\mu(C_k \cap B_j)}{\mu(B_j)}$$

Now apply convexity on $x \log x$.

Then

$$\frac{\mu(A_i \cap B_j)}{\mu(B_j)} \log \left[\frac{\mu(A_i \cap B_j)}{\mu(B_j)}\right] \leq \sum \frac{\mu(C_k \cap B_j)}{\mu(B_j)} \cdot \frac{\mu(A_i \cap C_k)}{\mu(C_k)} \log \left[\frac{\mu(A_i \cap C_k)}{\mu(C_k)}\right] \tag{24.2}$$

Now multiply (24.2) by $\mu(B_j)$ and sum over i and j .

$$\sum_{\substack{i\in I\\ j\in J}} \mu(A_i \cap B_j) \log \left[\frac{\mu(A_i \cap B_j)}{\mu(B_j)}\right] \leq \sum_{\substack{i\in I\\ k\in K}} \mu(A_i \cap C_k) \log \left[\frac{\mu(A_i \cap C_k)}{\mu(C_k)}\right]$$

which reads

$$H(\mathcal{A}|\mathcal{B}) \geq H(\mathcal{A}|\mathcal{C})$$

The similarity of this proof to that of eqn. 23.22 is apparent. Here is another relation

$$H(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) \leq H(\mathcal{A}|\mathcal{C}) + H(\mathcal{B}|\mathcal{C}) \tag{24.3}$$

Proof:

Use $H(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) = H(\mathcal{A}|\mathcal{C}) + H(\mathcal{B}|\mathcal{A} \vee \mathcal{C})$ (24.4)

Then the decreasing property (relation (24.1)) implies (24.3). We now relate this notion of entropy to that of a measure-preserving transformation. The idea in its essence goes back to Shannon.

Definition:

If $\mathcal{A}$ is a finite algebra of measurable sets (finite measurable partition), then so is $T^{-k}\mathcal{A}$ (k=1, 2, 3,....) where T is a measure-preserving transformation. Furthermore, $\mathcal{A} \vee T^{-1}\mathcal{A} \vee \cdots \vee T^{-(n-1)}\mathcal{A}$ is also a finite measurable partition and $h(\mathcal{A} \mid T)$, the entropy of T relative to $\mathcal{A}$ is defined by

$$h(\mathcal{A} \mid T) = \limsup_{n \to \infty} \frac{1}{n} H(\mathcal{A} \vee T^{-1}\mathcal{A} \vee \ldots T^{-(n-1)}\mathcal{A}) \tag{24.5}$$

Comment:

$h(\mathcal{A},T)$ is usually defined as the ordinary limit which is then shown to exist; the lim sup , however, always exists and we will prove that it equals the lim . Hence in the final analysis we obtain the same thing as in the more conventional treatment.

Definition (Kolmogorov-Sinai):

$h(T)$, the entropy of T itself is defined by

$$h(T) = \sup_{\mathcal{A}} h(\mathcal{A},T) \tag{24.6}$$

where the sup is taken over all finite subalgebras.

We shall now try to outline a motivation for Shannon's definition of $H(\mathcal{A},T)$. The original article can be found in a book Shannon and Weaver published by the University of Illinois Press. Shannon started with the following:

Problem:

Measure the information put out by a sender which sends forever letters chosen from a fixed finite alphabet one after the other.

Solution:

Introduce the space X of all messages, i.e., the space of bilateral sequences with elements in the alphabet. In order to specify the sender, you give the probability of the cylinder sets which consist of those messages for which the $j_1 \dots j_n$'th letters sent are $\alpha_{j_1} \dots \alpha_{j_n}$ respectively. Then by a general theorem (Kolmogorov, 1932) you get a unique probability measure on X. Then from what we have seen before, the average information conveyed by the j'th letter is

$$-\sum_{\alpha \in \text{alphabet}} \mu(\{x | x_j = \alpha\}) \log \mu(\{x | x_j = \alpha\}) \tag{24.7}$$

This is a useful number but includes no "information" about correlations between letters. The quantity

$$-\sum_{\alpha, \alpha' \in \text{alphabet}} \mu(\{x | x_j = \alpha, x_{j+1} = \alpha'\}) \log \mu(\{x | x_j = \alpha, x_{j+1} = \alpha'\}) \tag{24.8}$$

on the other hand contains "information" about correlations between the j'th and $j+1$'th letters. In fact, it is the average information conveyed in the sending of th j'th and $j+1$'th letters. If we want the information per letter sent, we must divide by 2. For n successive letters sent, beginning with the j'th, the average information conveyed per letter sent is

$$-\frac{1}{n}\sum_{\alpha_1 \dots \alpha_n} \mu(\{x | x_j = \alpha_1 \dots x_{j+n-1} = \alpha_n\}) \log \mu(\{x | x_j = \alpha_1 \dots x_{j+n-1} = \alpha_n\}) \tag{24.9}$$

We can rewrite this more compactly by introducing

$$(Tx)_j = x_{j+1} \tag{24.10}$$

and $\mathcal{Q}$ as the partition into the sets

$$\left\{ \{x \mid x_j = \alpha\} \quad \alpha \in \text{alphabet} \right\}$$

The claim is that then the expression (24.9) can be written as

$$\frac{1}{n} H(\mathcal{Q} \vee T^{-1}\mathcal{Q} \ldots T^{-(n-1)}\mathcal{Q})$$

Here is an example (actually it is more of an analogue). Consider the phase space for a set of particles whose motion is described by a system of differential equations. Partition the phase space into sets. In this case T is no longer the shifting operator (24.10) but serves effectively to push these sets around so that we will tend to get refinements of the partition. The sup then assures us of getting the "best" partition under this operation. Actually it is not obvious why we want the "best" partition in statistical mechanics; it may even be argued that we want the "worst" partition and should therefore take the inf. The big problem, however, is how to get out relaxation times for a finite time of pushing the sets around.

At this point we list two more pertinent references.

A. N. Kolmogorov - A New Invariant for Transitive Dynamical Systems, DAN 119, 861-864 (1958).

A N. Kolmogorov - Entropy per Second as a Metric Invariant of Automorphisms, DAN 124, 754-755 (1959).

Properties of $h(\mathcal{Q},T)$ and $h(T)$.

In the definition of $h(\mathcal{Q},T)$ lim sup = lim.

Proof:

$$H(\mathcal{Q} \mid \bigvee_{k=1}^{\ell} T^{-k}\mathcal{Q}) = H(\bigvee_{k=0}^{\ell} T^{-k}\mathcal{Q}) - H(\bigvee_{k=1}^{\ell} T^{-k}\mathcal{Q}) \tag{24.11}$$

$$= H\left(\bigvee_{k=0}^{\ell} T^{-k}\mathcal{a}\right) - H\left(\bigvee_{k=0}^{\ell-1} T^{-k}\mathcal{a}\right) \tag{24.12}$$

since the measure and entropy are invariant under T. Now sum (24.12) over ℓ; the right side will telescope and leave only two terms.

$$\sum_{\ell=1}^{n-1} H\left(\mathcal{a} \Big| \bigvee_{k=1}^{\ell} T^{-k}\mathcal{a}\right) = H\left(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{a}\right) - H(\mathcal{a}) \tag{24.13}$$

$\therefore$

$$\frac{1}{n}\sum_{\ell=1}^{n-1} H\left(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{a}\right) = \frac{1}{n} H(\mathcal{a}) + \frac{1}{n}\sum_{\ell=1}^{n-1} H\left(\mathcal{a} \Big| \bigvee_{k=1}^{\ell} T^{-k}\mathcal{a}\right) \tag{24.14}$$

Now $H\left(\mathcal{a} \Big| \bigvee_{k=1}^{\ell} T^{-k}\mathcal{a}\right)$ is monotonically decreasing in ℓ and since all terms are positive, the sequence in ℓ has a limit that equals the limit of the Cesāro mean occurring on the right side of (24.14). So we have found that

$$h(\mathcal{a},T) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{a}\right) = \lim_{n\to\infty} H\left(\mathcal{a} \Big| \bigvee_{k=1}^{n} T^{-k}\mathcal{a}\right) \tag{24.15}$$

This last equation may be interpreted in the context of information theory to mean: Take all the information available in n steps backwards leading up to the -1'st term in a message and ask for the increase in information that can be gotten in the step leading to the 0'th term.

There is also an equation of the form (24.15) for steps forward from 0 . Thus, we also have

$$h(\mathcal{a},T) = \lim_{n\to\infty} H(T^{-n}\mathcal{a} \mid \bigvee_{i=0}^{n-1} T^{-i}\mathcal{a}) \tag{25.1}$$

This equation tells us to start at 0 and to refine the partition by T to the n-1'th step and ask for the increase in information in the n'th step. The validity of (25.1) follows from the following:

<u>Proof:</u>

$$\sum_{\ell=1}^{n} H(\mathcal{a} \mid \bigvee_{k=1}^{\ell} T^{-k}\mathcal{a}) = H(\bigvee_{k=0}^{n} T^{-k}\mathcal{a}) - H(\mathcal{a}) \tag{24.13}$$

$$\sum_{\ell=1}^{n-1} H(\mathcal{a} \mid \bigvee_{k=1}^{\ell} T^{-k}\mathcal{a}) = H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{a}) - H(\mathcal{a}) .$$

Subtracting, we get

$$H(\mathcal{a} \mid \bigvee_{k=1}^{n} T^{-k}\mathcal{a}) = H(\bigvee_{k=0}^{n} T^{-k}\mathcal{a}) - H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{a}) = H(T^{-n}\mathcal{a} \mid \bigvee_{k=0}^{n-1} T^{-k}\mathcal{a})$$

If T is invertible we can get still another form. I.e,

$$h(\mathcal{a},T) = \lim_{n\to\infty} H(\mathcal{a} \mid \bigvee_{i=1}^{n} T^{i}\mathcal{a}) \tag{25.2}$$

This follows from applying T^n to both the variables $T^{-n}\mathcal{a}$ and $T^{-i}\mathcal{a}$ in (25.1).

As regards taking the inf rather than the sup, for strictly positive entropy systems we get irreversibility and the conjecture is that such systems are always K-systems (a concept that will be defined in the near future). In fact, the conjecture is that if $h(\mathcal{a},T) > 0$ for all non trivial $\mathcal{a}$, we get a K-system.

More Identities.

Equation (25.1) implies immediately that

$$h(\mathcal{A},T) \leq h(\mathcal{B},T) \qquad \text{if } \mathcal{A} \subseteq \mathcal{B} \tag{25.3}$$

because this relation holds for the conditional entropies which were $H(\mathcal{A}|\mathcal{C})$ and $H(\mathcal{B}/\mathcal{C})$, and taking an appropriate $\mathcal{C}$ and passing to the limit we get (25.3). However, a still sharper statement holds and contains (25.3) as a special case. We shall give the proof in detail.

$$h(\mathcal{A},T) \leq h(\mathcal{B},T) + H(\mathcal{A}/\mathcal{B}) \tag{25.4}$$

Proof:

$$H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}) \leq H(\{\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}\} \vee \{\bigvee_{j=0}^{n-1} T^{-j}\mathcal{B}\}) = H(\bigvee_{j=0}^{n-1} T^{-j}\mathcal{B}) + H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}/\bigvee_{j=0}^{n-1} T^{-j}\mathcal{B})$$

Now use the subadditivity of H in the first argument, i.e.,

$$H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}/\bigvee_{j=0}^{n-1} T^{-j}\mathcal{B}) \leq \sum_{k=0}^{n-1} H(T^{-k}\mathcal{A}/\bigvee_{j=0}^{n-1} T^{-j}\mathcal{B}) \leq \sum_{k=0}^{n-1} H(T^{-k}\mathcal{A}/T^{-j}\mathcal{B}) \qquad \forall_j$$

since the second argument of H is decreasing. Picking $j = k$ we get.

$$H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}) \leq \sum_{k=0}^{n-1} H(T^{-k}\mathcal{A}/T^{-k}\mathcal{B}) + H(\bigvee_{j=0}^{n-1} T^{-j}\mathcal{B})$$

$$\therefore \quad \frac{1}{n} H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}) \leq H(\mathcal{A}/\mathcal{B}) + \frac{1}{n} H(\bigvee_{j=0}^{n-1} T^{j}\mathcal{B})$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$h(\mathcal{A},T) \qquad\qquad\qquad h(\mathcal{B},T)$$

so we obtain (25.4) in the limit. One more identity:

Let r,s be integers with $n \leq s$ and if T has no inverse let $r \geq 0$.

Consider the following identity.

$$\bigvee_{i=0}^{n-1} T^{-i}\left(\bigvee_{j=r}^{s} T^{-j}\mathcal{A}\right) = T^{-r}\bigvee_{i=0}^{n+s-r-1} T^{-i}\mathcal{A}$$

$$\therefore \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\left(\bigvee_{j=r}^{s} T^{-j}\mathcal{A}\right)\right) = \frac{n+s-r}{n}\left(\frac{1}{n+s-r}\right) H\left(\bigvee_{i=0}^{n+s-r-1} T^{-i}\mathcal{A}\right)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad (25.5)$$

$$h\left(\bigvee_{j=r}^{s} T^{-j}\mathcal{A}, T\right) = h(\mathcal{A}, T)$$

Intuitive meaning of (25.5)

Any refinement resulting from a finite number of operations being taken separately or together yields no new information. Thus, in the context of information theory, the information obtained per letter sent does not change if we take finite groups of letters rather than individual letters as the basic unit.

Special cases of (25.5)

$s = r$

$$h(T^{-1}\mathcal{A}, T) = h(\mathcal{A}, T)$$

$r = 0$

$$h\left(\bigvee_{j=0}^{s} T^{-j}\mathcal{A}, T\right) = h(\mathcal{A}, T)$$

One last consequence of this line of argument. We start with the identity

$$\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} (T^k)^{-i}\left(\bigvee_{j=0}^{k-1} T^{-j}\mathcal{A}\right)\right) = k\left(\frac{1}{nk}\right) H\left(\bigvee_{u=0}^{nk-1} T^{-u}\mathcal{A}\right)$$

$$(25.6)$$

$$h\left(\bigvee_{j=0}^{k-1} T^{-j}\mathcal{A}, T^k\right) = k\, h(\mathcal{A}, T)$$

Theorem (Sinai)

If T is a measure-preserving transformation of a measure space (X, Σ, μ) and $\mathcal{A}$ is a finite algebra of measurable sets such that the σ-algebra, $\bigvee_{n=0}^{\infty} T^{-n}\mathcal{a}$ generated by the family $T^{-n}\mathcal{a}\,(n=0, 1, \ldots)$ is Σ, then

$$h(\mathcal{a},T) = h(T)$$

If T is invertible as well, the same conclusion holds under the weaker hypothesis that

$$\bigvee_{n=-\infty}^{\infty} T^{-n}\mathcal{a} = \Sigma$$

Discussion

This theorem is very useful because it permits us to calculate entropies in an easy manner. Thus we can pick $\mathcal{a}$ to be the same as in the previous example in information theory. In these simple cases when we write $h(\mathcal{a},T)$ as the limit of conditional entropies, all events are statistically independent and we get in the case of the shift (24.10)

$$h(\mathcal{a},T) = -\sum_{j=0}^{n} p_j \log p_j$$

However, this statistical independence does not hold in the standard examples of statistical mechanics.

Before proving the Sinai theorem we need the following

Lemma

Let α be a finite algebra of sets such that each element of α differs by a set of measure zero from some element of the σ-algebra $\sum$ generated by an algebra $\sum_0$. Let P be a probability measure defined on $\sum$. Then for each $\epsilon > 0$ there is a finite subalgebra $\mathcal{B}$ of $\sum_0$ such that $H(\mathcal{a} | \mathcal{B}) < \epsilon$.

