PORTFOLIO DOMINANCE AND OPTIMALITY IN INFINITE SECURITIES MARKETS

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Abstract

The most natural way of ordering portfolios is by comparing their payoffs. If a portfolio has a payoff higher than the payoff of another portfolio, then it is greater than the other portfolio. This order is called the portfolio dominance order. An important property that a portfolio dominance order may have is the lattice property. It requires that the supremum and the infimum of any two portfolios are well-defined. The lattice property implies that such portfolio investment strategies as portfolio insurance or hedging an option’s payoff are well-defined.

The lattice property of the portfolio dominance order plays an important role in the optimality and equilibrium analysis of markets with infinitely many securities with simple (i.e., arbitrary finite) portfolio holdings. If the portfolio dominance order is a lattice order and has a Yudin basis, then optimal portfolio allocations and equilibria in securities markets do exist. A Yudin basis constitutes a system of mutual funds of securities such that trading mutual funds provides the same spanning opportunities, and that the restriction of no short sales of mutual funds is equivalent to the restriction of non-negative wealth.

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1 Introduction

In competitive securities markets investors choose portfolios of securities so as to maximize their preferences given prices of the securities. Since investors' objects of choice are portfolios, the space of portfolios in a model of securities markets plays a role similar to the role of a commodity space in the standard model of competitive commodity markets. Equilibrium theory of securities markets exploits this similarity and relies on the methods of the Arrow–Debreu equilibrium theory. An equilibrium model of securities markets with finitely many securities is due to Hart [18] (see also Hammond [16], Nielsen [26], Page [27], and Werner [32] among others) and includes the classical Capital Asset Pricing Model as a special case. An extensive discussion of the relationship between the Hart’s model and the Arrow–Debreu model can be found in Milne [23] (see also Milne [24]).

An important difference between a portfolio space and a commodity space is in the order structure of the spaces. While the usual component-wise order is the most relevant for the commodity space, it is of secondary importance for the portfolio space. Far more important is an order induced by the payoff of a portfolio.

Each portfolio is associated with a payoff an investor expects to receive when holding the portfolio. Typically, the payoff is a random consumption stream—an element of a payoff space. The mapping that associates a payoff with a portfolio is the payoff operator. The usual order of the payoff space induces via the payoff operator an order on the portfolio space. According to that order one portfolio is greater than another portfolio, if its payoff is higher in every state of the world than the payoff of the other portfolio. We call that order the portfolio dominance order. In general, the portfolio

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dominance order differs from the component-wise order of the portfolio space. Typically there are portfolios with negative holdings of some securities that have positive payoff in every state.

The cone of positive portfolios (i.e., the positive orthant) under the portfolio dominance order is the set of portfolios with positive payoffs—the most natural set of investor's feasible portfolios. Monotonicity of investors' preferences with respect to the portfolio dominance order is an expression of the desirability of higher consumption.

The portfolio dominance order is a lattice order if for any two portfolios there is a well-defined supremum portfolio and an infimum portfolio. The supremum of two portfolios is the least upper bound with respect to the portfolio dominance order, i.e., a portfolio with the lowest payoff that is higher than the payoffs of both portfolios. The infimum is the greatest lower bound with respect to the portfolio dominance order, i.e., a portfolio with the highest payoff that is lower than the payoffs of both portfolios.

There is an interesting connection between the lattice operations and some important portfolio investment strategies. The supremum of a reference portfolio and a portfolio consisting of \( k \) shares of a riskless security represents a portfolio insurance. It is a portfolio with the least payoff larger than the payoff of the reference portfolio and the floor \( k \). If full portfolio insurance is possible so that there is a portfolio with payoff equal to the payoff of the reference portfolio whenever it is above the floor \( k \) and equal to \( k \) otherwise, then that portfolio equals the supremum of the reference portfolio and \( k \) shares of the riskless security relative to the portfolio dominance order.

For a portfolio obtained from a reference portfolio and negative (short position) \( k \) shares of the riskless security, its positive part, i.e., the supremum of it and zero, is a portfolio with the least payoff higher than the payoff of the call option on the reference portfolio with striking price \( k \). If such call option is traded in the markets, then it is the positive part of the reference portfolio less \( k \) shares of the riskless security.

The property of the portfolio dominance order being a lattice order is therefore of fundamental nature. However, not every set of securities in the Hart's model of securities markets generates a portfolio dominance order which is a lattice order. The portfolio dominance is a lattice order if and only if the asset span (i.e., the subspace of payoffs of all portfolios) is a lattice subspace of the payoff space. A characterization of payoffs that span a lattice-subspace can be found in Abramovich, Aliprantis and Polykrakis [1], and Polykrakis [29]. In a companion paper, Aliprantis, Brown and Werner [7], we provide a detailed analysis of finite securities markets with the asset span being a lattice-subspace. The focus of that paper is on portfolio hedging strategies.

In this paper, we consider the case when there are infinitely many securities available for trade. The securities markets model with infinitely many securities is a framework for general equilibrium analysis of such asset pricing models as the Arbitrage Pricing Theory of Ross [30]; see Brown and Werner [10]. We assume that an investor can choose a portfolio consisting of an arbitrary but finite subset of securities. Such portfolios might
be called simple portfolios in analogy with the simple trading strategies in the continuous time securities market model of Harrison and Kreps [17]. The asset span with the simple portfolios is the linear span of the securities payoffs in the payoff space. As in the case of finite markets, the portfolio dominance order is a lattice order if and only if the asset span is a lattice-subspace of the payoff space.

When an infinite number of securities is available for trade, it is possible that optimal (simple) portfolio allocations do not exist. In Section 5 we present an example of securities markets where there are infinitely many Arrow securities and a riskless bond available for trade, and there are no optimal portfolio allocations for two risk-averse expected utility maximizing investors.