Proof

Let the minimal elements of $\mathcal{a}$ be $A_1, \ldots, A_r$, $0 < \rho(A_i)$. Since $-x \log x$ is continuous on $[0,1]$ and vanishes on the ends we can find a δ_0, $0 < \delta_0 < 1$ such that $-x \log x \leq \epsilon/r$ if $0 \leq x \leq \delta_0$ or $1-\delta_0 \leq x \leq 1$. If we can find a subalgebra $\mathcal{B}$ of $\sum_0$ whose minimal elements $B_1, \ldots, B_r$ satisfy

$$P(A_i | B_i) > 1-\delta_0, \quad i=1, \ldots, r \tag{26.1}$$

then since $\sum_i P(A_i | B_j) = 1$ we also have

$$P(A_i | B_j) > \delta_0 \qquad i \neq j \tag{26.2}$$

and

$$H(\mathcal{a} | \mathcal{B}) = -\sum_{j=1}^{r} P(B_j) \sum_{i=1}^{r} P(A_i | B_j) \log P(A_i | B_j) \leq$$

$$\leq \sum_{j=1}^{r} P(B_j) \epsilon \leq \epsilon. \tag{26.3}$$

The problem therefore reduces to finding a $\mathcal{B}$ satisfying (26.1)

Suppose we can find a $\mathcal{B}$ satisfying

$$P(A_i \triangle B_i) < \delta = \min_{1 \leq i \leq r} \delta_0 \frac{P(A_i)}{2}, \tag{26.4}$$

then we have

$$P(A_i) \le P(B_i) + \delta \le P(B_i) + \frac{P(A_i)}{2} \Rightarrow$$

$$\Rightarrow \frac{P(A_i)}{2} \le P(B_i) \Rightarrow$$

$$\Rightarrow P(B_i) - P(A_i \cap B_i) < \delta < \delta_0 P(B_i) \Rightarrow$$

$$\Rightarrow P(A_i | B_i) < 1 - \delta_0. \tag{26.5}$$

We are therefore reduced to finding a $\mathcal{B}$ satisfying (26.4).

Since by the hypothesis of the lemma each A_i differs by a set of measure zero from an element of $\sum$ and this element in turn can be approximated in measure arbitrarily well by an element $\sum_0$ we have for each $\lambda > 0$ a collection of sets $B_i' \in \sum_0$, $i = 1,\ldots,4$ such that

$$P(A_i \triangle B_i') < \lambda, \; i = 1,\ldots,r. \tag{26.6}$$

Now $P(A_i \cap A_j) = 0$, $i \neq j$ and making use of properties (18.11) of the symmetric difference we get for $i \neq j$

$$(B_i' \triangle A_i) \cup (B_j' \triangle A_j) = (B_i'^c \triangle A_i^c) \cup (B_j'^c \triangle A_j^c) \supset$$

$$\supset (B_j'^c \cup B_j'^c) \triangle (A_i^c \cup A_j^c) = (B_i' \cap B_j')^c \triangle (A_i \cap A_j)^c =$$

$$= (B_i' \cap B_j') \triangle (A_i \cap A_j) \Rightarrow$$

$$\Rightarrow (B_i' \cap B_j') \subset (B_i' \triangle A_i) \cup (B_j' \triangle A_j) \bmod 0. \tag{26.7}$$

Therefore

$$P(B_i' \cap B_j') \le P(B_i' \triangle A_i) + P(B_j' \triangle A_j) \le 2\lambda. \tag{26.8}$$

If

$$N = \bigcup_{i \neq j} (B_i' \cap B_j') \tag{26.9}$$

then

$$P(N) \leq r(r-1)\,\lambda \tag{26.10}$$

Define

$$B_i = \begin{cases} B_i' - N, & i = 1,\dots, r-1 \\ X - \bigcup_{j<r} B_j, & i = r, \end{cases} \tag{26.11}$$

then $\{B_1,\dots, B_r\}$ generates a finite subalgebra of $\sum_0$ and for $i \leq r-1$

$$\begin{aligned} P(A_i \triangle B_i) &= P(A_i \triangle B_i' \triangle B_i' - B_i) \leq \\ &\leq P(A_i \triangle B_i') + P(B_i' - B_i) \leq \\ &\leq P(A_i \triangle B_j') + P(N) \leq \\ &\leq \lambda + r(r-1)\lambda. \end{aligned} \tag{26.12}$$

Also using (18.11) again

$$\begin{aligned} P(A_r \triangle B_r) &= P\big((\bigcup_{i<r} A_i)^c \triangle (\bigcup_{j<r} B_j)^c\big) = \\ &= P\big((\bigcup_{i<r} A_i) \triangle (\bigcup_{j<r} B_j)\big) \leq P\big(\bigcup_{i<r} (A_i \triangle B_i)\big) \leq \\ &\leq \sum_{i<r} P(A_i \triangle B_i) = (r-1)(\lambda + r(r-1)\lambda). \end{aligned} \tag{26.13}$$

If λ is small enough (26.4) is satisfied and the lemma is proved. Q.E.D.

<u>Proof of Sinai's Theorem</u>

It suffices to prove that if $\mathcal{B}$ is any finite subalgebra of $\sum$ that

$$h(\mathcal{B},T) \leq h(\mathcal{A},T) \tag{26.14}$$

since then

$$h(T) \geq \sup_{\mathcal{B}} [h(\mathcal{B}, T)] \leq h(\mathcal{A},T) \leq h(T). \tag{26.15}$$

Let

$$\mathcal{A}_n = \begin{cases} \bigvee_{j=0}^{n} T^{-j}\mathcal{A}, & T \text{ not invertible} \\ \bigvee_{j=-n}^{n} T^{-j}\mathcal{A}, & T \text{ invertible} \end{cases} \tag{26.16}$$

then by (25.5)

$$h(\mathcal{A}_n, T) = h(\mathcal{A}, T). \tag{26.17}$$

Also by (25.4)

$$h(\mathcal{B}, T) \leq h(\mathcal{A}_n, T) + H(\mathcal{B}|\mathcal{A}_n) =$$

$$= h(\mathcal{A}, T) + H(\mathcal{B}|\mathcal{A}_n) \tag{26.18}$$

therefore it suffices to prove $H(\mathcal{B}|\mathcal{A}_n) \to 0$ as $n \to \infty$.

Since $\mathcal{A}_n$ is an increasing family of algebras, $\sum_0 = \bigcup_{n=0}^{\infty} \mathcal{A}_n$ is an algebra By hypothesis of the theorem the σ-algebra it generates is $\sum$. There if $\epsilon > 0$ there is by the lemma just proved a finite subalgebra of $\sum_0$ such that $H(\mathcal{B}|\mathcal{C}) < \epsilon$. Since $\mathcal{C}$ lies in some $\mathcal{A}_N$ we have by (24.1) $\forall_n \geq N$.

$$H(\mathcal{B}|\mathcal{A}_n) \leq H(\mathcal{B}|\mathcal{A}_N) \leq H(\mathcal{B}|\mathcal{C}) < \epsilon \tag{26.19}$$

$$\therefore \lim_{u \to \infty} H(\mathcal{B}|\mathcal{A}_n) = 0. \tag{26.20}$$

Q.E.D.

Sinai also proved the following:

(i) If n is a positive integer

$$h(T^n) = nh(T) \tag{26.21}$$

(ii) If T is **invertible**

$$h(T) = h(T^{-1}). \tag{26.22}$$

If T is **invertible** then combining (i) and (ii) we get for all integers

$$h(T^n) = |n| h(T) \tag{26.23}$$

Proof

By (25.3) and (25.6) we get

$$h(\mathcal{A}, T^n) \le h(\bigvee_{J=0}^{n-1} T^{-j}\mathcal{A}, T^n) = nh(\mathcal{A}, T) \Rightarrow$$

$$\Rightarrow h(T^n) = \sup_{\mathcal{A}} h(\mathcal{A}, T^n) \le n \sup_{\mathcal{A}} h(\mathcal{A}, T) = nh(T) \tag{26.24}$$

On the other hand

$$h(T^n) \ge h(\bigvee_{J=0}^{n-1} T^{-j}\mathcal{A}, T^n) = nh(\mathcal{A}, T) \Rightarrow$$

$$\Rightarrow h(T^n) \ge n \sup_{\mathcal{A}} h(\mathcal{A}, T) = nh(T) \tag{26.25}$$

Combining (26.24) and (26.25) we get (26.21).

If T is **invertible** then

$$H(\bigvee_{k=0}^{n-1} (T^{-1})^{-k}\mathcal{A}) = H(T^{n-1}\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}) =$$

$$= H(\bigvee_{k=0}^{n-1} T^{-k}\mathcal{A}) \Rightarrow h(\mathcal{A}, T^{-1}) = h(\mathcal{A}, T) \Rightarrow$$

$$\Rightarrow \quad h(T^{-1}) = h(T) \tag{26.26}$$

This proves (26.22) Q.E.D.

Another result is:

Theorem (Sinai)

If T is an **invertible** measure-preserving transformation of (X, Σ, μ) and $\mathcal{A}$ is a finite algebra such that

$$\bigvee_{n=0}^{\infty} T^{-n}\mathcal{A} = \Sigma \qquad (26.27)$$

then $h(T) = h(\mathcal{A},T) = 0$.

Intuitively this theorem says that in the invertible case if knowledge of the future (the left hand side of 26.27) determines everything (i.e. Σ) including the past then the system has no entropy; intuitively not "random" or has infinite relaxation time.

Proof

Since $\mathcal{A} \subset \Sigma = T^{1}\Sigma = \bigvee_{n=1}^{\infty} T^{-n}\mathcal{A}$ we can apply the lemma to get that the algebra $\bigcup_{n=1}^{\infty} \bigvee_{j=1}^{n} T^{-j}\mathcal{A}$ contains for all $\varepsilon>0$ a finite subalgebra $\mathcal{B}$ such that $H(\mathcal{A}\,|\,\mathcal{B}) < \varepsilon$.

For sufficiently large n we have

$$\mathcal{B} \subset \bigvee_{j=1}^{n} T^{-j}\mathcal{A} \Rightarrow H(\mathcal{A}\,|\,\bigvee_{j=1}^{n} T^{-j}\mathcal{A}) \leq H(\mathcal{A}\,|\,\mathcal{B}) \leq \varepsilon \qquad (26.28)$$

by (24.15) and Sinai's theorem we conclude since ε is arbitrary that

$h(T) = h(\mathcal{A},T) = 0$. Q.E.D.

For a flow T_t we have immediately for rational t that

$$h(T_t) = |t|\,h(T_1) \qquad (26.29)$$

since writing $t = n/d$, n,d integers we have

$$|d|\,h(T_{n/d}) = h(\,(T_{n/d})^{d}) = h(T_n) = h(\,(T_1)^{n}) =$$

$$= |n|\,h(T_1) \Rightarrow h(T_{n/d}) = \left|\frac{n}{d}\right|\,h(T_1). \qquad (26.30)$$

To prove (26.29) for all t it is necessary to prove the continuity of $h(T_t)$ in t, for which a deep theorem on the representation of flows is required. By (26.29) if t is measured in seconds the characteristic time associated with the flow is

$$\frac{1}{h(T_{1sec})}.$$

Let X be the space of all bilateral sequences,,.., $x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ where the x_j belong to the finite set $\{\alpha_1, \ldots, \alpha_r\}$. Let $\sum$ be the σ-algebra generated by the cylinder sets:

$$\{x \mid x_k = \alpha_{i_0}, x_{k+1} = \alpha_{i_1}, \ldots, x_{k+n} = \alpha_{i_n}\} \tag{27.1}$$

where $-\infty < k < \infty$ and $n \geq 0$. The shift transformation T is defined as before:

$$(Tx)_j = x_{j+1} \tag{27.2}$$

Let P be a probability measure on Σ such that the coordinate functions define a stationary random process, that is, T is measure-preserving. We fix a coordinate j and define the finite algebra $\mathcal{a}$ to be the one generated by the finite partition

$$\{\{x \mid x_j = \alpha_s\},\ s=1,\ldots,r\}\ , \tag{27.3}$$

then

$$\sum = \bigvee_{n=-\infty}^{\infty} T^{-n} \mathcal{a} \tag{27.4}$$

and Sinai's theorem applies. We have therefore by Sinai's theorem and (25.2)

$$h(T) = h(\mathcal{a}, T) = \lim_{n\to\infty} H(\mathcal{a} \mid \bigvee_{k=1}^{n} T^{-k} \mathcal{a}). \tag{27.5}$$

Now

$$H(\mathcal{a} \mid \bigvee_{k=1}^{n} T^{-k}\mathcal{a}) = \sum_{i_1,\ldots i_n=1}^{r} P(x_{j+1}=\alpha_{i_1},\ldots,x_{j+n}=\alpha_{i_n}) \times$$

$$\times \sum_{i_0=1}^{r} P(x_j = \alpha_{i_0} \mid x_{j+1}=\alpha_{i_1},\ldots,x_{j+n}=\alpha_{i_n}) \times$$

$$\times \log P(x_j = \alpha_{i_0} \mid x_{j+1}=\alpha_{i_1},\ldots,x_{j+n}=\alpha_{i_n}) \tag{27.6}$$

Using the fact that P is a probability measure and (23.5) we get the bound

$$H(\alpha | \bigvee_{k=1}^{n} T^{-k} \alpha) \leq \log r \Rightarrow$$

$$h(T) \leq \log r . \qquad (27.7)$$

A Bernoulli shift is defined by a probability distribution $p_i, i=1,\ldots r$ on the α's and the probability measure

$$P(x_k=\alpha_{i_0}, x_{k+1}=\alpha_{i_1}, \ldots, x_{k+n} = \alpha_{i_n}) = P_{i_0} P_{i_1} \cdots P_{i_n} \qquad (27.8)$$

on X. In this case we have

$$P(x_j=\alpha_{i_0} | x_{j+1} = \alpha_{i_1}, \ldots, x_{j+n} = \alpha_{i_n}) = P_{i_0} \Rightarrow$$

$$\Rightarrow H(\quad | \bigvee_{k=1}^{n} T^{-n} \quad) = - \sum_{i=1}^{r} p_i \log p_i \Rightarrow$$

$$\Rightarrow h(T) = - \sum_{i=1}^{r} p_i \log p_i . \qquad (27.8)$$

Notice that the entropies of the Bernoulli shifts (1/2, 1/2) and (1/3,1/3,1/3) are log 2 and log 3 respectively. The two shifts are therefore not isomorphic although all Bernoulli shifts have the same spectral invariants (see Billingsley pp. 73-77). The question of the isomorphism of the above two shifts was an open problem for a long time until it was solved by Kolmogorov by the introduction of entropy.

The Markov shift is defined by the set of numbers p_i, p_{ij} $i,j=1,\ldots,r$ having the properties

$$\begin{aligned} &(i) && p_i \geq 0 \quad \sum p_i = 1 \\ &(ii) && p_{ij} \geq 0 \quad \sum_j p_{ij} = 1 \qquad (27.9) \\ &(iii) && \sum_i p_i p_{ij} = p_j \end{aligned}$$

and the probability measure

$$P(x_k = \alpha_{i_o}, x_{k+1} = \alpha_{i_1}, \ldots, x_{k+n} = \alpha_{i_n}) = P_{i_o} P_{i_o i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}, n > 0$$

(27.10)

$$P(x_k = \alpha_j) = P_j$$

The P_{ij} are **intuitively** thought of as transition probabilities for the transition $i \to j$ and the P_i by virtue of (27.9(iii)) as a stationary probability distribution. In the Markov case one has by (27.9), (27.10) and (22.5)

$$P(x_j = \alpha_{i_o} | x_{j-1} = \alpha_{i_1}, \ldots, x_{j-n} = \alpha_{i_n}) = P_{i_1 i_o} \Rightarrow$$

$$\Rightarrow H(\mathcal{A} | \bigvee_{k=1}^{n} T^k \mathcal{A}) = -\sum_{i_1, \ldots i_n = 1}^{r} P_{i_n} P_{i_n i_{n-1}} P_{i_{n-1} i_{n-2}} \cdots P_{i_2 i_1} \times$$

$$\times \sum_{i_o = 1}^{r} P_{i_1 i_o} \log P_{i_1 i_o} =$$

$$= -\sum_{ij} P_i P_{ij} \log P_{ij} \Rightarrow$$

(27.11)

$$\Rightarrow h(T) = -\sum_{i,j} P_i P_{ij} \log P_{ij}$$

In order to deal with more complicated examples we need a theorem which allows us to disregard certain sets of measure zero. (see Billingsley pp. 87-90).

If $\mathcal{A}$ and $\mathcal{B}$ are subfamilies of $\sum$ we shall write $\mathcal{A} = \mathcal{B}$ mod 0 to mean that each set in $\mathcal{A}$ differs from some set in $\mathcal{B}$ by a set of measure zero with respect to a given measure on $\sum$.

<u>Theorem</u>

Let $\{\mathcal{G}_n, n=1,2,\ldots\}$ be a nondecreasing sequence of algebras of measurable sets. If T is a measure-preserving transformation and if

$$\bigvee_{n=1}^{\infty} \bigvee_{i=-\infty}^{\infty} \lim T^{-i} \mathcal{A}_n = \sum \bmod 0 \qquad (27.13)$$

then

$$h(T) = \lim_{n\to\infty} \sup_{\mathcal{a}\subset\mathcal{A}_n} h(\mathcal{a},T) \qquad (27.14)$$

where $\mathcal{a}$ runs over finite subalgebras.

Proof

It is clear that the limit on the right hand side of (27.14) exists; since $h(\mathcal{a},T)$ is nondecreasing as a function of $\mathcal{a}$ and $\mathcal{A}_n$ is a nondecreasing family we are lead to a limit of a nondecreasing sequence of numbers which always exists.

We shall prove the theorem in the case (27.13) holds, the other case being completely similar.

If $\mathcal{H}_n$ is the algebra generated by $\bigcup_{i=0}^{n} T^{-i}\mathcal{A}_n$ and $\sum_0$ is the algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{H}_n$ then every set of $\sum$ differs by a set of measure zero from a set in the σ-algebra generated by $\sum_0$. By the lemma of Lectrue 26 if $\mathcal{B}$ is any finite subalgebra of $\sum$ and ϵ any positive number there is a finite subalgebra ζ of $\sum_0$ such that $H(\mathcal{B}|\zeta) < \epsilon$. Since ϵ is arbitrary

$$h(\mathcal{B},T) \leq h(\zeta,T) + H(\mathcal{B}|\zeta) \Rightarrow$$

$$\Rightarrow h(T) = \sup_{\zeta\subset\sum_0} h(\zeta,T). \qquad (27.15)$$

Now since $\zeta \subset \sum_0$, $\zeta \in \mathcal{H}_n$ for some n, and since ζ is a finite algebra it has some minimal elements $B_1,\ldots B_k$ of the form

$$B_u = \bigcup_{v=1}^{\ell} \bigcap_{i=0}^{n} T^{-i} G_{iuv}, u=1,\ldots,k \qquad (27.16)$$

where $G_{iuv} \in \mathcal{A}_n$. If $\mathcal{Q}$ is the algebra generated by the G_{iuv} then $\mathcal{Q}$ is finite, $\mathcal{Q} \in \mathcal{G}_n$ and $\mathcal{E} \subset \bigvee_{i=0}^{n} T^{-i}\mathcal{Q}$. We then have

$$h(\mathcal{C},T) \leq h(\bigvee_{i=0}^{n} T^{i}\mathcal{Q},T) = h(\mathcal{Q},T) \leq$$

$$\leq \sup_{\mathcal{Q} \subset \mathcal{A}_n} h(\mathcal{Q},T). \tag{27.17}$$

The last term is increasing in n, therefore

$$h(\mathcal{C},T) \leq \lim_{n\to\infty} \sup_{\mathcal{Q} \subset \mathcal{A}_n} h(\mathcal{Q},T) \tag{27.18}$$

and by (27.15)

$$h(T) \leq \lim_{n\to\infty} \sup_{\mathcal{Q} \subset \mathcal{A}_n} h(\mathcal{Q},T) \tag{27.19}$$

The opposite inequality is obviously true since $h(\mathcal{Q},T) \leq h(T)\ \forall\, \mathcal{Q}$. Q.E.D.

<u>Remarks</u>

1) It may happen that the $\mathcal{A}_n$ are themselves σ-algebras and $T^{-1}\mathcal{A}_n \subset \mathcal{A}_n$. Then if T_n and μ_n are the respective restrictions of T and μ to $\mathcal{A}_n$ we get an alternative statement of the theorem:

If $\mathcal{A}_n$ is a nondecreasing sequence of σ-algebras of measureable sets and $\bigvee_{n=1}^{\infty} \mathcal{A}_n = \sum$ mod 0 then

$$h(T) = \lim_{n\to\infty} h(T_n) \tag{27.20}$$

2) If $\mathcal{A}_n = \mathcal{Q}$ a fixed finite algebra then the theorem reduces to Sinai's theorem with a slightly weaker hypothesis in that $\bigvee_{i=0}^{\infty} T^{-i}\mathcal{A}_n$ is required to be $\sum$ only modulo sets of measure zero.

Corolliaries

1) If $\{\mathcal{A}_n, n=1,2,\dots\}$ is a nondecreasing sequence of finite algebras such that

$$\bigvee_{n=1}^{\infty} \mathcal{A}_n = \sum \mod 0 \qquad (27.21)$$

then

$$h(T) = \lim_{n\to\infty} h(\mathcal{A}_n, T). \qquad (27.22)$$

2) If $\mathcal{B} \subset \bigvee_{i=0}^{\infty} T^{-i}\mathcal{A}$ or if T is invertible and $\mathcal{B} \subset \bigvee_{i=-\infty}^{\infty} T^{-i}\mathcal{A}$ then

$$h(\mathcal{B}, T) \leq h(\mathcal{A}, T) \qquad (27.23)$$

Proof

Let T_o be the transformation T considered as a transformation of the space $(X, \sum_o, \mu_o)$ where $\sum_o$ is $\bigvee_{i=0}^{\infty} T^{-i}\mathcal{A}$ or $\bigvee_{i=-\infty}^{\infty} T^{-i}\mathcal{A}$ as is appropriate and μ_o is the restriction of μ to $\sum_o$. Then $h(\mathcal{B}, T) = h(\mathcal{B}, T_o)$ and $h(\mathcal{A}, T) = h(T_o)$ and the corollary follows.

Corollaries

3) If $\mathcal{A}$ is an algebra and if either $\bigvee_{i=1}^{\infty} T^{-i}\mathcal{A} = \sum$ mod 0 or if T is invertible $\bigvee_{i=-\infty}^{\infty} T^{-i}\mathcal{A} = \sum$ mod 0 then

$$h(T) = \sup_{\mathcal{Q} \subset \mathcal{A}} h(\mathcal{Q}, T) \tag{28.1}$$

4) If $\sum_0$ is an algebra which generates $\sum$ then

$$h(T) = \sup_{\mathcal{Q} \subset \sum_0} h(\mathcal{Q}, T)$$

where $\mathcal{Q}$ runs over finite algebras.

Theorem

Let T_1 and T_2 be measure preserving transformations acting on $(X_1, \sum_1, \mu_1)$, and $(X_2, \sum_2, \mu_2)$ respectively. Then

$$h(T_1 \times T_2) = h(T_1) + h(T_2). \tag{28.2}$$

Proof

Let $\mathcal{Q}_i$, $i=1,2$ be a finite subalgebra of $\sum_i$. We denote by 2_i the trivial algebra $\{\phi, X_i\}$. We have

$$\bigvee_{i=0}^{n-1} (T_1 \times T_2)^{-i}(\mathcal{Q}_1 \times \mathcal{Q}_2) =$$

$$= \left[\bigvee_{i=0}^{n-1} (T_1^{-i}\mathcal{Q}_1 \times 2_2)\right] \vee \left[\bigvee_{i=0}^{n-1} (2_1 \times T_2^{-i}\mathcal{Q}_2)\right] \tag{28.3}$$

The two members of the right hand side are independent algebras in the sense that if M belongs to one and N to the other then $\mu(M \wedge N) = \mu(M)\,\mu(N)$. This fact implies

$$H\Big(\bigvee_{i=0}^{n-1} (T_1 \times T_2)^{-i}(\mathcal{Q}_1 \times \mathcal{Q}_2)\Big) =$$

$$= H\Big(\bigvee_{i=0}^{n-1} (T_1^{-i}\mathcal{Q}_1 \times 2_2)\Big) + H\Big(\bigvee_{i=0}^{n-1} (2_1 \times T_2^{-i}\mathcal{Q}_2)\Big) =$$

$$= H\Big(\bigvee_{i=0}^{n-1} T_1^{-i}\mathcal{Q}_1\Big) + H\Big(\bigvee_{i=0}^{n-1} T_2^{-i}\mathcal{Q}_2\Big) \tag{28.4}$$

Dividing by n and passing to the limit we get

$$h(\mathcal{Q}_1 \times \mathcal{Q}_2, T_1 \times T_2) = h(\mathcal{Q}_1, T_1) + h(\mathcal{Q}_2, T_2) \tag{28.5}$$

By corollary 4) and the fact that the algebra generated by the family $\{\mathcal{Q}_1 \times \mathcal{Q}_2, \mathcal{Q}_1 \subset \sum_1, \mathcal{Q}_2 \subset \sum_2\}$ generates the σ-algebra generated by $\sum_1 \times \sum_2$ we conclude

$$h(T_1 \times T_2) = \sup_{\mathcal{Q}_1 \subset \sum_1, \mathcal{Q}_2 \subset \sum_2} h(\mathcal{Q}_1 \times \mathcal{Q}_2, T_1 \times T_2) =$$

$$= h(T_1) + h(T_2) . \qquad \text{Q.E.D.} \tag{28.6}$$

Example: Cascades and Flows on the n-torus $\mathcal{T}^n$

Since the n-torus is the direct product of n circles it is sufficient to consider the case n = 1. Let T be the transformation $x \to x + \alpha \bmod 1$. Let $\mathcal{Q}$ be the algebra generated by th partition $\{[0,1/2), [1/2,1)\}$, then $T^{-n}\mathcal{Q}$ contains two intervals of length 1/2 and beginning at $-n\alpha$ and $-n\alpha + 1/2$. If α is irrational we know that $n\alpha \bmod 1$ is arbitrarily close to any preassigned real number in $[0,1)$ for a suitable choice of n. Thus by intersecting $T^{-n}[0,1/2)$ and $T^{-m}[0,1/2)$ we can get an interval which approximates arbitrarily closely any subinterval of $[0,1)$ of length less than 1/2. The σ-algebra generated by $\bigvee_{n=0}^{\infty} T^{-n}\mathcal{Q}$ is all Borel sets and since T is invertible $h(T) = 0$. If α is rational then $T^n = I$, the identity, for some integer n. In this case $nh(T) = h(T^n) = h(I) = 0$ and therefore $h(T) = 0$ for all α.

Example: Two hard discs on the two-torus $\mathcal{T}^2$

Consider two hard discs of radius R moving freely and with elastic collisions on the torus $\mathcal{T}^2$. The positions of the centers will be denoted by $\vec{q}_1$ and $\vec{q}_2$ where $\vec{q}_1 = (q_{11}, q_{12}), \vec{q}_2 = (q_{21}, q_{22})$ and each coordinate is taken mod 1. The velocities will be denoted by $\vec{v}_1$ and $\vec{v}_2$ whose coordinates are of course unrestricted. There are three integrals of motion:

$$E = \frac{m}{2}\left[\vec{v}_1^{\,2} + \vec{v}_2^{\,2}\right] \tag{28.7}$$

$$\vec{P} = m\left[\vec{v}_1 + \vec{v}_i\right].$$

The equations of motion when not under collision are

$$\frac{d\vec{q}_1}{dt} = v_1 , \quad \frac{d\vec{q}_i}{dt} = \vec{v}_i \tag{28.8}$$

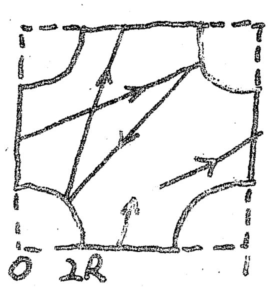
$$\frac{d\vec{v}_1}{dt} = 0, \quad \frac{d\vec{v}_i}{dt} = 0$$

and the collisions are elastic.

As $\vec{q}_1$ and $\vec{q}_2$ vary over the torus the quantities $\vec{q}_1 + \vec{q}_i$ and $\vec{q}_1 - \vec{q}_2$ also vary over a torus. Now $\frac{d}{dt}(\vec{q}_1 + \vec{q}_2) = \vec{v}_1 + \vec{v}_2$ a constant of the motion. The configuration space therefore factors into the torus of $\vec{q}_1 + \vec{q}_2$ and th torus of $\vec{q}_1 - \vec{q}_2$ and in the $\vec{q}_1 + \vec{q}_2$ space the motion is identical to the toroidal flow of the preceding example with the corresponding entropy zero. In the $\vec{q}_1 - \vec{q}_2$ torus there is a restriction since interpenetration is forbidden:

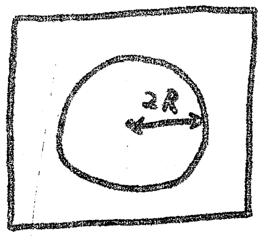
$$|\vec{q}_1 - \vec{q}_2|^2 \geq 4R^2 \tag{28.9}$$

where the left hand side is interpreted as the toroidal length squared; that is, the square of the shortest toroidal distance from the origin to $\vec{q}_1 - \vec{q}_2$. We have the following picture of the $\vec{q}_1 - \vec{q}_2$ factor of configuration space, which we shall call the reduced configuration space.



(28.10)

The motion of the discs induces a motion of the configuration point which is seen to satisfy periodic boundary conditions at the "edges" and elastic reflections from the forbidden region. By a change of origin we get another useful picture of the reduced configuration space:



(28.11)

where the induced motion again satisfies periodic boundary conditions and elastic reflections from the centrally placed forbidden region. The magnitude of the velocity of the configuration point is a constant of the motion:

$$|\vec{v}_1 - \vec{v}_2|^2 = \frac{4}{m} E - \vec{P}^2 \qquad (28.12)$$

At each point of the reduced configuration space we therefore have only the direction of the velocity at our disposal. The phase space on which we shall consider our flow is therefore the direct product of the reduced configuration space and the unit circle representing the direction of the velocity.

Reference:

Ya. Sinai Vestnik Moskow University 1963 pp. 6-12

Lecture 32

Remark

Concerning condition I of the definition of K-systems one may ask if it is possible that $\mathcal{M}^{t_1} = \mathcal{M}^{t_2}$ for $t_1 > t_2$. This however contradicts the other two conditions and the nontriviality of $\sum$ according to the following argument. Since $\mathcal{M}^t = T^t \mathcal{M}^0$, $\mathcal{M}^{t_1} = \mathcal{M}^{t_2} \Rightarrow T^{t_1-t_2} \mathcal{M}^0 = \mathcal{M}^0$, but because $\mathcal{M}^{t'} \geq \mathcal{M}^t$ for $t' \geq t$ we must have $T^t \mathcal{M}^0 = \mathcal{M}^0$ for $0 \leq t \leq t_1 - t_2$. By the group property of the flow we must therefore have $T^t \mathcal{M}^0 = \mathcal{M}^0 \; \forall t$, hence $\bigwedge_{-\infty}^{\infty} \mathcal{M}^t = \mathcal{M}^0$ and by condition II $\mathcal{M}^0 = \{\emptyset, X\}$ mod 0. This however contradicts condition III since $\bigvee_{-\infty}^{\infty} \mathcal{M}^t = \mathcal{M}^0 = \sum$ mod 0 but $\sum$ is presumed nontrivial.

K - flows and K-cascades have the following three fundamental properties.

A) The U^t corresponding to the T^t has denumerably multiple homogeneous Lebesgue spectrum (to be defined later) on the complement of the constants.

B) The entropy of a K-system is strictly positive.

C) Every K-system is mixing of all orders.