A crucial property of the portfolio dominance order turns out to be that the positive cone of the portfolio space under the portfolio dominance order has a Yudin basis, or equivalently, that the positive cone of the asset span has a Yudin basis. Yudin basis of the positive cone of the asset span is a set of positive payoffs such that each payoff in the asset span is a linear combination of the payoffs of the basis and that a payoff is positive if and only if it is a positive linear combination of the payoffs of the basis. For example, the Arrow securities form a Yudin basis in the positive cone of their asset span.

If the asset span has a Yudin basis, optimal portfolio allocations do exist. Furthermore, there exist equilibria in securities markets under the standard continuity and convexity assumptions. Our result establishing the existence of optimal portfolio allocations is a significant contribution to the recent literature on equilibrium models of infinite securities markets. In Brown and Werner [10] and Dana, Le Van and Magnien [14] the existence of optimal portfolio allocations (more precisely, the closedness of the utility set) is assumed, not derived from primitive assumptions on agent’s preferences or on the securities payoffs. When markets are implicitly incomplete, as is the case of their model and the model of this paper, the existence of optimal portfolio allocations has only been verified in special cases. Chichilnisky and Heal [12], Cheng [11], and Dana, and Le Van [15] assumed that the payoff space is a Sobolev space and restricted the class of agents’ utility functions. Connor [13] and Werner [33] for a class of securities payoffs having a factor structure, show that optimal portfolio allocations lie in a finite dimensional subspace of the portfolio space.

The property of the asset span having a Yudin basis has a simple interpretation. Each payoff of the Yudin basis is a payoff of some simple portfolio. These portfolios can be thought of as mutual funds. Trading mutual funds provides investors with the same spanning opportunities as trading securities. Since investors are only interested in positive payoffs, they have no need to sell short mutual funds. In other words, trading securities is equivalent to trading mutual funds under the no short sales restriction.

The paper is organized as follows: In Section 2 we present basic facts about the mathematical notions of a Yudin basis and a lattice-subspace. In Section 3 we explore the portfolio dominance order of the space of simple portfolios and its connections with portfolio insurance and options. Optimal portfolio allocations and equilibria in securities
markets are studied in Sections 4, 5 and 6. In Section 7 we show a fundamental invariance of our results to a change of security payoffs as long as the asset span remains the same. Using that invariance results we derive the mutual funds interpretation of the Yudin basis in Section 8. Section 9 presents some interesting results concerning a duality between portfolio allocations and consumption allocations in our model of securities markets.

2 Yudin Bases and Lattice-Subspaces

As mentioned in the introduction, this work is based heavily on the mathematical notions of a Yudin basis and a lattice-subspace. We shall discuss here briefly the basic properties of these concepts. For details and proofs we refer the reader to [6]. We follow the notation and terminology of the monographs [4, 8, 9, 20].

An order relation $\geq$ on a vector space $X$ is said to be a **linear order** if, in addition to being reflexive, antisymmetric and transitive, it is also compatible with the algebraic structure of $X$ in the sense that $x \geq y$ implies:

a. $x + z \geq y + z$ for each $z$, and
b. $\alpha x \geq \alpha y$ for all $\alpha \geq 0$.

A vector space equipped with a linear order is called a **partially ordered vector space** or simply an **ordered vector space**. In a partially ordered vector space $(X, \geq)$ any vector satisfying $x \geq 0$ is known as a **positive vector** and the collection of all positive vectors $X^+ = \{x \in X : x \geq 0\}$ is referred to as the **positive cone** of $X$. A linear operator $T : X \to Y$ between two partially ordered vector spaces is **positive** if $Tx \geq 0$ for each $x \geq 0$ (i.e., if $T(X^+) \subseteq Y^+$).

A subset $C$ of a vector space $X$ is said to be a **cone** if:

1. $C + C \subseteq C$,
2. $\lambda C \subseteq C$ for each $\lambda \geq 0$, and
3. $C \cap (-C) = \{0\}$.

Notice that (1) and (2) guarantee that every cone is a convex set. An arbitrary cone $C$ of a vector space $X$ defines a linear order on $X$ by letting $x \geq y$ if $x - y \in C$, in which case $X^+ = C$. On the other hand, if $(X, \geq)$ is an ordered vector space, then $X^+$ is a cone in the above sense. These show that the linear order relations and cones correspond in one-to-one fashion.

A partially ordered vector space $X$ is said to be a **vector lattice** or a **Riesz space** if it is also a lattice. That is, a partially ordered vector space $X$ is a vector lattice if for
every pair of vectors \( x, y \in X \) their supremum (least upper bound) and infimum (greatest lower bound) exist in \( X \). Any cone of a vector space that makes it a Riesz space will be referred to as a lattice cone. As usual, the supremum and infimum of a pair of vectors \( x, y \) in a vector lattice are denoted by \( x \vee y \) and \( x \wedge y \) respectively. In a vector lattice, the element \( x^+ = x \vee 0 \), \( x^- = (-x) \vee 0 \) and \( |x| = x \vee (-x) \) are called the positive, negative, and absolute value of \( x \). We always have the identities

\[
x = x^+ - x^- \quad \text{and} \quad |x| = x^+ + x^-.
\]

A vector subspace \( Y \) of a vector lattice \( X \) is said to be:

1. a vector sublattice if for each \( x, y \in Y \) we have \( x \vee y \) and \( x \wedge y \) in \( Y \); and
2. an ideal if \( |y| \leq |x| \) and \( x \in Y \) imply \( y \in Y \).

An ideal is always a vector sublattice but a vector sublattice need not be an ideal.

**Definition 1** A vector subspace \( Y \) of a partially ordered vector space \( X \) is said to be a lattice-subspace if \( Y \) under the induced ordering from \( X \) is a vector lattice in its own right. That is, \( Y \) is a lattice-subspace if for every \( x, y \in Y \) the least upper bound of the set \( \{x, y\} \) exists in \( Y \) when ordered by the cone \( Y \cap X^+ \).