The original proof of these properties is found in Ya. G. Sinai, <u>Dynamical Systems with Denumerably Multiple Lebesgue Spectrum I</u>, Izvest. Akad.Nauk <u>25</u> 899-924 (1961) and which has been translated by the A.M.S. in Translation Series 2 Vol. 39 pp. 83-110 (1964). In the above paper Sinai uses Lebesgue spaces for the measure space X and refers to an article of Rohlin <u>Selected Topics in the Metric Theory of Dynamical Systems</u> (V.A. Rohlin, Uspekhi. Mat. Nauk, (1949)) A.M.S. Translations No. 49-1966) for their properties.

Definitions

A measure μ on the measure space $(X, \sum, \mu)$ is said to be <u>complete</u> if whenever $S \subset G \subset T$, $S, T \in \sum$ and $\mu(S) = \mu(T)$ then $G \in \sum$. If μ is not complete one can always extend it to a complete measure by adjoining the above sets G to $\sum$ and defining $\mu(G)$ by $\mu(G) = \mu(S) = \mu(T)$.

A countable family $\{B_i ; i \in I\}$ of measurable sets is called a <u>basis</u> for $\sum$ if:

(a) The σ-algebra generated by $\{B_i\}$ is $\sum$ mod 0.

(b) for any x, y $\in$ X there is a B_j such that one but not both of these points belong to it.

A Lebesgue space is a finite measure space with a complete measure and a countable basis and in which the basis satisfies the additional property that every intersection of the form $\bigcap_{i \in I} E_i$, where each E_i is either a B_i or an $X-B_i$, is not empty.

Theorem

Let T^t be a measurable flow in a Lebesgue space $(X, \sum, \mu)$. If there exists σ-subalgebra $\mathcal{Q}$ of $\sum$ such that

$$t_1 < t_2 \Rightarrow T^{t_1}\mathcal{Q} \subset T^{t_2}\mathcal{Q}, \tag{32.1}$$

(For the purposes of this theorem $\subset$ shall denote strict inclusion even when sets of measure zero can be disregarded) then the flow has strictly positive entropy.

Before giving the proof of the theorem we must first prove the

Lemma

If $(X, \sum, \mu)$, T^t and $\mathcal{Q}$ are as in the hypothesis of the theorem then there is a finite subalgebra $\mathcal{B}$ of $\mathcal{Q}$ such that

$$\bigvee_{\tau \le t_1} T^{\tau}\mathcal{B} \subset \bigvee_{\tau \le t_2} T^{\tau}\mathcal{B} \quad \text{for } t_1 < t_2 \tag{32.2}$$

Proof of Lemma

The proof is by reductio ad absurdum. Assume that for every finite subalgebra $\mathcal{B}$ of $\mathcal{Q}$ there exists a pair of real numbers t_1, t_2, $t_1 < t_2$ such that

$$\bigvee_{\tau \le t_1} T^{\tau}\mathcal{B} = \bigvee_{\tau \le t_2} T^{\tau}\mathcal{B}, \tag{32.3}$$

which implies

$$\bigvee_{\tau \leq 0} T^{\tau}\mathcal{B} = T^{(t_2-t_1)} \bigvee_{\tau \leq 0} T^{\tau}\mathcal{B} . \tag{32.4}$$

But for $t' \geq t$ we have

$$T^{t'} \bigvee_{\tau \leq 0} T^{\tau} = \bigvee_{\tau \leq t'} T^{\tau}\mathcal{B} \geq \bigvee_{\tau \leq t} T^{\tau}\mathcal{B} = T^{t} \bigvee_{\tau \leq 0} T^{\tau}\mathcal{B} \tag{32.5}$$

and combining this with (32.4) we see that $\bigvee_{\tau \leq 0} T^{\tau}\mathcal{B}$ must be invariant under T^t for $0 \leq t \leq t_2 - t_1$ and by the group property of the flow it must therefore be invariant under $T^t (\forall t.)$

Since $\sum$ is generated by a countable family of sets we can find an increasing sequence $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$ of finite subalgebras such that

$$\bigvee_{n=1}^{\infty} \mathcal{B}_n = \mathcal{Q} \bmod. 0 \tag{32.6}$$

Since by hypothesis each B_j satisfies (32.3) we have that $\bigvee_{\tau \leq 0} T^{\tau}\mathcal{B}_j$ is invariant under T^t. Also by condition (32.1) $\mathcal{Q} = T^0 \mathcal{Q} \supset T^{\tau}\mathcal{Q}$ for $\tau \leq 0 \Rightarrow \mathcal{Q} = \bigvee_{\tau \leq 0} T^{\tau}\mathcal{Q}$. We now have

$$T^t\mathcal{Q} = T^t \bigvee_{\tau \leq 0} T^{\tau}\mathcal{Q} = T^t \bigvee_{\tau \leq 0} T^{\tau} \bigvee_{n=1}^{\infty} \mathcal{B}_n =$$

$$= T^t \bigvee_{\tau \leq 0} \bigvee_{n=1}^{\infty} T^{\tau}\mathcal{B}_n = \bigvee_{n=1}^{\infty} T^t \bigvee_{\tau \leq 0} T^{\tau}\mathcal{B}_n =$$

$$= \bigvee_{n=1}^{\infty} \bigvee_{\tau \leq 0} T^{\tau}\mathcal{B}_n = \bigvee_{\tau \leq 0} \bigvee_{n=1}^{\infty} T^{\tau}\mathcal{B}_n = \bigvee_{\tau \leq 0} T^{\tau}\mathcal{Q} = \mathcal{Q} \tag{32.7}$$

which contradicts condition (32.1). Q.E.D.

<u>Proof of Theorem</u>

Let $\mathcal{B}$ be the finite subalgebra whose existence is asserted by the previous lemma. We then have

$$h(T^1) \geq H(T^1 \mathfrak{B} \mid \bigvee_{k=0}^{\infty} T^{-k} \mathfrak{B}) \geq$$

$$\geq H(T^1 \mathfrak{B} \mid \bigvee_{\tau \leq 0} T^{\tau} \mathfrak{B}) > 0 \qquad (32.8)$$

The last inequality follows since by the strict increasing property of $\mathfrak{B}$ under T the algebra $T^1 \mathfrak{B}$ is not obtainable from $\bigvee_{\tau \leq 0} T^{\tau} \mathfrak{B}$ by adjunction of sets of measure zero. Q.E.D.

That the theorem proves property B) is clear since $\mathfrak{M}^0$ can play the role of $\mathfrak{A}$.

Flows Under a Function; the Ambrose-Kakutani Representation Theorem

The construction is a generalization to general measure spaces of one occurring in the work of Poincaré and Birkhoff in the theory of differentiable flows. In a special case it occurs in the work of von Neumann; the general theorem is due in the ergodic case to Ambrose and in the general case to Ambrose and Kakutani.

G. D. Birkoff, Dynamical Systems pp. 143-144, 150-158

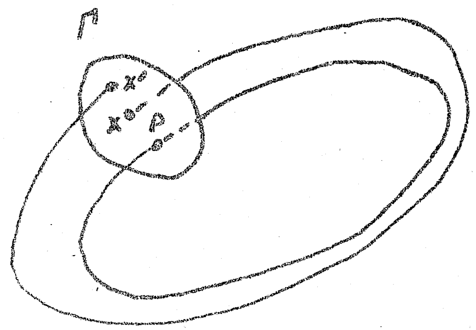
J. von Neumann, Ann. of Math. 33 (1932) 587-642

W. Ambrose, Ann. of Math. 42 (1441) 723-734

W. Ambrose and Kakutani, Duke Math. Jrnl. 9 (1942) 25-42

V. A. Rohlin, Uspekhi Mat. Nauk 4 (1949) 57-178 last section.

The intuitive idea can be illustrated by considering a differentiable flow with a periodic orbit.



At a point P of the periodic orbit construct a hypersurface element Γ which is not tangent to the flow. By continuity of the flow the orbits starting from a point x on Γ and sufficiently close to P will again intersect Γ. Call x' the first such intersection after the orbit leaves x. The map $S: x \longrightarrow Sx = x'$ is called the Poincaré map and completely describes the behaviour of the flow near the periodic orbit. We clearly have $SP=P$ and all other periodic orbits lying close to the given one are given by points x on Γ satisfying $S^k x = x$ for some integer k. This point of view enables one to prove the existence of periodic orbits for a large variety of general dynamical systems. The generalization of the above concepts is that to a flow under a function.

Definition

Let $(X, \sum, \mu)$ be a finite measure space with a complete measure μ. Let S be an invertible measure-preserving transformation of X into itself and f a real positive integrable function on X We assume $\forall x$

$$\sum_{n=0}^{\infty} f(S^n(x)) = \sum_{n=1}^{\infty} f(S^{-n}(x)) = +\infty. \tag{33.1}$$

Consider the product space of X with $\mathbb{R}^1$ **(equipped with Lebesgue measure) and the** complete product measure $\bar{\mu}$ on $X \times \mathbb{R}^1$. Let $\bar{X}$ be the portion of $X \times \mathbb{R}^1$ under f

$$\bar{X} = \left\{(x,t) \mid 0 \leq t \leq f(x)\right\} . \tag{33.2}$$

and $\bar{\sum}$ the set of all measurable subsets of $\bar{X}$. Then $(\bar{X}, \bar{\sum}, \bar{\mu})$ is a measure space and we define the flow T^t by

$$T^t(x,u) = \begin{cases} (x, u+t) \text{ if } -u < t \leq -u + f(x), \\ (S^n x,\ u+t - f(x) - \cdots - f(S^{n-1}(x))) \text{ if } \\ \qquad -u + \sum_{k=0}^{n-1} f(S^k(x)) \leq t \leq -u + \sum_{k=0}^{n} f(S^k(x)), \\ (S^{-n}x,\ u+t+f(S^{-1}(x)) + \cdots + f(S^{-n}(x))) \text{ if } \\ \qquad -u - \sum_{k=1}^{n} f(S^{-k}(x)) \leq t \leq -u - \sum_{k=0}^{n-1} f(S^{-k}(x)) \end{cases} \tag{33.3}$$

This complicated definition is pictorially very simple:

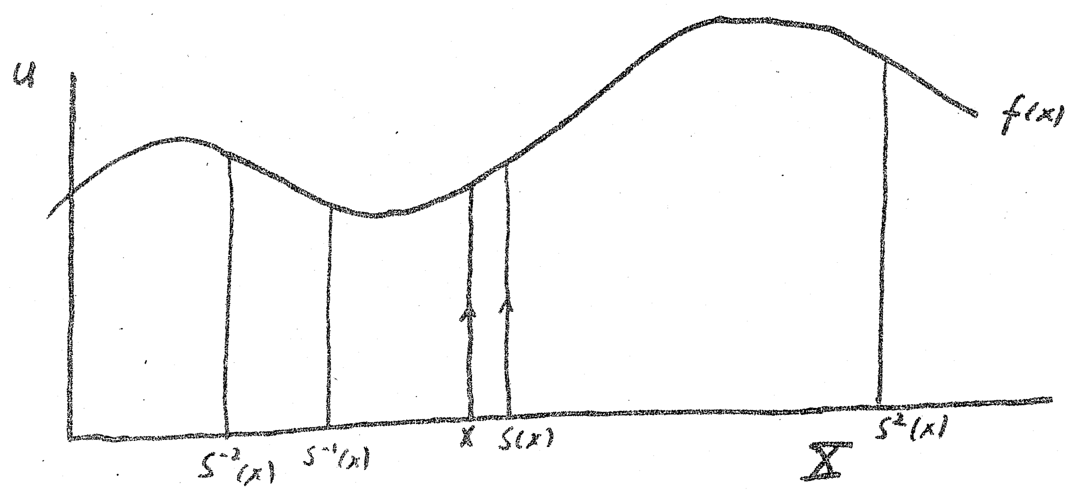
The flow continues uniformly upward until it reaches the graph of f whereupon it jumps to (Sx,o) if it reached the graph at (x,f(x)), there it again resumes uniform upward translation. This T^t is called the flow built under the function f by S.

If X is not a finite measure space the same construction can be carried through but one assumes in addition that $\bar{\mu}(\bar{X}) = \int_X f(x)\,\mu(dx) < \infty$ The object is then called a generalized flow built under a function.

Theorem (Ambrose-Kakutani)

Every measurable flow on a Lebesgue space is isomorphic to a flow built under a function f with $f(x) \geq \gamma > 0$ a.e.

The advantage of this theorem is that it reduces the study of the flow T^t to that of the cascade S.



LECTURE 34

SPECTRUM OF K-FLOWS

Definition:

A continuous one parameter unitary group $\{ U^t, -\infty < t < \infty \}$ defined on a Hilbert space $\mathcal{H}$ is said to have a simple Lebesgue spectrum if there is an isomorphism $V: \mathcal{H} \to L^2(\mathbb{R}^1, dx)$ such that if $h \in \mathcal{H}$,

$$(V\, U^t\, h)\,(x) = (V\, h)\,(x - t) = (\mathcal{U}^t\, V\, h)\,(x) \tag{34.1}$$

where $\mathcal{U}^t$ is the unitary translation group on $L^2(\mathbb{R}^1)$. In other words, U^t has simple Lebesgue spectrum if it is unitarily equivalent to the group of translations of $\mathbb{R}^1$ in its action on $L^2(\mathbb{R}^1)$.

Under the Fourier transformation $\mathcal{F}$ the translation operation $\mathcal{U}^t$ goes over into multiplication by e^{ikt}, that is if $F \in L^2(\mathbb{R}^1)$

$$(\mathcal{F}\, F)\,(k) = \frac{1}{\sqrt{2\pi}} \int e^{ikx}\, F(x)\, dx \tag{34.2}$$

and

$$(\mathcal{F}\, \mathcal{U}^t\, F)\,(k) = \frac{1}{\sqrt{2\pi}} \int e^{ikx}\, (\mathcal{U}^t\, F)\,(x)\, dx = e^{ikt}\, (\mathcal{F}\, F)\,(k) \tag{34.3}$$

which is the traditional form of an operation with simple Lebesgue spectrum.

One of course has Parseval's theorem:

$$\int \left| (\mathcal{F}\, F)\,(k) \right|^2 dk = \int |F(x)|^2\, dx \;. \tag{34.4}$$

Definition:

The continuous one parameter group U^t acting on $\mathcal{H}$ has homogeneous Lebesgue spectrum of multiplicity κ (a finite integer or an infinite cardinal) if $\mathcal{H}$ can be decomposed into a direct sum of κ summands, $\mathcal{H} = \bigoplus_j \mathcal{H}_j$ such that

$U^t \mathcal{H}_j \subset \mathcal{H}_j$ and on each $\mathcal{H}_j$, U^t has simple Lebesgue spectrum.

From this definition we see that if U^t has homogeneous Lebesgue spectrum of some multiplicity κ then there is a subspace $\mathcal{H}_o$ of $\mathcal{H}$ such that

$$\text{I}')\quad \mathcal{H}^{t_1} \subset \mathcal{H}^{t_2} \text{ for } t_1 < t_2 \text{ where } \mathcal{H}^t = U^t \mathcal{H}_o$$

$$\text{II}')\quad \bigvee_{-\infty}^{\infty} \mathcal{H}^t = \mathcal{H} \tag{34.5}$$

$$\text{III}')\quad \bigwedge_{-\infty}^{\infty} \mathcal{H}^t = \{0\}$$

in fact one can take for $\mathcal{H}_o$ the subspace $\bigoplus_j V_j^{-1} L_o^2$ where L_o^2 is the subspace of $L^2(\mathbb{R}^1)$ consisting of functions vanishing for $x > 0$ and V_j is the isomorphism in the definition of simple Lebesgue spectrum.

Definition:

Let $\{E^t, -\infty < t < \infty\}$ be a family of projection operators in a Hilbert space $\mathcal{H}$ satisfying

$$\text{a})\quad E^{t_1} \leq E^{t_2} \text{ for } t_1 \leq t_2$$

$$\text{b})\quad \lim_{t\to\infty} E^t = I \tag{34.6}$$

$$\lim_{t\to-\infty} E^t = 0$$

then the family is called a <u>spectral family</u> of projections.

Definition:

A spectral family has homogeneous Lebesgue spectrum of multiplicity κ on a Borel set B of $\mathbb{R}^1$ if $\mathcal{H}$ is isomorphic to a direct sum of κ copies of $L^2(B,dx)$ and in each of which E^t acts as the projection onto the subspace of functions defined on B and vanishing for $x > t$.