If \( X \) is a vector lattice, then every vector sublattice of \( X \) is automatically a lattice-subspace but a lattice-subspace need not be a vector sublattice. For details about lattice-subspaces see [1; 25, 28, 29].

A normed space which is also a partially ordered vector space is a partially ordered normed space. A norm \( \| \cdot \| \) on a vector lattice is said to be a lattice norm if \( |x| \leq |y| \) implies \( \|x\| \leq \|y\| \). A normed vector lattice is a vector lattice equipped with a lattice norm. A complete normed vector lattice is called a Banach lattice.

The next result of B. Z. Vulikh [31, Theorem I.7.1, p. 13] presents a simple condition for a vector space to be a Riesz space. This result in connection with Theorem 6 will provide a variety of interesting results concerning lattice cones.

**Lemma 2 (Vulikh)** Let \( T: X \to Y \) be a linear isomorphism between two vector spaces, i.e., \( T \) is a surjective one-to-one linear operator. If \( Y \) (resp. \( X \)) is a Riesz space and we define the linear ordering \( \geq \) on \( X \) (resp. on \( Y \)) by letting \( u \geq v \) whenever \( Tu \geq Tv \) (resp. \( T^{-1}u \geq T^{-1}v \)), then \( X \) (resp. \( Y \)) is a Riesz space and \( T: X \to Y \) is a (surjective) lattice isomorphism.

And now we are ready to introduce formally the notion of a Yudin basis.

**Definition 3** A cone \( C \) of a vector space is called a Yudin cone if there exists a family \( \{e_i\}_{i \in I} \) of vectors of \( C \) such that each \( x \in C \) has a unique representation of the form
Let $x = \sum_{i \in I} \lambda_i e_i$, where $\lambda_i \geq 0$ and $\lambda_i = 0$ for all but finitely many $i$. Any such a family $\{e_i\}_{i \in I}$ of vectors of $C$ is called a Yudin basis of $C$. A partially ordered vector space has a Yudin basis if its cone has a Yudin basis.

Clearly, every Yudin basis of a cone $C$ is a family of linearly independent vectors and is a Hamel basis for its linear span $M = C - C$. In addition, a given cone can have (essentially) at most one Yudin basis [6, Lemma 2.7].

An arbitrary cone need not have a Yudin basis even if it is a lattice cone. The next result provides some examples; for a proof, see [6, Theorem 2.8].

**Theorem 4** If an infinite dimensional partially ordered vector space has an order unit, then its positive cone does not have a Yudin basis.

Here is a connection between lattice-subspaces and Yudin bases.

**Lemma 5** A finite dimensional vector subspace $M$ of a partially ordered vector space $X$ is a lattice-subspace if and only if $M^+ = M \cap X^+$ is a Yudin cone generating $M$.

An important Riesz space for our economic model is the space $\phi$ of all eventually zero real sequences. That is,

$$\phi = \{ \theta = (\theta_1, \theta_2, \theta_3 \ldots) \in \mathbb{R}^{\infty}: \theta_n = 0 \text{ for all but a finite number of } n \},$$

where $\mathbb{R}^{\infty}$ denotes the Riesz space of all real sequences. The ordering and the lattice operations in $\phi$ are the pointwise ones. Moreover, $\phi$ equipped with the sup norm, defined by $||\theta||_{\infty} = \sup_n |\theta_n|$, is a Dedekind complete normed Riesz space. The standard cone of $\phi$ is the cone

$$\phi^+ = \{ \theta \in \phi: \theta_n \geq 0 \text{ for all } n \}.$$

The following fundamental result describes the lattice structure of a vector space generated by a cone with a countable Yudin basis.

**Theorem 6** Let $C$ by a cone in a vector space $X$ having a countable Yudin basis $\{e_n\}$ and let $M = C - C$ be the linear span of $C$. For each $n$ let $M_n$ denote the vector subspace generated by the finite set $\{e_1, \ldots, e_n\}$. That is,

$$M_n = \{ x \in X: \exists \lambda_1, \ldots, \lambda_n \text{ such that } x = \sum_{i=1}^n \lambda_i e_i \}.$$

Then we have the following properties.

1. The partially ordered vector space $(M, C)$ is a Dedekind complete Riesz space. If $x = \sum_{n=1}^{\infty} \lambda_n e_n$ and $y = \sum_{n=1}^{\infty} \mu_n e_n$ are arbitrary elements of $M$, then $x \geq y$ (i.e.,
\(x - y \in C\) is equivalent to \(\lambda_n \geq \mu_n\) for each \(n\) and the lattice operations of \((M, C)\) are given by

\[
x \vee y = \sum_{n=1}^{\infty} (\lambda_n \vee \mu_n) e_n \quad \text{and} \quad x \wedge y = \sum_{n=1}^{\infty} (\lambda_n \wedge \mu_n) e_n.
\]

2. The vector space \(M\) is also a normed Riesz space under the lattice norm

\[
\|x\|_M = \left\| \sum_{n=1}^{\infty} \lambda_n e_n \right\| = \max_n |\lambda_n|, \quad x = \sum_{n=1}^{\infty} \lambda_n e_n.
\]

Moreover, the “payoff” operator \(R: \phi \to M\), defined by \(R(\theta) = \sum_{n=1}^{\infty} \theta_n e_n\), is a surjective lattice isometry.

3. Each subspace \(M_n\) is a finite dimensional ideal of \(M\) (and also of each \(M_k\) for \(k \geq n\)) and a (Dedekind complete) Banach lattice under the \(\| \cdot \|_M\) norm.