Theorem:

If f is an element of the Hilbert space $\mathcal{H}$ such that $\widehat{(f, U^t f)} = 0 \;\forall |t| > \tau_o$

then U^t has simple Lebesgue spectrum in the cyclic subspace of f .

Proof:

Recall that the cyclic subspace of f is the subspace spanned by vectors of the form $\Sigma\, c_k\, U^{t_k} f$.

By the spectral theorem

$$(f,\, U^t f) = \int e^{i\lambda t}\,(f,\, E(d\lambda) f) \quad . \tag{34.7}$$

Since the left hand side is continuous and has compact support

$$\int e^{-it\lambda}\,(f,\, U^t f)\, dt \tag{34.8}$$

is an entire function of λ . This can be seen by substituting $\lambda + i\eta$ for λ in (34.8), seeing that the integral still converges by the compactness of the support of $(f,\, U^t f)$ and then showing that the resulting function of the complex variable $\lambda + i\eta$ has a derivative with respect to it. We can now conclude that

$$(f,\, E(d\lambda) f) = r_f(\lambda)\, d\lambda \tag{34.9}$$

where $r_f(\lambda)$ is an entire function. The equality

$$(\Sigma\, c_k\, U^{t_k} f,\, \Sigma\, d_e\, U^{t_e} f) = \Sigma\, \bar{c}_k\, d_e \int e^{i(t_e - t_k)\lambda}\, r_f(\lambda)\, d\lambda \tag{34.10}$$

establishes a correspondence

$$\Sigma\, c_k\, U^{t_k} f \leftrightarrow \psi(\lambda) = \Sigma\, c_k\, e^{i\lambda t_k} \tag{34.11}$$

and the inner product

$$(\varphi,\, \psi) = \int_{-\infty}^{\infty} \varphi(\lambda)\, \psi(\lambda)\, r_f(\lambda)\, d\lambda \; . \tag{34.12}$$

under this correspondence.

$$U^t \sum c_k U^{t_k} f \longleftrightarrow e^{i\lambda t} \sum c_k e^{i\lambda t_k} \tag{34.13}$$

thus the unitary operator U^t is transformed into multiplication by $e^{i\lambda t}$. Let $\psi \in L^2(\mathbb{R}^1)$ and $\tilde{\psi} = \mathcal{F}\psi$. Define V by

$$(V^{-1}\psi)(\lambda) = \frac{\tilde{\psi}(\lambda)}{\sqrt{r_f(\lambda)}} \tag{34.14}$$

Even though $r_f(\lambda)$ may vanish at some points if the inner product (34.12) is used, integrability is still preserved, in fact

$$\int \left|(V^{-1}\psi)(\lambda)\right|^2 r_f(\lambda)\, d\lambda = \int \left|\tilde{\psi}(\lambda)\right|^2 d\lambda = \int \left|\psi(t)\right|^2 dt \tag{34.15}$$

and V is an isometry. Moreover

$$(V^{-1}\mathcal{U}^t\psi)(\lambda) = e^{i\lambda t}\frac{\tilde{\psi}(\lambda)}{\sqrt{r_f(\lambda)}} = (U^t V^{-1}\psi)(\lambda)$$

$$\therefore \quad V^{-1}\mathcal{U}^t V = U^t \tag{34.16}$$

and U therefore has simple Lebesgue spectrum. QED.

<u>Criterion for Homogeneous Lebesgue Spectrum.</u>

<u>Theorem:</u>

Let $\{U^t, -\infty < t < \infty\}$ be a continuous unitary group on a Hilbert space $\mathcal{H}$. A necessary and sufficient condition that it have a homogeneous Lebesgue spectrum is that there exist a spectral family $\{E^t\}$ such that

$$U^t E^\tau U^{-t} = E^{t+\tau} \tag{34.17}$$

Moreover, $\{E^t\}$ also has homogeneous Lebesgue spectrum with the same multiplicity as $\{U^t\}$.

Before proving the theorem stated last time, we shall prove the Uniqueness Theorem of von Neumann which shall make the proof of the necessity trivial. von Neumann - Die Eindeutigkeit der Schrödingerschen Operatoren - Mathematische Annalen 104 570-578 (1931).

Theorem:

Every representation of the Weyl relations

$$U^{t_1} U^{t_2} = U^{t_1 + t_2} \tag{35.1a}$$

$$V^{t_1} V^{t_2} = V^{t_1 + t_2} \tag{35.1b}$$

$$U^{t_1} V^{t_2} = e^{it_1 t_2} V^{t_2} U^{t_1} \tag{35.1c}$$

is a direct sum of irreducible representations, and each irreducible representation is equivalent to the Schrödinger representation

$$(U^t f)(x) = f(x-t) \tag{35.2a}$$

$$(V^t f)(x) = e^{itx} f(x) \tag{35.2b}$$

on $L^2 (\mathbb{R}^1)$.

Proof:

Define $S(t,\tau) = e^{-\frac{1}{2}it\tau} U(t) V(\tau) = e^{\frac{1}{2}it\tau} V(\tau) U(t)$ (35.3)

Then the equations (35.1) can be combined into the equation

$$S(t_1, \tau_1) S(t_2, \tau_2) = e^{\frac{1}{2}i(t_1\tau_2 - \tau_1 t_2)} S(t_1 + t_2, \tau_1 + \tau_2) \tag{35.4}$$

Consequently $S(o,o) = 1$

and hence $S(-t, -\tau) = S(t,\tau)^{-1} = S(t,\tau)^*$

We now consider linear combinations of these S's. These are defined as follows: Let $a(t,\tau)$ be an absolutely integrable function over the whole t, τ plane then by Schwarz's inequality

$$|(f, S(t,\tau)g)| < \| f \| . \| S(t,\tau)g\| = \|f\| . \| g \|$$

and hence

$$\int\int a(t,\tau) (f,S(t,\tau)g) \, dt d\tau \leq \int\int |a(t,\tau)| \, dt d\tau . \| f \| . \| g \|$$

Furthermore the integral is linear in g and antilinear in f. Thus, we know that there exists for each fixed g, some g^* such that

$$\iint a(t,T)\ (f,\ S(t,T)g)dt,dT = (f,g^*) \tag{35.5}$$

and

$$\| g^* \| \leq \iint | a(t,T) | dtdT\ \| g \|$$

g^* is determined linearly in terms of g and hence we can define a linear operator A by

$$Ag = g^* \tag{35.6}$$

We write symbolically

$$A = \iint a(t,T)\ S(t,T)dtdT \tag{35.7a}$$

which means

$$(f,Ag) = \iint a(t,T)\ (f,\ S(t,T)g)dtdT \tag{35.7b}$$

We call $a(t,T)$ the kernel of A.
To solve our uniqueness problem we consider the kernel

$$a(t,T) = e^{-\frac{1}{4} t^2 - \frac{1}{4} T^2} \tag{35.8}$$

which will give us the projection onto the ground state for the simple harmonic oscillator. It is easily verified that A is self-adjoint and different from zero. Furthermore with this kernel we get by direct computation that

$$A\ S(t,T)\ A = 2\Pi\ e^{-\frac{1}{4} t^2 - \frac{1}{4} T^2}\ A \tag{35.9}$$

setting $t = T = 0$ we get:

$$A^2 = 2\Pi\ A \tag{35.10}$$

Now consider the solutions of the equation

$$Af = 2\Pi f \tag{35.11}$$

since A is linear and bounded, these span a closed linear subspace $\mathcal{M}$.
Also each

$f \varepsilon \mathcal{M}$ is of the form Ag where $g = \frac{1}{2\Pi} f$ and conversely each $Ag \in \mathcal{M}$ since $A(Ag) = A^2 g = 2\Pi(Ag)$

If

$$f, g \in \mathcal{M}$$

Then

$$(S(t_1, T_1) f, S(t_2, T_2)g) = \frac{1}{4\Pi^2} (S(t_1, T_1)Af, S(t_2, T_2)Ag)$$

$$= \frac{1}{4\Pi^2} (Af, S(-t_1, -T_1) S(t_2, T_2)Ag) = \frac{1}{4\Pi^2} e^{\frac{1}{2}i(T_1 t_2 - t_1 T_2)}$$

$$(f, AS(t_2 - t_1, T_2 - T_1)Ag)$$

$$= e^{\frac{1}{2}i(T_1 t_2 - t_1 T_2) - \frac{1}{4}(t_2 - t_1)^2 - \frac{1}{4}(T_2 - T_1)^2} (f,g)$$

Now let ϕ_1, ϕ_2 --- be a complete orthonormal set in $\mathcal{M}$. (If the underlying Hilbert space is separable, this set is necessarily countable; furthermore the existence of such a set follows from Zorn's Lemma even if the Hilbert space is non-separable.)

From $(\phi_m, \phi_n) = \delta_{m,n}$

we get

$$(S(t_1, T_1)\phi_m, S(t_2, T_2) \phi_n) = e^{\frac{1}{2}i(T_1 t_2 - t_1 T_2) - \frac{1}{4}(t_2 - t_1)^2 - \frac{1}{4}(T_2 - T_1)^2} \delta_{m,n} \tag{35.12}$$

Let $\mathcal{P}_n$ be the closed linear subspace spanned by $S(t,T)\phi_n$ for fixed n, and with t, T varying.

From (35.12) we get $\mathcal{P}_n \perp \mathcal{P}_m$ if $n \neq m$. Call $\mathcal{S} = \bigoplus_n \mathcal{P}_n$ (finite, countable or uncountable sum).

It is clear that each $\mathcal{P}_n$ and consequently $\mathcal{S}$ and $\mathcal{S}^\perp$ is invariant under (S(t,T). Since $\mathcal{S}$ includes all $\mathcal{P}_n$, that is $\mathcal{M}$, it follows that $\mathcal{S}^\perp \subset \mathcal{M}^\perp$

$$\therefore \forall \; f \in \mathcal{S}^\perp$$

$$Af = 0$$

But all our considerations for A are true in $\mathcal{S}^\perp$ because the S(t,T) can be considered as operators in $\mathcal{S}^\perp$ since $\mathcal{S}^\perp$ is invariant under S(t,T). Since in $\mathcal{S}^\perp$ A = 0 it is impossible for any $f \in \mathcal{S}^\perp$ to be different from zero. That is

$$\mathcal{S}^\perp = \{0\}$$

and hence $\{\mathcal{P}_n\}$ span the whole Hilbert space. If in $\mathcal{P}_n$ we call

$$S(t,T)\, \phi_n = f_{t,T} \tag{35.13}$$

Then we get (restricted to $\mathcal{P}_n$)

$$S(t_1,T_1)\, f_{t,T} = e^{\frac{1}{2}i(t_1T - T_1t)} \quad f_{t+t_1\,,\,T+T_1}$$

$$(f_{t_1\,T_1},\, f_{t,T}) = e^{-\frac{1}{4}(t-t_1)^2 - \frac{1}{4}(T-T_1)^2 + \frac{1}{2}i(t_1T - T_1t)}$$

for which on returning to U^t, V^t we get:

$$U^{t_1}\, f_{t,T} = e^{\frac{1}{2}it_1T}\; f_{t_1+t_1T}$$

$$V^{T_1}\, f_{t,T}\;\; e^{-\frac{1}{2}itT_1}\; f_{t,\,T+T_1}$$

We have now shown that every irreducible representation of the Weyl relations is unitary equivalent to one of the form

$$U^{t_1} f_{t,\tau} = e^{\frac{1}{2}it_1\tau} f_{t+t_1,\tau}$$

$$V^{\tau_1} f_{t,\tau} = e^{-\frac{1}{2}it\tau_1} f_{t,\tau+\tau_1}$$

It follows that the Schrödinger representation, being irreducible, is equivalent to a representation of this form. Hence, any representation of the Weyl relations is equivalent to a direct sum of Schrödinger representations.
This completes the proof of von Neumann's theorem. We shall now give the proof of the theorem giving a criterion for homogeneous Lebesgue spectrum

Proof:

1. Sufficiency:

Given $U^t E^\tau U^{-t} = E^{t+\tau}$ (35.14a)

with $E^{t_1} \leq E^{t_2}$ for $t_1 \leq t_2$ (35.14b)

and $\lim_{t\to-\infty} E^t = 0 \qquad \lim_{t\to+\infty} E^t = 1$ (35.14c)

Then we write

$$V^t = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE^\lambda \tag{35.15a}$$

which means

$$(\phi, V^t\psi) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d(\phi, E^\lambda\psi) \tag{35.15b}$$

From these equations it is a simple matter to verify that U^t and V^t satisfy the Weyl relations and that they therefore have, by von Neumann's theorem, homogeneous Lebesgue spectrum with the same multiplicity.

2. Necessity:

Thus, given U^t has homogeneous Lebesgue spectrum, then U^t is defined on a direct sum of copies of $L^2(R^1)$ and restricted to any one of these spaces U^t acts as follows

Define V^t by

$$(V^t f)\ (x) = e^{ixt}\ f\ (x)$$

So V^t is clearly unitary and has simple Lebesgue spectrum. So if we write

$$V^t = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE^{\lambda}$$

which is possible since V^t is unitary, then equations (35.14) will hold on each $L^2(R^1)$

Forming the direct sum of all these copies of L^2 to get back to the full space for U^t, the equations will still remain valid and furthermore V^t will now have the same multiplicity as U^t.

We now apply this as a theorem for K-systems.

Theorem:

Let T $(-\infty < t < \infty)$ be a measurable flow in a Lebesgue space (X, Σ, μ). Suppose there exists a sub σ - algebra $\mathcal{M}^o$ of Σ such that

1. $\mathcal{M}^{t_1} = T^{t_1} \mathcal{M}^o \subset \mathcal{M}^{t_2}$ for $t_1 < t_2$

Here $\subset$ signifies proper inclusion as contrasted with $\subseteq$.

2. $\bigvee_{-\infty}^{\infty} \mathcal{M}^t \overset{*}{=} \Sigma$ 3. $\bigwedge_{-\infty}^{\infty} \mathcal{M}^t \overset{*}{=} \emptyset, X$

Let $\mathcal{H}^T$ be the subspace of $L^2\ (X,\mu)$ composed of those functions which are measurable relative to $\mathcal{M}^T$. Let

$$\underline{\mathcal{H}} = \bigwedge_{-\infty}^{\infty} \mathcal{H}^T$$

and $\mathcal{H}_T = \mathcal{H}^T - \underline{\mathcal{H}}$

Then $U^t\, \mathcal{H}^T = \mathcal{H}^{t+T}$

$U^t\, \underline{\mathcal{H}} = \underline{\mathcal{H}}$

and U^t has homogeneous Lebesgue spectrum on $\mathcal{H} - \underline{\mathcal{H}}$ where $\mathcal{H} = L^2\ (X,\mu)$

That $\mathcal{H}^t$ is really a subspace is a consequence of the most basic properties of measurable functions.

Also

$$\mathcal{H}^{t_1} \subseteq \mathcal{H}^{t_2} \quad \text{if } t_1 \leq t_2 \tag{35.16}$$

Define E^T to be the projection on $\mathcal{H}^T$

Then (35.16) implies

$$E^{t_1} \leq E^{t_2} \text{ if } t_1 \leq t_2$$

Furthermore from the definition we get

$$U^t E^T U^{-t} = E^{t+T}$$

and from condition 2. $\lim_{t \to \infty} E^t = 1$

Also $\lim_{t \to -\infty} E^t = \underline{E}$ the projection operator onto $\underline{\mathcal{H}}$. So on $\mathcal{H} - \underline{\mathcal{H}}$ the E's are a spectral family satisfying the conditions of von Neumann's uniqueness theorem and hence U^t has homogeneous Lebesgue spectrum on $\mathcal{H} - \underline{\mathcal{H}}$.

<u>Corollary</u>:

For a K-flow U^t has homogeneous Lebesgue spectrum in the complement of the subspace of constant functions.