4. The order intervals of \(M\) lie in finite dimensional subspaces—and hence they are norm compact.

From the preceding theorem, a moment’s thought reveals that any Riesz space with a countable Yudin basis is basically a copy of \(\phi\).

The inductive limit topology on a vector space \(M\) is the finest locally convex topology \(\xi_M\) on \(M\) such that for each finite dimensional vector subspace \(F\) of \(M\) equipped with its Euclidean topology the natural embedding \(i: F \hookrightarrow (M, \xi_M)\) is continuous. The inductive limit topology on \(\phi\) will be denoted by \(\xi\).

The next remarkable result indicates how one can use a countable family of independent positive vectors to “twist” the standard lattice ordering of \(\phi\). This is a basic result for our work here.

**Theorem 7** Let \(\{x_n\}\) be a sequence of linearly independent positive vectors in a partially ordered vector space \(X\) and let \(M\) denote the span of \(\{x_n\}\). Also, let \(R: \phi \to M\) be the “payoff” operator defined by \(R(\theta) = \sum_{n=1}^{\infty} \theta_n x_n\). If the cone \(M^+ = M \cap X^+\) has a Yudin basis (which must be necessarily countable), then we have the following.

1. \(M\) is a lattice-subspace of \(X\).
2. The vector space \(\phi\)-equipped with the cone

\[
\phi_R^+ = R^{-1}(M^+) = \{\theta \in \phi: R(\theta) \geq 0\}
\]

is a Dedekind complete Riesz space.
3. The cone \(\phi_R^+\) is a Yudin cone.
4. Each order interval of \((\phi, \phi^+_R)\) is \(\xi\)-compact and lies in a finite dimensional vector subspace.

5. The inductive limit topology \(\xi\) on the Riesz space \((\phi, \phi^+_R)\) is Hausdorff, locally convex-solid and order continuous and the operator \(R: (\phi, \phi^+_R, \xi) \rightarrow (M, M^+, \xi_M)\) is a (surjective) topological lattice isomorphism.

6. The Riesz space \(\mathbb{R}^\infty\) coincides with the topological, algebraic and order dual of \((\phi, \phi^+_R, \xi)\). Moreover, \(\mathbb{R}^\infty\) equipped with the dual cone

\[
(\phi^+_R)' = \{ q = (q_1, q_2, \ldots) \in \mathbb{R}^\infty: q \cdot \theta = \sum_{n=1}^{\infty} q_n \theta_n \geq 0 \ \forall \ \theta = (\theta_1, \theta_2, \ldots) \in \phi^+_R \}
\]

is a Dedekind complete Riesz space.

7. If \(E = (\phi, \phi^+_R)\) and \(E' = (\mathbb{R}^\infty, (\phi^+_R)')\), then \((E, E')\) is a symmetric Riesz dual system.

8. In case \(\{x_n\}\) is itself a Yudin basis, then \(\phi^+_R = \phi^+\) (the standard positive cone of \(\phi\)) and \(R\) is essentially the identity operator.

3 Portfolio Dominance

We consider a two-period securities market model. There are countably many securities traded at date 0 labelled by the natural numbers 1, 2, \ldots. Securities are described by their payoffs at date 1. The payoff of security \(n\) is \(x_n\), an element of a payoff space \(X\). Typically, the space \(X\) is a vector space of state-contingent consumption plans modeled as real-valued random variables on some underlying probability measure space \((\Omega, \Sigma, P)\). Examples are the \(L^p(\Omega, \Sigma, P)\)-spaces for \(1 \leq p \leq \infty\).

Securities can be combined in portfolios. A portfolio is a sequence of share holdings \(\theta = (\theta_1, \theta_2, \ldots)\), where \(\theta_n\) is the number of shares of security \(n\). In the case of a short position in security \(n\), the holding \(\theta_n\) is negative. In other words, we assume that each \(\theta_n\) can be any real number. Throughout this paper we restrict our attention to portfolios with non-zero holdings of only finitely many securities. Thus, each portfolio is formed from a finite subset of securities.¹ The space of all portfolios is the vector space \(\phi\) of all eventually zero sequences, and will be referred to as the portfolio space.

The payoff of a portfolio \(\theta \in \phi\) is

\[
R(\theta) = \sum_{n=1}^{\infty} \theta_n x_n . \tag{\ast}
\]

¹Such portfolios might be called simple portfolios in analogy with the simple trading strategies in the continuous time securities market model of Harrison and Kreps [17].
Clearly, \( R(\theta) \in X \). It should be clear that formula (\( \ast \)) defines a linear operator \( R: \phi \to X \), which we shall refer to as the **payoff operator**. The payoff vectors \( x_1, x_2, \ldots \) are assumed to be linearly independent (non-redundant securities), so that the payoff operator \( R \) is always one-to-one.

The payoff space \( X \) is assumed to be a partially ordered vector space. The positive cone of \( X \) is the set of all positive payoffs and, as usual, is denoted by \( X^+ \). The (partial linear) order of the payoff space \( X \) induces a (partial linear) order \( \geq_R \) on the portfolio space \( \phi \) via the payoff operator \( R \) by

\[
\theta \geq_R \theta' \quad \text{whenever} \quad R(\theta) \geq R(\theta').
\]

The order \( \geq_R \) will be called the **portfolio dominance** order.\(^2\) The positive cone under the portfolio dominance order is

\[
\phi_R^+ = \{ \theta \in \phi: \theta \geq_R 0 \} = \{ \theta \in \phi: R(\theta) \geq 0 \} = R^{-1}(X^+)
\]

and is precisely the set of all portfolios with positive payoff. The cone \( \phi_R^+ \) will be referred to as the cone of **positive payoff portfolios**. As usual, \( \theta \geq_R \theta' \) if and only if \( \theta - \theta' \in \phi_R^+ \).

We shall assume that the portfolio dominance order \( \geq_R \) is a lattice order. That is, for any two portfolios \( \theta, \theta' \in \phi \) we have a well-defined supremum portfolio \( \theta \vee_R \theta' \), and an infimum portfolio \( \theta \wedge_R \theta' \). The supremum \( \theta \vee_R \theta' \) is the least upper bound of \( \theta \) and \( \theta' \) with respect to \( \geq_R \), i.e., a portfolio with the lowest payoff that is higher than the payoffs of \( \theta \) and \( \theta' \). The infimum \( \theta \wedge_R \theta' \) is the greatest lower bound of \( \theta \) and \( \theta' \) with respect to \( \geq_R \), i.e., a portfolio with the highest payoff that is lower than the payoffs of \( \theta \) and \( \theta' \).

There is an interesting connection between lattice operations and some important portfolio investment strategies. Suppose that the payoff space is \( X = L_p(\Omega, \Sigma, P) \), and that security 1 is a riskless security with payoff \( x_1(\omega) = 1 \) for every \( \omega \in \Omega \). Let \( e_1 = (1, 0, 0, \ldots) \in \phi \) denote the portfolio of one share of the riskless security. For any reference portfolio \( \theta \) and any positive number (“floor”) \( k \), the portfolio \( \theta \vee_R ke_1 \) is the portfolio with the lowest payoff higher than the payoff of \( \theta \) and the floor \( k \). Thus the portfolio \( \theta \vee_R ke_1 \) represents portfolio insurance. If there exists a portfolio the payoff of which equals the payoff of \( \theta \) whenever the letter is above the floor \( k \) and equals \( k \), otherwise, then that portfolio equals \( \theta \vee_R ke_1 \), and we have full portfolio insurance.

The positive part of portfolio \( \theta - ke_1 \), i.e., \( (\theta - ke_1)^+ = (\theta - ke_1) \vee_R 0 \), is a portfolio with the lowest payoff that is positive and higher than the payoff of \( \theta - ke_1 \). If a call option on the reference portfolio \( \theta \) with striking price \( k \) is traded in the markets, then this option equals \( (\theta - ke_1)^+ \). Otherwise, if the option is not traded, the portfolio \( (\theta - ke_1)^+ \) is the portfolio with the lowest payoff higher than the payoff of the call option. Similarly, the portfolio \( (ke_1 - \theta)^+ \) equals the put option on portfolio \( \theta \), if the put option is traded, or has the lowest payoff higher than the payoff of the put option, if the option is not traded.

\(^2\)We use the subscript \( R \) to distinguish this linear ordering from the standard pointwise ordering \( \geq \) on \( \phi \) defined by \( \theta \geq \theta' \) whenever \( \theta_n \geq \theta'_n \) for each \( n \).
The well-known (see Leland [19]) equivalence of portfolio insurance and the strategy of holding the reference portfolio and buying a put option on the portfolio takes the form of an elementary identity of the lattice operations:

\[ \theta + (ke_1 - \theta)^+ = \theta \vee_R ke_1. \]

The range \( M = R(\phi) \subseteq X \) of the payoff operator \( R \) is the subspace of payoffs of all portfolios. We shall refer to \( M \) as the asset span of the securities—it is also known as the space of marketed securities. By Lemma 2, for \( \geq_R \) to be a lattice ordering it is necessary and sufficient that \( M \) is a Riesz space under the induced ordering from \( X \). That is, we have the following result.

**Lemma 8** The portfolio dominance \( \geq_R \) ordering on \( \phi \) is a lattice order (or, equivalently, \( \phi_R^+ \) is a lattice cone of \( \phi \)) if and only if asset span \( M \) is a lattice-subspace of the payoff space \( X \).

So, assuming that \( \phi_R^+ \) is a lattice cone is equivalent to asserting that \( M^+ = M \cap X^+ \) is a lattice cone of the asset span \( M \). Conditions under which a subspace is a lattice-subspace can be found in Abramovich, Aliprantis and Polyrakis [1], and Polyrakis [29]. In a companion paper, Aliprantis, Brown and Werner [7], we provide a detailed analysis of finite securities markets with the asset span being a lattice-subspace.

A price of security \( n \) is simply a real number \( q_n \). Any vector \( q = (q_1, q_2, \ldots) \in \mathbb{R}^\infty \) will be called a security price system—or simply a vector of security prices. The market value of portfolio \( \theta \in \phi \) at security prices \( q \) is then the real number

\[ q \cdot \theta = \sum_{n=1}^{\infty} q_n \theta_n. \]

The portfolio space \( \phi \) and the space of security prices \( \mathbb{R}^\infty \) form a dual system \( \langle \phi, \mathbb{R}^\infty \rangle \), the portfolio-price duality. By Theorem 7, we know that \( \langle \phi, \mathbb{R}^\infty \rangle \) is a symmetric Riesz dual system. The dual cone \( (\phi_R^+)' \) of \( \phi_R^+ \) is defined by

\[ (\phi_R^+)' = \{ q = (q_1, q_2, \ldots) \in \mathbb{R}^\infty: q \cdot \theta = \sum_{n=1}^{\infty} q_n \theta_n \geq 0 \quad \forall \theta = (\theta_1, \theta_2, \ldots) \in \phi_R^+ \}. \]

The standard concepts of arbitrage and strong arbitrage portfolios can be easily expressed using the portfolio dominance. A strong arbitrage under prices \( q \) is a portfolio \( \theta \in \phi \) that dominates the zero portfolio \( (\theta \geq_R 0) \) and has negative value \( (q \cdot \theta < 0) \), i.e., a portfolio with negative value and positive payoff. An arbitrage under prices \( q \) is a portfolio \( \theta \in \phi \) such that \( \theta >_R 0 \) and \( q \cdot \theta \leq 0 \), i.e., a portfolio with zero or negative value and positive payoff. A security price system that excludes strong arbitrage (resp. arbitrage) is weakly arbitrage-free (resp. arbitrage-free). Clearly, every arbitrage-free price is also weakly arbitrage-free. The set of weakly arbitrage-free prices is the dual cone \( (\phi_R^+)' \).
Since \(\{(\phi, \phi_R^+), (\mathbb{R}^\infty, (\phi_R^+)'\}\) is a Riesz dual system, it follows that

\[
\theta \geq_R \theta' \iff q \cdot \theta \geq q \cdot \theta'
\]

for each arbitrage-free price vector \(q\).