Proof:

3. says $\bigwedge_{-\infty}^{\infty} \mathcal{M}^t \doteq \{\emptyset, X\}$

which implies that $\underline{\mathcal{H}}$ consists of constant functions. To complete the proof of property A for K-systems we must show that the spectrum of U^t has denumerable multiplicity. We now proceed to outline the proof for this as given by Sinai.

<u>Outline</u>:

Sinai looks for sufficient conditions that a flow built under a function have a portion of its spectrum Lebesgue with infinite miltiplicity. Then he uses the Ambrose Kakutani Theorem to realize the K-flow as a flow built under a function and he shows the function can be so chosen that the sufficiency criterion is in fact satisfied. In this connection he makes use of the following two theorems which we quote without proof.

Suppose that for the flow T^t in a measure space $\{\bar{X}, \bar{\Sigma}, \bar{\mu}\}$ built under the function f by the measure-preserving transformation S of (X, Σ, μ) there exists a σ - subalgebra $\mathcal{A}$ of Σ such that

$$\mathcal{A}^1 = S\,\mathcal{A} \supset \mathcal{A} \qquad (35.17)$$

and that f is measurable with respect to $\mathcal{A}$. Then in $L^2(\bar{X}, \bar{\mu})$ there is a subspace invariant under U^t in which U^t has homogeneous Lebesgue spectrum of denumerable multiplicity.

The proof of this theorem proceeds in two stages:

1. Given $\mathcal{A}$ we construct an $\bar{\mathcal{A}}$ with property (35.17)
2. Then we use $\bar{\mathcal{A}}$ to construct a family of orthogonal invariant subspaces of $L^2(\bar{X}, \bar{\mu})$ on each of which U^t has simple Lebesgue spectrum.

<u>Theorem:</u>

Let T^t $(-\infty < t < \infty)$ be a measurable flow on a measure space (X, Σ, μ). Suppose there exists a σ-subalgebra of Σ such that

I $\quad \mathcal{A}^{t_1} = T^{t_1}\mathcal{A} < \mathcal{A}^{t_2} \qquad t_1 < t_2$

II $\quad \bigvee_{-\infty}^{\infty} \mathcal{A}^t \overset{*}{=} \Sigma$

Then according to a previous Lemma there exists a finite subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that

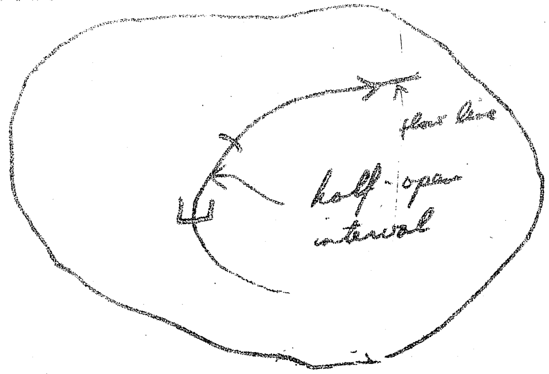
$$\bigvee_{\tau \le t_1} \mathcal{B}^\tau = \bigvee_{\tau \le t_1} T^\tau \mathcal{B} \subset \bigvee_{\tau \le t_2} \mathcal{B}^\tau \qquad t_1 < t_2$$

Let $\mathcal{B}^{-\infty} = \bigcap_{t=-\infty}^{0} \bigvee_{\tau \le t} \mathcal{B}^\tau$

Then $\hat{T}^t$, the quotient flow induced in $X/\mathcal{B}^{-\infty}$ is isomorphic to a flow built under a function which satisfies the hypotheses of preceding theorem.

<u>Outline of Proof</u>

We first introduce the notion of a <u>regular partition</u> of (X, Σ, μ) with respect to a flow.



Its sets consist of segments of orbits under the flow, closed at the earlier end and open at the later end, such that the function t(x) (which is the time taken to flow from the earlier end of the segment on which x lies to x) is a measurable function of x.

We must show first that such a partition exists.

The complete proofs can all be found in Ja. G. Sinai - Dynamical Systems with Countably - Multiple Lebesque Spectrum. I AM. MATH. SOC. TRANSL. 2 39 83-110.

LECTURE 36

Asymptotic Orbits, Transversal Fields, U Systems

Hadamard 1896 - examined the nature of geodesics on a surface of constant negative curvature.

Artin 1924 - gave the first example of "regional (topological) transitivity." I.e., one dense orbit.

Definition:

A flow has <u>topometric transitivity</u> if the union of all dense orbits has a complement of measure zero.

2. **A flow has <u>metric transitivity</u> (defined before in Lecture 8) if the only measur-** able subsets of the whole measure space X invariant under the flow are either of measure zero or differ from X by sets of measure zero. I.e., ergodic.

Hedlund 1934 - **proved metric transitivity for the above systems (constant negative** curvature)

Hopf 1939 - extended the proof to n dimensions with constant negative curvature

Sinai 1961 - proved that compact geodesic flows with variable curvatures are K-systems

Anosov 1963 - extended this to n dimensions and varying curvature. Proved that every U system is a K system.

Consider the unit disc $D = \{ z \,|\, |z| < 1, z = \chi^1 + i\chi^2 \}$ equipped with the metric

$$g_{11} = g_{22} = \frac{4}{(1-|z|^2)^2}$$

$$g_{12} = g_{21} = 0$$

Then

$$ds^2 = g_{\mu\nu}\, d\chi^\mu\, d\chi^\nu = \frac{4[(d\chi^1)^2 + (d\chi^2)^2]}{(1-|z|^2)^2} \tag{36.1}$$

Angles are ordinary and if we compute as usual we get that volume element is

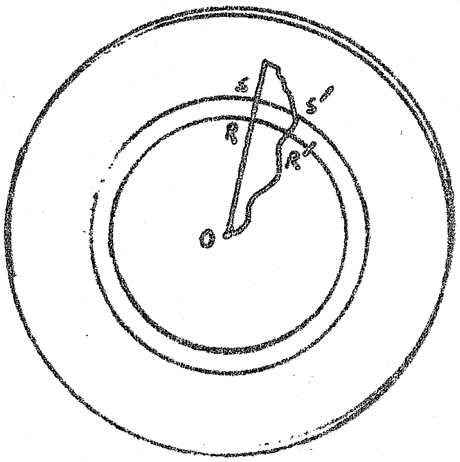
$$dv = \sqrt{\det g}\; d\chi^1\, d\chi^2 = \frac{2\, d\chi^1\, d\chi^2}{(1-|z|^2)^2} \tag{36.2}$$

This metric arises in a fairly natural fashion if one considers the Möbius transformation.*

$$\frac{z-\alpha}{1-\bar{\alpha}z} = \frac{w-\beta}{1-\bar{\beta}w} \tag{36.3}$$

which maps the unit disc in the z plane onto the unit disc in the w plane in a conformal and one-to-one manner with the point α going into the point β. It is easily verified that circles and straight lines go into circles and straight lines.

If we now let $\alpha \to z$, that is we let



$$|z-\alpha|^2 = (\chi^1-\alpha^1)^2 + (\chi^2-\alpha^2)^2 \to 0$$

then

$$|w-\beta|^2 = (w^1-\beta^1) + (w^2-\beta^2)^2 \to 0$$

and we get

$$\frac{\sqrt{(d\chi^1)^2 + (d\chi^2)^2}}{1-|z|^2} = \frac{\sqrt{(dw^1)^2 + (dw^2)^2}}{1-|w|^2}$$

so that this metric is preserved under the Möbius transformation.

To get the geodesics we first show that the geodesic between any point and the center is a straight line. For consider any other path and 2 concentric circles with radii differing by an infinitesimal amount. The denominator $1-z^2$ is essentially constant between these circles and so we get the ordinary Euclidean metric with $RS \leq R'S'$.

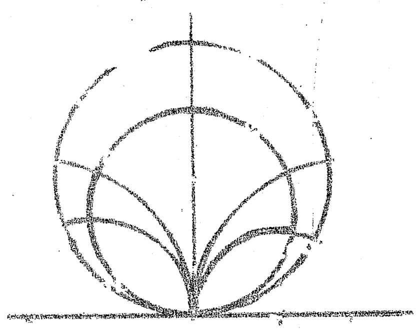
Now consider any 2 points P, Q. Then we map P into the origin using a

* The discussion in the following section was pointed out by L. Schulman.

Möbius transformation. The geodesic will then be a straight line which intersects the unit circle normally. After mapping back this can only become a circle or straight line passing through P and Q and intersecting the unit circle normally (since the mapping is conformal).

There is a particular family of great importance for the theory of geodesics, the horocycles.

A horocycle consists of circles tangent to the unit circle and lying inside it.



P

In this case if P is the point at which asymptotic orbits converge, the horocycle tangent at P intersects the orbits normally.

Consider the transformation $T \in GL(2,c)$ where $GL(2,c)$ is the group of nonsingular 2x2 matrices. Let T act on z by

$$z \to Tz = \frac{az+b}{cz+d} \qquad ad - bc \neq 0 \tag{36.4}$$

T^{-1} is again of this form.

Definition:

Consider any subgroup Γ which is

1) properly discontinuous--that is, $\{Tz \mid T \in \Gamma\}$ is not dense for any z.

2) Any $T \in \Gamma$ maps D onto itself.

Such a Γ is call a Fuchsian group.

We now introduce the following equivalence relation. Two points are equivalent if they are mapped into one another by Γ. (An elementary account of this is given in chapters I and II of L. R. Ford's Automorphic Functions.)

Definition:

A fundamental domain O for Γ is a connected subset of D, containing no two equivalent points, such that $\{TO \mid T\epsilon\Gamma\}$ covers D with the possible exception of a set of measure zero.

O turns out to be bounded by geodesics with some boundary points identified to get various Riemannian manifolds because the metric (36.1) is invariant under Γ. The fundamental domains fall into 2 classes.

I. No segments of ∂D (the unit circle) form a portion of the boundary of O.

II. Segments of ∂D form portions of the boundary of O.

We shall now draw some typical examples.

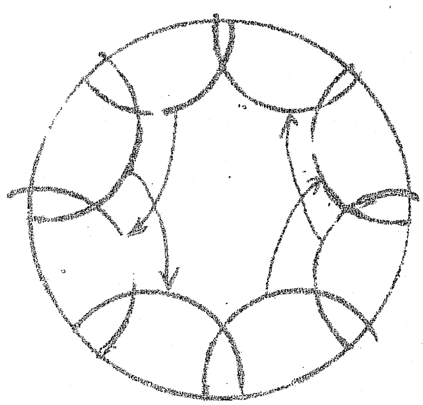


Fig. 1

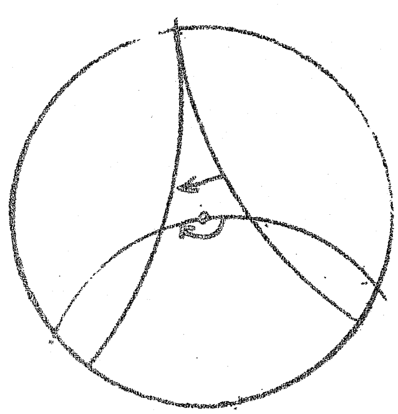


Fig. 2

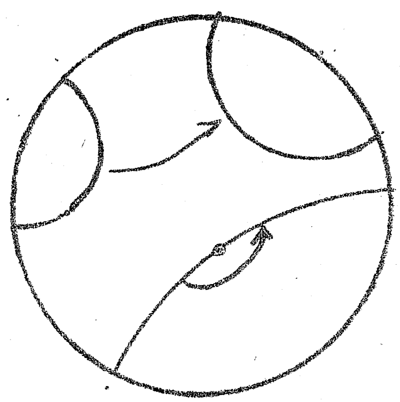
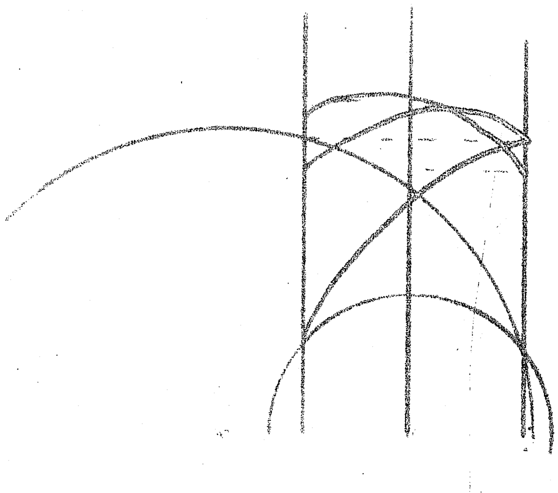


Fig. 3

Figures 1 and 2 belong to class I, figure 3 belongs to class II. Note that via this process we obtain only surfaces of constant negative curvature. As a matter of fact, in this manner one obtains all surfaces of constant negative curvature.

Figure 2 gives the example used by Artin. However, he worked in the upper half plane instead of on the unit disc and so he had:



modular $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$ad - bc = 1$

a, b, c, d integers.

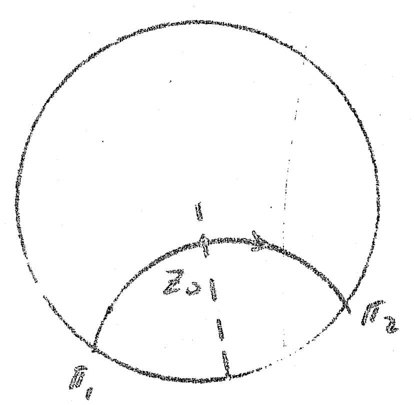
Artin's procedure:

Label a geodesic by its base points (ξ,η) $\xi \geq \eta$ Identify (ξ,η) with (η,ξ).

Artin asked whether this is physically realizable. He modified the model slightly and considered also $z \to -\bar{z}$, reflection in the imaginary axis. This then yields a flow that is equivalent to the flow of a free particle colliding elastically with the boundary of a triangle lying on the surface of revolution of the tractrix.

In the last lecture we described a geodesic flow on a disc and indicated how via Fuchsian groups this leads to other geodesic flows. We shall now give a more quantitative example.

We introduce the coordinates (π_1, π_2, s) to replace the coordinates $(z,\varphi) = (x_1, x_2, \varphi)$ as coordinates in phase space. This can be done since we always have the energy integral and therefore can pick units such that the constant velocity is 1. Then we need only specify the direction φ of the velocity at the point (x_1, x_2).



We now indicate the derivation of the relation between the 2 sets of coordinates as given in Hopf.

If

$$[z_1, z_2, z_3, z_4] = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \tag{37.1}$$

the cross ratio between the 4 points, then

$$s = \text{geodesic distance from } z_0 \text{ to } z = \log([\pi_1, \pi_2, z, z_0]) \tag{37.2}$$

Here z_0 is the midpoint of the oriented geodesic connecting π_1 and π_2. It is clear that $|\pi_1| = |\pi_2| = 1$.

Also

$$z_0 = \frac{\pi_1 + \pi_2}{2 + (\pi_1 - \pi_2)} \tag{37.3}$$

And

$$\varphi = \arg\left\{\frac{(z - \pi_1)(z - \pi_2)}{\pi_1 - \pi_2}\right\} \tag{37.4}$$

Then in the (π_1, π_2, s) coordinates (36.4) becomes

$$T^t(\pi_1, \pi_2, s) = (\pi_1, \pi_2, t+s) \qquad (37.5$$

The measure

$$d\mu = \frac{4\, d\chi_1\, d\chi_2\, d\varphi}{(1 - |z|^2)^2} = \frac{d\theta_1\, d\theta_2}{|\pi_1 - \pi_2|^2}\, ds \qquad (37.6)$$

where

$$\pi_1 = e^{i\theta_1} \quad \text{and} \quad \pi_2 = e^{i\theta_2} \quad \text{and} \quad ds = \frac{2\sqrt{(d\chi^1)^2 + (d\chi^2)^2}}{1 - |z|^2}$$

Note that the transformation from (z, φ) to (π_1, π_2, s) is single-valued.

Theorem:

A geodesic flow with Fuchsian group Γ having a finite basis and $\int d\mu < \infty$ for the fundamental domain is metrically transitive (ergodic) if and only if every set A on the (π_1, π_2)-torus with positive torus measure

$$\iint_A d\theta_1\, d\theta_2 > 0 \qquad (37.7)$$

and invariant under Γ is of measure equal to that of the torus.

Proof: (Hopf)

It is only necessary to prove ergodicity since the converse is clear. Now consider a measurable set B in the phase space that is invariant under the flow and of positive measure. To this set there corresponds a cylinder set

$$C = \left\{ (\pi_1, \pi_2, s) \,\middle|\, \pi_1 \text{ and } \pi_2 \text{ fixed} \right\} \text{ in the } (\pi_1, \pi_2, s) \text{ space.}$$

This set is measureable and invariant under the flow (37.5). The projection A of this set onto the (π_1, π_2)-torus has positive measure

$$\iint_A \frac{d\theta_1\, d\theta_2}{|\pi_1 - \pi_2|^2}$$

and hence of positive torus measure (37.7). According to the assumptions of the theorem, the complement of A on the torus has measure zero. Hence we get that the complement of B on the phase space is of measure zero.

Metric transitivity for the geodesic flow in the above case in which $\mathbf{F}$ has a finite basis and the total $\int d\mu < \infty$ for the fundamental domain was first established in special cases by Hedlund using Symbolic Dynamics (see G. Hedlund-- Topological Dynamics) and then in general by Hopf.

Theorem:

Under the assumptions of the preceding theorem, metric transitivity of the geodesic flow is equivalent to the following statement.

A bounded function in the product $D \times D$ of the unit disc with itself and harmonic in each variable

i.e.,

$$\Delta_z U(z,w) = 0 = \Delta_w U(z,w) \tag{37.8}$$

is a constant if it is invariant under $\mathbf{F}$.

I.e.,

$$U(Tz, Tw) = U(z,w) \qquad \forall T \in \mathbf{F} \tag{37.9}$$

Proof: (Hopf)

Again we need only prove ergodicity or show that the assumptions of this theorem lead to those of the previous theorem. To this effect let $U(\pi_1, \pi_2)$ be the characteristic function of the set A of the previous theorem. U is measureable on the torus and $\forall T \in \mathbf{F}$ we have

$$U(T\pi_1, T\pi_2) = U(T_1, T_2) \tag{37.10}$$

We have to show that $U = 0$ on a set of torus measure zero.

Now the Poisson integral

$$U(z,\gamma) = \frac{1}{2\pi} \int_{|\zeta|=1} U(\zeta,\gamma) \frac{1 - z\bar{z}}{|\zeta - z|^2} |d\zeta| \tag{37.11}$$

represents a harmonic function in z for almost all γ with $|\gamma| = 1$. This function is bounded above by 1 and is a measureable function of γ on $|\gamma| = 1$ for all z. Furthermore

$$U(z,w) = \frac{1}{2\pi} \int_{|\gamma|=1} U(z,\gamma) \frac{1 - w\bar{w}}{|\gamma - w|^2} |d\gamma| \tag{37.12}$$

is separately harmonic in z and w and bounded for $|z| < 1$ $|w| < 1$.

The invariance (37.10) then implies (37.9) and hence according to the assumptions of the theorem if $U = 0$ then the theorem is proved. Note that the function on the torus $U(\zeta,\gamma)$ is specified by the harmonic function $U(z,w)$ in the following sense

$$\lim_{\substack{r \to 1 \\ \rho \to 1}} \int_{|\zeta|=1} \int_{|\gamma|=1} \left\{ U(r\zeta, \rho\gamma) - U(\zeta,\gamma) \right\}^2 |d\zeta| \, |d\gamma| = 0 \tag{37.13}$$

In passing, note the work of W. Seidel, "On A Metric Property of Fuchsian Groups," Proc. Nat. Acad. of Sci. 21, 475 (1935), in which he gives an F with an infinite number of generators so that the flow is regionally transitive but not metrically transitive. This proves the insufficiency of the "quasi-ergodic" hypothesis for getting time averages equal to phase space averages.

Hopf later found a proof whose ideas were the inspiration of Sinai's and Anosov's proofs: E. Hopf, Sächsische Akad. der Wissenshaft (Leipzig), 91, 261-304 (1939).

The idea consists of finding a quantitative formulation of the notion that two geodesics are asymptotic to the same point.

Lemma (Hopf's Principle)

Let r and r' be two points of the phase space whose orbits are asymptotic at $+\infty$. Then there exists a real number a such that

$$\lim_{t \to +\infty} \rho(T^{t+a} r, T^t r') = 0 \tag{37.14}$$

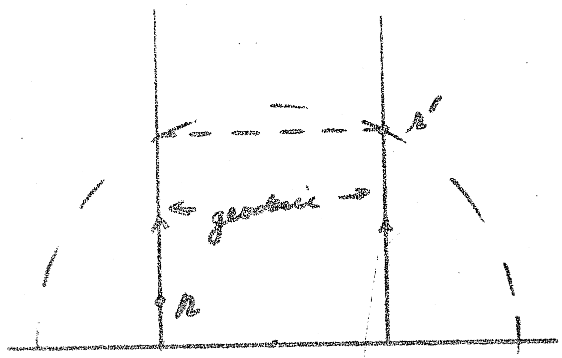
where ρ is the distance in phase space measured in the metric

$$\frac{4\,[(dx^1)^2 + (dx^2)^2]}{(1 - |z|^2)^2} + (d\varphi)^2 \quad \Bigg\} \text{ inf of this integrated along all curves between the two points.}$$

An analogous statement holds with a possibly different a for geodesics asymptotic at $-\infty$.

Proof (asymptotic at $+\infty$):

Map the disc into the upper half plane with the asymptotic point at ∞. Then



$$ds = \frac{[(d\chi^1)^2+(d\chi^2)^2]^{1/2}}{\chi^2}$$

The geodesic distance along a vertical line from ξ^2 to $\hat{\xi}^2$ is given by:

$$\int_{\xi^2}^{\hat{\xi}^2} \frac{d\chi^2}{\chi^2} = \log\left(\frac{\hat{\xi}^2}{\xi^2}\right)$$

Take $a = \log \frac{\chi'^2}{\chi^2}$ where χ^2 and χ'^2 are the 2 coordinates of r and r', respectively. Then $T^{t+a}r$ and $T^t r'$ will have the same 2 coordinates. So the distance $\rho\,(T^{t+a}r, Tr')$ is bounded by some constant $\frac{|\chi^1-\chi'^1|}{\chi^2}$ for sufficiently large t. And hence, in the limit as $t \to \infty$, $\rho \to 0$. The same statement holds on the manifold obtained by identifying points using Γ.

... the fundamental

domain has finite volume. Since we are always working with unit velocity the velocity coordinates in phase space reduce to a single angular variable and the volume of phase space is also finite.

<u>Theorem</u>: (Hopf)
(Sächs Akad Leipzig Berichte, <u>91</u>, 261-304 (1939)).

If the phase space X has finite volume, $\mu(X) < \infty$, then the geodesic flow is ergodic. A stronger statement holds: If f is a uniformly continuous and integrable function on phase space and if m is a point of the fundamental domain belonging to a point P of phase space, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(T^t P)dt = \frac{1}{\mu(X)}\int f(P)\,\mu(dP) \tag{38.1}$$

for almost all directions through m.

<u>Proof</u>:

To prove ergodicity it is sufficient to show that the following limits:

$$\begin{aligned} f_+^*(P) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f(P^t)\,dt \\ f_-^*(P) &= \lim_{T\to\infty} \frac{1}{T}\int_{-T}^0 f(P^t)\,dt \end{aligned} \tag{38.2}$$

whose existence is assured by Birkhoff (Lecture 15) are constant a.e.,

Let

$$S_\pm(\alpha) = \left\{ P \,\middle|\, f_\pm^*(P) \geq \alpha \right\} \tag{38.3}$$

we must show that for all α either

$$\mu(S_\pm) = 0 \qquad \text{or} \qquad \mu(S_\pm) = \mu(X).$$

(We drop the α for brevity).

If P and P' are asymptotic at $+\infty$ then

[illegible]

for some real number a. By an already familiar argument

$$\lim_{T \to} \frac{1}{T} \int_0^T f(P^{t+a})\, dt = f_+^*(P) . \tag{38.5}$$

Doing the same for negative times and combining the result with the uniform continuity of f we get

$$f_\pm^*(P) = f_\pm^*(P') \tag{38.6}$$

whenever P and P' are asymptotic at $\pm\infty$. Let $\hat{S}_\pm$ be the projection of $S_\pm$ onto the torus defined by the coordinates (π_1, π_2). If P and P' are asymptotic at $\pm\infty$ then from (38.6) it follows that

$$\begin{aligned} (\pi_1, \pi_2) \in \hat{S}_+ &\Leftrightarrow (\pi_1', \pi_2) \in \hat{S}_+ \\ (\pi_1, \pi_2) \in \hat{S}_- &\Leftrightarrow (\pi_1, \pi_2') \in \hat{S}_- \quad . \end{aligned} \tag{38.7}$$

The sets $\hat{S}_\pm$ are therefore cylinders

$$\begin{aligned} \hat{S}_+ &= \Phi_1 \times A_+ \\ \hat{S}_- &= A_- \times \Phi_2 \end{aligned} \tag{38.8}$$

where the Φ_i are the unit circles determined by the coordinates π_i. Now by Birkhoff $f_+^* = f_-^*$ a. e. Therefore using the lemma that an invariant set R has zero measure if and only if its projection $\hat{R}$ has zero toroidal measure we conclude that $\hat{S}_+$, $\hat{S}_-$ and their intersection $A_- \times A_+$ differ by sets of measure zero. In particular if by μ and θ we denote the toroidal and respectively the circular measures we have

$$\begin{aligned} \hat{\mu}(A_-^c \times A_+) &= \theta(A_-^c)\, \theta(A_+) = 0 \\ \hat{\mu}(A_- \times A_+^c) &= \theta(A_-)\, \theta(A_+^c) = 0 \quad . \end{aligned} \tag{38.9}$$

Therefore in each product at least one of the circular measures must vanish, and we deduce

$$\theta(A_-^c) = 0 \Rightarrow \theta(A_-) = 2\pi \Rightarrow \theta(A_+^c) = 0 \Rightarrow \theta(A_+) = 2\pi \Rightarrow \hat{\mu}(\hat{S}_+) = \hat{\mu}(\hat{S}_-) = (2\pi)^2 \tag{38.10}$$

$$\mu(A_+) = 0 \Rightarrow \mu(A_+^c) = [illegible] \Rightarrow \mu(A_-) = 0 \Rightarrow \mu(S_+) = \mu(S_-) = 0 . \qquad (38.11)$$

Invoking the lemma once more we conclude that $\mu(S_+)$ is either zero or $\mu(X)$ thereby proving ergodicity.

To see the last statement of the theorem notice that since (38.1) holds for almost all points of phase space and the phase volume is $4\, d\chi_1\, d\chi_2\, d\varphi/(1-|z|^2)$, it must be true for almost all points (χ_1, χ_2) (relative to the measure $d\chi_1\, d\chi_2/(1-|z|^2)$) that the limit exists for almost all φ (relative to the measure $d\varphi$). Now look at the situation in the variables (π_1, π_2, s). We know that if (38.1) holds for one such point it also holds for all with the same π_2 and varying π_1 and s. Now if (38.1) holds for a point $(\chi_1, \chi_2, \varphi)$ with χ_1, χ_2 fixed for almost all φ, it holds for all (π_1, π_2, s) with the exception of a set of points whose π_2 coordinates form a set of measure zero. This implies that for any χ_1, χ_2, (38.1) holds for almost all φ. QED.

For fundamental domains of infinite volume there is a result due to Hadamard: if for all points there is a set of positive measure of directions about the point such that the geodesics in these directions go to ∞ (i.e., eventually leave every bounded set containing the given point) then the geodesics go to ∞ for almost all directions, furthermore if the above situation holds about any point it holds about all.

Sinai (Sinai, DAN, 131, 752-755 (1960, translated in SOVMAT. DOK. 1, 335ff (1960)) proved a stronger result: geodesic flows on two-dimensional manifolds of constant negative curvature and finite volume are K-systems.

Extension to Higher Dimensions:

The approach presented in the last three lectures can be extended to the higher dimensional case; here one introduces the open unit ball in $\mathbb{R}^n$:

with the metric

$$ds^2 = \frac{4}{(1 - |x|^2)^2} |dx|^2 \quad . \tag{38.13}$$

The geodesics are now Euclidian circles which intersect the unit sphere $\{|x| = 1\}$ normally (in the Euclidian sense). Practically everything is generalizable to this case including a group of isometric transformations (Kugelverwandschaften) under which identifications can be made. It is however not known whether all higher dimensional manifolds of constant negative curvature can be obtained in this fashion. In the paper mentioned in the previous paragraph Sinai proved the following:

Theorem:

If T^t is the geodesic flow of speed w on an n-dimensional compact Riemannian manifold of negative constant sectional curvature $-k$ and volume V then

$$h(T^1) = \frac{w\sqrt{k}}{\sqrt[n]{V}\,\omega_{n-1}} \log_2 e \tag{38.14}$$

where ω_{n-1} is the surface volume of the unit $n-1$ sphere.

Extension to Variable Curvature:

In 1939 Hopf extended his results to the case in which the curvature of a two-dimensional manifold satisfies the conditions

$$-a < k < -b < 0 \tag{38.15}$$

$$\left|\frac{dk}{ds}\right| \text{ bounded along geodesics} \quad .$$

The qualitative features of the geodesic flow are the same as before but one has to make extensive use of differential geometry. (See A. Grant, Duke Math. J., 5, 202-229 (1939)) Hopf proved ergodicity and Sinai (SOVMAT. DOK. 2, 106-109 (1961))

[illegible]

and liberated the results from the necessity of negative curvature. Instead of condition (38.15) one requires that the curvature k be bounded and that every solution of the equation

$$\frac{d^2 n}{ds^2} + k\, n = 0 \tag{38.16}$$

along any geodesic should satisfy any of a number of exponential growth conditions such as:

$$\frac{n'(S)}{n(S)} \geq c \tag{38.17}$$

or

$$\frac{n(S')}{n(S)} \geq c e^{d(s'-s)}, \quad s' > s \tag{38.18}$$

Equation (38.16) is the two dimensional case of the well known <u>equation of geodesic deviation</u> the solutions to which describe to first order how the separation between neighboring geodesics behaves. Conditions (38.17) and (38.18) therefore insist that this behavior be exponential. In this case Hopf was able to show that the flow is mixing and ergodic.

D. V. Anosov (SOVMAT. DOK. 4, 1153-1156 (1963)) generalized the situation and introduced the concept of the U-system.

<u>Definitions:</u>

Let $\mathcal{M}$ be an n-dimensional differentiable manifold, $P \in \mathcal{M}$, and $x^1, \cdots, x^n$ a system of local coordinates in a neighborhood of P. By a tangent vector X_p at P we shall mean any "directional derivative" of the form

$$X_p f = \sum_{i=1}^{n} \xi^i(p) \frac{\partial f}{\partial x^i}(x(p)) \tag{38.19}$$

[illegible] called the components of X_p relative to $(x^1, \cdots, x^n)$ and the set of tangent vectors $\partial/\partial x^1|_p, \cdots, \partial/\partial x^n|_p$ span an n-dimensional linear space $T(p)$ called the tangent space at p.

The set $T(\mathcal{M}) = \bigcup_{p \in \mathcal{M}} T(p)$ can be given a 2n-dimensional differentiable structure by insisting that whenever $(x^1, \cdots, x^n)$ are local coordinates in a neighborhood $U \subset \mathcal{M}$ that $(x^1, \cdots, x^n, \xi^1, \cdots, \xi^n)$ be local coordinates for the set $\bigcup_{p \in U} T(p)$. The manifold $T(\mathcal{M})$ is called the tangent bundle of $\mathcal{M}$.

A differentiable vector field on $\mathcal{M}$ is a differentiable map $X: \mathcal{M} \to T(\mathcal{M})$ such that $\mathcal{M} \ni p \to X_p \in T(p)$.