Thus, a portfolio \(\theta\) dominates another portfolio \(\theta'\) if and only if \(\theta\) is more expensive than \(\theta'\) under every weakly arbitrage-free price. The insured portfolio \(\theta^+ = \theta \lor_R 0\) is the cheapest no-loss portfolio the payoff of which dominates the payoff of \(\theta\).

4. Equilibrium in Securities Markets

As mentioned in the previous section, the portfolio-price duality in our securities markets model is described by the Riesz dual system \((\phi, \mathbb{R}^\infty)\). In this duality, \(\phi\) is understood as a Riesz space with the lattice cone \(\phi_R^+\) and \(\mathbb{R}^\infty\) as a Riesz space equipped with the dual cone \((\phi_R^+)'\). Unless otherwise stated, the portfolio space \(\phi\) will be understood equipped with its inductive limit topology \(\xi\).

There are \(m\) investors indexed by \(i\), i.e., \(i = 1, \ldots, m\). Each investor \(i\) has:

1. The cone of positive payoff portfolios \(\phi_R^+\) as her feasible portfolio set.
2. An initial portfolio \(\bar{\theta}^i \in \phi_R^+\). The aggregate portfolio \(\bar{\theta} = \sum_{i=1}^m \bar{\theta}^i\) is called the market portfolio.
3. A utility function \(\hat{u}_i: \phi_R^+ \to \mathbb{R}\) such that
   1. \(\hat{u}_i\) is quasi-concave and \(\xi\)-continuous,
   2. \(\hat{u}_i\) is monotone with respect to \(\geq_R\), i.e., \(\theta \geq_R \theta'\) implies \(\hat{u}_i(\theta) \geq \hat{u}_i(\theta')\), and
   3. the market portfolio \(\bar{\theta}\) is desirable in the sense that \(\hat{u}_i(\theta + \alpha \bar{\theta}) > \hat{u}_i(\theta)\) for all \(\theta \in \phi_R^+\) and each \(\alpha > 0\).

If an investor \(i\) has a preference over the state-contingent consumption plans described by a utility function \(u_i: X^+ \to \mathbb{R}\), then we shall assume that the portfolio utility function \(\hat{u}_i\) is the indirect utility given by \(\hat{u}_i(\theta) = u_i(R(\theta))\). When \(X = L_p(\Omega, \Sigma, P)\), a typical example of a utility function \(u_i: X^+ \to \mathbb{R}\) is a separable utility function given by

\[
u_i(x) = \int_{\Omega} v_i(x(\omega), \omega) dP(\omega),
\]

where the kernel \(v_i: \mathbb{R}_+ \times \Omega \to \mathbb{R}\) satisfies certain concavity and measurability properties; see [3] for details about separable utility functions.

If a consumption utility function \(u_i: X^+ \to \mathbb{R}\) is quasi-concave, monotone, and the payoff \(R(\bar{\theta})\) is desirable for \(u_i\), then the indirect utility function \(\hat{u}_i\) is quasi-concave, monotone, and \(\bar{\theta}\) is desirable for \(\hat{u}_i\). Moreover, if \(u_i\) is continuous for a Hausdorff locally
convex topology of the payoff space $X$, then $\hat{u}_i$ is continuous in the inductive limit topology $\xi$ of the portfolio space $\phi$.

The **portfolio budget set** $B_i(q)$ of an investor $i$ at prices $q \in \mathbb{R}^\infty$ is

$$B_i(q) = \{ \theta \in \phi_R^+ : q \cdot \theta \leq q \cdot \theta^i \}.$$  

An **optimal portfolio** for investor $i$ at prices $q$ is a portfolio $\theta^i \in B_i(q)$ that maximizes the utility $\hat{u}_i$ over all portfolios in the budget set $B_i(q)$.

A **portfolio allocation** is any $m$-tuple $(\theta^1, \ldots, \theta^m)$, where $\theta^i$ is a feasible portfolio (i.e., $\theta^i \in \phi_R^+$) for investor $i$ and $\sum_{i=1}^m \theta^i = \sum_{i=1}^m \theta^i = \theta$.

An equilibrium in securities markets is now defined as follows.

**Definition 9** A portfolio allocation $(\theta^1, \ldots, \theta^m)$ is said to be a **portfolio equilibrium** if there exists a non-zero security price system $q \in \mathbb{R}^\infty$ such that each $\theta^i$ is optimal for investor $i$ at prices $q$. Any price system $q$ that satisfies this property is called a **price supporting** $(\theta^1, \ldots, \theta^m)$.

The reader should notice immediately that a portfolio allocation $(\theta^1, \ldots, \theta^m)$ is an equilibrium with respect to a non-zero price $q \in \mathbb{R}^\infty$ if and only if

$$\theta \in \phi_R^+ \text{ and } \hat{u}_i(\theta) > \hat{u}_i(\theta^i) \implies q \cdot \theta > q \cdot \theta^i.$$  

For studying sufficient conditions for the existence of a portfolio equilibrium it is useful to introduce the notion of a portfolio quasiequilibrium.