A differentiable vector field X is said to be <u>complete</u> if there is a differentiable flow $F^t: \mathcal{M} \to \mathcal{M}$ such that

$$\lim_{t \to 0} \frac{1}{t} [f(F^t p) - f(p)] = X_p f \tag{38.20}$$

for all p and all differentiable function f. A vector field on a compact manifold is always complete.

If $F: \mathcal{M} \to \mathcal{M}'$ is a differentiable map of one manifold into another, there is a natural induced map $\tilde{F}: T(\mathcal{M}) \to T(\mathcal{M}')$. $\tilde{F}$ takes $T(p)$ into $T(F(p))$ and if $(x^1, \cdots, x^n)$ are local coordinates at $p \in \mathcal{M}$ and $(y^1, \cdots, y^m)$ are local coordinates at $F(p) \in \mathcal{M}'$ then the action of $\tilde{F}$ can be defined by:

$$\tilde{F} \left(\sum_{i=1}^{n} \xi^i(p) \left. \frac{\partial}{\partial x^i} \right|_p \right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \xi^i(p) \frac{\partial y^j}{\partial x^i}(p) \right) \left. \frac{\partial}{\partial y^j} \right|_{F(p)} \tag{38.21}$$

thus

$$\sum_{i=1}^{n} \xi^i(p) \frac{\partial y^j}{\partial x^i}(p) \quad \text{are the components of } \tilde{F} X_p \text{ at } F(p) .$$

We are now in position to define U-systems.

[illegible] a compact Riemannian manifold with a differentiable vector field.
Let F^t be the associated differentiable flow and $\tilde{F}^t$ the flow induced in the tangent bundle. The vector field is called a U-system if the following two conditions are satisfied.

U-1) The vector field has no critical points (i.e., does not vanish anywhere),

U-2) Each tangent space $T(p)$ has a vector space decomposition as a sum

$$T(p) = T_1(p) + T_2(p) + T_3(p) \tag{38.22}$$

where $T_3(p)$ is one dimensional and in the direction of the flow and if $X \in T_1(p)$, $Y \in T_2(p)$ then

$$\begin{aligned} \|\tilde{F}^t X\| &\leq a \| X \| e^{-ct}, \quad t \geq 0 \\ \|\tilde{F}^t X\| &\geq b \| X \| e^{-ct}, \quad t \leq 0 \end{aligned} \tag{38.23}$$

and

$$\begin{aligned} \|\tilde{F}^t Y\| &\leq a \| Y \| e^{ct}, \quad t \leq 0 \\ \|\tilde{F}^t Y\| &\geq b \| Y \| e^{ct}, \quad t \geq 0 \end{aligned} \tag{38.24}$$

where the norms are taken with respect to the Riemannian metric and a, b, c are positive constants.

Intuitively if for X and Y we take an "infinitesimal displacement vector" to neighboring orbits then neighboring orbits lying in a direction taken from T_1 approach the given one asymptotically for $t \to \infty$ and those lying in a direction taken from T_2 approach the given one asymptotically for $t \to -\infty$. The similarity of condition U-2) to condition (38.18) should also be noted.

Anosov proved an important theorem about U-systems: every U-system is a K-system. The proof of this result is scheduled to appear in the 1966 Proceedings of the Steklov Institute.

A measurable essentially bounded real function f defined on a Lebesgue space X with measure μ and on which there acts an ergodic measurable flow T^t, $-\infty < t < \infty$ is said to obey the central limit theorem if $\forall\alpha$, $-\infty < \alpha < \infty$

$$\lim_{t\to\infty} \mu\left\{x \,\Big|\, \frac{\int_0^t f(T^\tau x)\,d\tau - t\bar{f}}{\sqrt{R_t(f)}} \le \alpha\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-u^2/2}\,du , \quad (39.1)$$

where

$$\bar{f} = \int f(x)\,\mu(dx) \quad (39.2)$$

and

$$R_t(f) = \int_X [\int_0^t f(T^\tau x)\,d\tau]^2\,\mu(dx) \quad (39.3)$$

The physical significance of the statement is that in the limit of long time intervals the fluctuations in the time average of f about the phase average is normally distributed with a variance squared equal to the phase average of the square of the time average.

Sinai (DAN 133, 1303-1306 (1960) translated in SOVMAT. DOK. 1, 983-987 (1960)) gives sufficient conditions for a function f to obey the central limit theorem when the flow is the geodesic flow on Riemannian manifolds of constant negative curvature. These conditions constitute the first rigorous results in fluctuation theory.

In his second Izvestia paper (Izv. 1966, p. 15ff) Sinai uses the concept of transverse flows to study dynamical systems. An example of a transverse flow is the horocycle flow on the unit disc with the metric $ds^2 = 4|dz|^2/(1-|z|^2)^2$. The orbits of the flow are the family of horocycles all tangent to the same point on the unit circle. The velocity along each orbit can be taken to be constant (in the underlying metric) and such as to map the orthogonal family of geodesics (all mutually asymptotic at + or − ∞) into itself. Given a differentiable flow on a manifold a transverse flow is therefore one which is generated by a vector field nowhere tangent an orbit of the given flow.

The conditions for the existence of certain increasing families of subalgebras of the algebra of measurable sets is related to the existence of transverse flows with dilating or contracting properties. For a K-system one needs properly related dilating and contracting transverse flows.

Sinai applies these methods to C^2 vector fields on Riemannian manifolds with or without a piecewise smooth boundary. In the case where there is a boundary one must map the regions of the boundary where the vector field is pointing "out" onto those where it is pointing "in" in order to have an uninterrupted flow.

In conclusion of these lectures we return to some computations in simple statistical mechanical systems.

Consider again the mechanics of N particles in a box as studied in Lecture 6. This time we shall use a finite box, i.e., $V(\vec{x}_1, \cdots, \vec{x}_n) = \Sigma_{i=1}^{N} V(\vec{x}_i)$ where $V(\vec{x})$ is a C^∞ function bounded below and approaching $+\infty$ whenever $\vec{x}$ approaches the boundary of some regular open region (the interior of the box) in $\mathbb{R}^3$.

The Hamiltonian is as before

$$H(x,p) = \sum_{j=1}^{N} \frac{\vec{p}_j^{\,2}}{2m_j} + \frac{1}{2}\sum_{j\neq k} V_{jk}(\vec{x}_j - \vec{x}_k) + \sum_{j=1}^{N} V(\vec{x}_j) \qquad (39.4)$$

we choose a high enough energy so that the energy surface $H(x,p) = E$ exists and is non-singular.

<u>Computation of Time Averages</u>:

Hamilton's equations give:

$$\begin{aligned} \dot{\vec{p}}_j(t) &= \vec{F}_j(t) = -\nabla_j V(\vec{x}_j(t)) - \nabla_j \sum_k V_{jk}(\vec{x}_j(t) - \vec{x}_k(t)) \\ \dot{\vec{x}}_j(t) &= \frac{1}{m_j}(t) \end{aligned} \qquad (39.5)$$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{\vec{p}_j^{\,2}(t)}{2m_j} \, dt \quad . \tag{39.6}$$

We assume the flow is ergodic, therefore (39.6) is equal to

$$\frac{1}{V_E} \int d\vec{x}_1 \cdots d\vec{x}_N \, d\vec{p}_1 \cdots d\vec{p}_N \; \delta(E - H(x,p)) \; \frac{\vec{p}_j^{\,2}}{2m_j} \tag{39.7}$$

where

$$V_E = \int d\vec{x}_1 \cdots d\vec{x}_N \, d\vec{p}_1 \cdots d\vec{p}_N \; \delta(E - H(x,p)) \quad . \tag{39.8}$$

If we further assume that the Hamiltonian is symmetric under the interchange of particles then (39.7) is independent of j and therefore so is (39.6) and we arrive at the <u>equipartition theorem</u>: the time average (and therefore the space average) of the kinetic energy is the same for each of a set of indistinguishable particles. For more general equipartition laws see Huang, Statistical Mechanics, Wiley and Sons, p. 149ff.

We define

$$\rho(\vec{p}_1) = \frac{1}{V_E} \int d\vec{x}_1 \cdots d\vec{x}_N \, d\vec{p}_2 \cdots d\vec{p}_N \; \delta(E - H(x,p)) \tag{39.9}$$

then

$$\rho(\vec{p}_1) \geq 0 \quad , \quad \int \rho(\vec{p}_1) \, d\vec{p}_1 = 1 \; . \tag{39.10}$$

We can use (39.7) to define kT by

$$\int \rho(\vec{p}) \, \frac{\vec{p}^{\,2}}{2m} \, d\vec{p} = \frac{3}{2} \, kT \; . \tag{39.11}$$

Invoking Birkhoff and ergodicity once more we have

$$\lim_{T \to +\infty} \int_{-T}^{T} \nabla V(\vec{x}_j(t)) \, dt = \int g(\vec{x}_1) \, \nabla V(\vec{x}_1) \, d\vec{x}_1 \tag{39.12}$$

[illegible]

$$g(\vec{x}_1) \geq 0 \qquad \int g(\vec{x}_1)\, d\vec{x}_1 = 1 \quad . \tag{39.14}$$

Consider now the time average of the force. By the boundedness of $\vec{p}_j$ on the energy surface we have

$$\frac{1}{2T}\int_{-T}^{T} \vec{F}_j(t)\, dt = \frac{1}{2T}\int_{-T}^{T} \dot{\vec{p}}_j(t)\, dt = \frac{\vec{p}_j(T) - \vec{p}_j(-T)}{2T} \to 0 \tag{39.15}$$

as $T \to \infty$.

Now

$$\lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T} \sum_{k \neq j} \nabla_j V_{jk}(\vec{x}_j(t) - x_k(t))\, dt =$$

$$= \frac{1}{V_E}\int dx\, dp\, \delta(E - H(x,p)) \sum_{k \neq j} \nabla_j V_{jk}(\vec{x}_j - \vec{x}_k) = \tag{39.16}$$

$$= \sum_{k \neq j} \int d\vec{x}_j\, d\vec{x}_k\, \nabla_j V_{jk}(\vec{x}_j - \vec{x}_k)\, g(\vec{x}_j, \vec{x}_k)$$

where

$$g(\vec{x}_j, \vec{x}_k) = \int d\vec{x}_1 \ldots \widehat{d\vec{x}_j} \ldots \widehat{d\vec{x}_k} \ldots d\vec{x}_N\, d\vec{p}_1 \ldots d\vec{p}_N\, \delta(E - H(x,p)) \tag{39.17}$$

Since H was assumed symmetric under interchange $g(\vec{x}_j, \vec{x}_k)$ is even under interchange of $\vec{x}_j$ and $\vec{x}_k$. The right hand side of (39.16) therefore vanishes since it is an integral of a product of an even and an odd function. By (39.5) we have

$$\int g(\vec{x})\, \nabla V(\vec{x})\, d\vec{x} = 0 \tag{39.18}$$

If we assume that $V(\vec{x})$ contains a gravitational potential, i.e., [illegible] have by (39.18)

$$m\vec{g} = \int g(\vec{x}) \, \nabla V'(\vec{x}) \, d\vec{x} \tag{39.20}$$

and so the gravitational forces are just balanced by forces from the walls. This balancing effect can take place only near the "bottom" of the box and the particles will therefore tend to spend more time near the bottom, i.e., we have a "settling" phenomenon.

We can now examine the virial equation of state. We have:

$$\frac{1}{2T}\int_{-T}^{T} \sum_{i=1}^{N} \vec{x}_i(t) \cdot \dot{\vec{p}}_i(t) \, dt = \frac{1}{2T}\sum_{i=1}^{N} \vec{x}_i \cdot \vec{p}_i \Big|_{-T}^{T} - \frac{1}{2T}\int_{-T}^{T} \sum_{i=1}^{N} \dot{\vec{x}}_i(t) \cdot \vec{p}_i(t) \, dt \quad . \tag{39.21}$$

By the boundedness of $\vec{x}_i$ and $\vec{p}_i$ on the energy surface the first term on the right hand side vanishes in the limit $t \to \infty$. The second term by (39.5) and Birkhoff approaches minus twice the time average of the kinetic energy, i.e.,

$$\lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T} \sum_{i=1}^{N} \vec{x}_i(t) \cdot \dot{\vec{p}}_i(t) \, dt = -2 \langle K \, E \rangle_t \tag{39.22}$$

using (39.5) the left hand side is also equal to

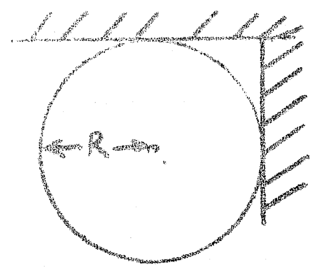
$$\lim_{T \to} \left[-\frac{1}{2T}\int_{-T}^{T} \sum_{i=1}^{N} \vec{x}_i \cdot \nabla_i V(\vec{x}_i) \, dt - \frac{1}{2T}\int_{-T}^{T} \sum_{i \neq j} \vec{x}_i \cdot \nabla_i V_{ij}(\vec{x}_i - \vec{x}_j) \, dt \right] \tag{39.23}$$

which by ergodicity is computed to be

$$-N \int g(\vec{x}) \, \vec{x} \cdot \nabla V(\vec{x}) \, d\vec{x} - \frac{N(N-1)}{2} \int g(\vec{x}_1, \vec{x}_2) \, (\vec{x}_1 - \vec{x}_2) \cdot \nabla V(\vec{x}_1 - \vec{x}_2) d\vec{x}_1 \, d\vec{x}_2 \quad .$$

Even though the first term is not manifestly invariant under change of origin [illegible] requires a [illegible] argument and [illegible]

[illegible] be [illegible]. One cannot in general assume that the pressure is uniform over the boundary of the box since for example if the particles have a finite size, say a sphere of radius R then as in the picture below there can be regions on the walls of the box inaccessible to collisions.



Transport Coefficients:

We confine ourselves to a few remarks about viscosity; similar remarks apply to the other transport phenomena.

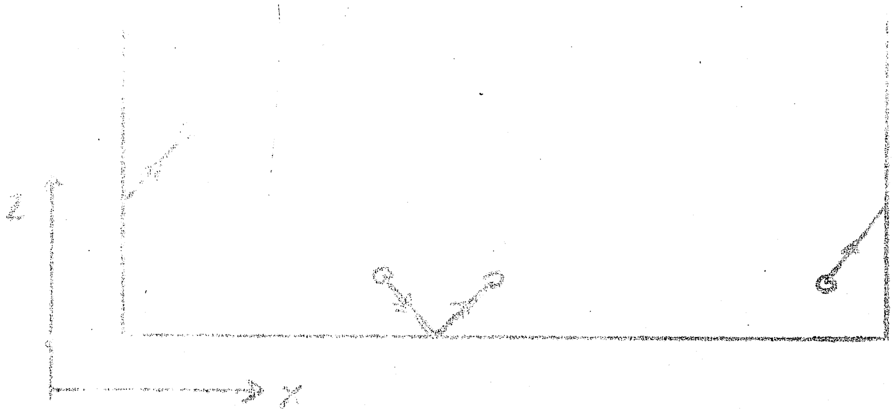
Practically one measures transport coefficients in a stationary (but not equilibrium) state. For example, viscosity is measured by observing the force on one of a pair of concentric cylindrical shells when the other one is rotated at constant angular velocity with a fluid filling the space in between. This number is then used in situations which may be quite different physically such as in the Navier-Stokes equation. With Sinai's approach one can get exact formulas for viscosity by setting up models that correspond to the physical situation at hand. For example, one can consider the motion of hard discs in a square with the following collision conditions:

(I) Elastic collisions between discs.

(II) Periodic boundary conditions between the right and left hand edge of the square.

(III) Elastic collision with the bottom edge.

(IV) Collision with the top edge resulting in an addition of a constant velocity parallel to the top edge.



In this model we have a wall pumping momentum into a gas and to calculate viscosity one must get a relation between $\partial \langle \vec{v}_x \rangle_t / \partial z$ and the momentum transport. With Sinai's method these quantities are in principle calculable, however a note of warning must be interjected: the approach describes everything in hydrodynamics; including turbulence. One must therefore devise methods for separating out the desired quantities.