**Definition 10** A portfolio quasiequilibrium is a portfolio allocation $(\theta^1, \ldots, \theta^m)$ for which there exists a non-zero price system $q \in \mathbb{R}^\infty$ such that

$$\theta \in \phi_R^+ \text{ and } \hat{u}_i(\theta) \geq \hat{u}_i(\theta^i) \text{ imply } q \cdot \theta \geq q \cdot \theta^i.$$  

The price system $q$ is called a price that **supports** $(\theta^1, \ldots, \theta^m)$.

Clearly, every equilibrium is a quasiequilibrium. Conversely, a quasiequilibrium in which the wealth of each agent is strictly positive is an equilibrium. We state this standard result below.

**Lemma 11** If a portfolio quasiequilibrium $(\theta^1, \ldots, \theta^m)$ is supported by a price $q$ which satisfies $q \cdot \theta^i > 0$ for each $i$, then $(\theta^1, \ldots, \theta^m)$ is a portfolio equilibrium supported by the price $q$.

It is important to keep in mind that prices supporting equilibria or quasiequilibria are arbitrage-free.
**Theorem 12** Every non-zero price supporting a quasiequilibrium is arbitrage-free.

**Proof:** Let \( q \in \mathbb{R}^\infty \) be a non-zero price supporting a quasiequilibrium \((\theta^1, \ldots, \theta^m)\). If \( \theta \in \phi_R^+ \), then \( \theta + \theta^1 \in \phi_R^+ \) and the monotonicity of \( \hat{u}_i \) implies \( \hat{u}_i(\theta + \theta^1) \geq \hat{u}_i(\theta^1) \). So, from the supportability of \( q \), we infer \( q \cdot (\theta + \theta^1) = q \cdot \theta + q \cdot \theta^1 \geq q \cdot \theta^1 \). Hence, \( q \cdot \theta \geq 0 \) and this shows that \( q \) is an arbitrage free price.

## 5 Optimal Portfolio Allocations

An optimal portfolio allocation is defined as follows.

**Definition 13** A portfolio allocation \((\theta^1, \ldots, \theta^m)\) is **optimal**, if there is no other portfolio allocation \((\theta'^1, \ldots, \theta'^m)\) satisfying \( \hat{u}_i(\theta'^i) \geq \hat{u}_i(\theta^i) \) for every \( i \) and \( \hat{u}_i(\theta'^i) > \hat{u}_i(\theta^i) \) for at least one \( i \).

By the First Welfare Theorem a portfolio equilibrium allocation is (under the standard assumptions) optimal. Clearly, it is also individually rational in the sense that it is weakly preferred to the initial portfolio allocation \((\overline{\theta}^1, \ldots, \overline{\theta}^m)\). The existence of an individually rational optimal portfolio allocation is a necessary condition for the existence of a portfolio equilibrium.

A useful concept for studying the existence of optimal portfolio allocations is the utility possibility set. The **utility possibility set** of security markets is a subset of \( \mathbb{R}^m \) consisting of utility levels of all portfolio allocations which are individually rational. It is defined by

\[
U = \{(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m: \exists \text{ some allocation } (\theta^1, \ldots, \theta^m) \text{ with } \hat{u}_i(\overline{\theta}^i) \leq \lambda_i \leq \hat{u}_i(\theta^i) \ \forall \ i \}.
\]

From the assumption that the portfolio utility functions are monotone it follows that the utility set is a bounded set. If it is closed, then the existence of optimal portfolio allocations is assured. Indeed, portfolio allocations that generate the highest (in the sense of the usual order of \( \mathbb{R}^m \)) utility levels are optimal—and so their utility allocations lie in the boundary of the utility possibility set.

If \((\theta^1, \ldots, \theta^m)\) is a feasible portfolio allocation, then \( 0 \leq_R \theta^i \leq_R \overline{\theta} \), i.e., each portfolio \( \theta^i \) lies in the order interval \([0, \overline{\theta}]_R \) of the portfolio dominance order. A sufficient condition for the closedness of the utility set that is well known from finite dimensional equilibrium theory is the weak compactness of the order interval \([0, \overline{\theta}]_R \); see [2, Problem 3.5.1, p. 145]. If the cone of positive payoff portfolios \( \phi_R^+ \) has a Yudin basis, then it follows from Theorem 7(4) that the interval \([0, \overline{\theta}]_R \) lies in the finite dimensional vector subspace—and hence is compact.
We can now state the following theorem regarding the closedness of the utility possibility set.

**Theorem 14** If the cone of positive payoff portfolios \( \phi^+ \) has a Yudin basis (or, equivalently, if the positive cone \( M^+ = M \cap X^+ \) of the asset span \( M \) has a Yudin basis), then the utility possibility set \( U \) is closed.

**Proof:** We consider \( \phi \) as a Riesz space under the order \( \succeq_R \). As mentioned above, by Theorem 7(4), the interval \([0, 0]_R\) is compact and lies in a finite dimensional subspace of \( \phi \).

Let a sequence \( \{\ell_n\} \subseteq U \) satisfy \( \ell_n \rightarrow \ell \) in \( \mathbb{R}^m \), where \( \ell_n = (\lambda^1_n, \ldots, \lambda^m_n) \). For each \( n \) pick a portfolio allocation \( (\theta^{1n}, \ldots, \theta^{mn}) \) such that \( \hat{u}_i(\bar{\theta}^i) \leq \lambda^i_n \leq \hat{u}_i(\theta^{in}) \) for each \( i \) and \( n \). Since \( 0 \leq_R 0 \leq_R \bar{\theta} \) for each \( i \) and \( n \) and the order interval \([0, \bar{\theta}]_R\) is compact, by passing to appropriate subsequences if necessary, we can assume that \( \theta^{in} \rightarrow \theta^i \) for each \( i \). Clearly, \((\theta^1, \ldots, \theta^m)\) is a portfolio allocation. The \( \xi \)-continuity of the utility functions imply

\[
\hat{u}_i(\bar{\theta}^i) \leq \lambda_i = \lim_{n \rightarrow \infty} \lambda^i_n \leq \lim_{n \rightarrow \infty} \hat{u}_i(\theta^{in}) = \hat{u}_i(\theta^i).
\]

This shows \( \ell = (\lambda_1, \ldots, \lambda_m) \in U \) so that \( U \) is a closed subset of \( \mathbb{R}^m \). \( \blacksquare \)

The next two examples illustrate the closedness of the utility possibility set.

**Example 15** Let the securities be the Arrow securities, i.e., \( x_n = e_n \) is the \( n \)th unit vector in the payoff space \( X = \ell_\infty \) of all bounded consumption plans on a countable state space. In this case the payoff operator \( R: \phi \rightarrow \ell_\infty \) is simply the natural embedding of \( \phi \) into \( \ell_\infty \) (i.e., \( R(\theta) = \theta \)) and the portfolio dominance order coincides with the usual order of the portfolio space \( \phi \), i.e.,

\[
\theta \geq_R \theta' \iff \theta_n \geq \theta'_n \text{ for each } n .
\]

The order interval \([0, \bar{\theta}]\) lies in a finite dimensional vector subspace of \( \phi \). Indeed, the set \( \mathcal{F} = \{n: \theta_n > 0\} \) is finite, and \( \theta \in [0, \bar{\theta}] \) implies that \( \theta_n = 0 \) for \( n \notin \mathcal{F} \). Consequently, \([0, \bar{\theta}]\) is compact and the utility possibility set \( U \) is closed. In fact, \( U \) is the same as the utility possibility set of an economy with the finite set of securities \( \{x_n: n \in \mathcal{F}\} \). Note that the positive cone \( \phi^+_R \) has a Yudin basis consisting of the unit vectors.

More generally, if the payoffs \( x_n \) are such that for each security \( n \) there is a state \( \omega_n \in \Omega \) in which \( x_n(\omega_n) > 0 \) and \( x_k(\omega_n) = 0 \) for every security \( k \neq n \), then the portfolio dominance order coincides with the usual order of \( \phi \). The utility possibility set is closed in such a case. \( \triangle \)

\( ^3 \)The reader should notice that this proof also shows that the utility set remains closed if we assume that the utility functions are only \( \xi \)-upper semicontinuous rather than \( \xi \)-continuous.
Example 16 Let the securities \( \{x_1, x_2, \ldots \} \) be the Arrow securities (i.e., \( x_n = e_n \) for each \( n \)) and the riskless bond \( x_0 = e = (1, 1, \ldots) \), where \( e \) is the order unit in the payoff space \( X = \ell_\infty \). The payoff operator \( R: \phi \to \ell_\infty \) is given by

\[
R(\theta) = (\theta_0 + \theta_1, \theta_0 + \theta_2, \theta_0 + \theta_3, \ldots)
\]

for each \( \theta = (\theta_0, \theta_1, \theta_2, \ldots) \in \phi \). Consequently, the portfolio dominance order in this case is given by

\[
\theta \succeq_R \theta' \iff \theta_0 + \theta_n \geq \theta'_0 + \theta'_n \text{ for each } n.
\]

Suppose that the Arrow securities are in zero supply and that there is a strictly positive supply \( b \) of the bond. Then the market portfolio \( \overline{\theta} \) includes only the bond so that \( \overline{\theta} = (b, 0, 0, \ldots) \). The order interval \( [0, \overline{\theta}]_R \) is not \( \xi \)-compact. Indeed, if we let \( y_n = be_n \) for each \( n \), then \( y_n \) lies in the order interval \( [0, \overline{\theta}]_R \) for each \( n \), but the sequence \( \{y_n\} \) does not have any \( \xi \)-convergent subnet.

We shall specify now investors' utility functions for which the utility set is not closed. There are two risk averse expected utility maximizing investors. For each investor \( i = 1, 2 \) define the consumption utility function \( u_i: \ell_\infty^+ \to \mathbb{R} \) by

\[
u_i(x) = \sum_{\omega=1}^{\infty} P_i(\omega)v_i(x(\omega)) ,
\]

where \( v_i: [0, \infty) \to \mathbb{R} \) is assumed to be strictly increasing, strictly concave and continuous. The probability beliefs \( P_i \) on the state space \( \Omega = \{1, 2, \ldots\} \) are given by \( P_1(\omega) = 2^{-\omega} \) for every \( \omega \in \Omega \), and \( P_2(\omega) = 2^{1-\omega} \), if \( \omega \in A = \{2, 4, 6, \ldots\} \) and \( P_2(\omega) = 2^{-1-\omega} \), if \( \omega \in A^c = \{1, 3, 5, \ldots\} \). Note that investors assign the same conditional probabilities to every state, conditional on \( A \) and on \( A^c \), but they assign different probabilities to events \( A \) and \( A^c \).

The payoff \( R(\overline{\theta}) = bx_0 = (b, b, \ldots) \) of the market portfolio is state independent. We claim that all Pareto optimal consumption allocations with respect to \( X = \ell_\infty \) are state independent within \( A \) and within \( A^c \).

In order to prove this claim, consider a Pareto optimal consumption allocation \((y_1, y_2)\) and suppose that \( y_1(\omega_1) \neq y_1(\omega_2) \) for some \( \omega_1, \omega_2 \in A \). Define the consumption plans \( \overline{y}_1 \) and \( \overline{y}_2 \) by

\[
\overline{y}_1(\omega) = \begin{cases} y_i(\omega), & \text{if } \omega \notin \{\omega_1, \omega_2\} \\ z_i, & \text{if } \omega \in \{\omega_1, \omega_2\}, \end{cases}
\]

where \( z_i = P_1(\omega_1 | \{\omega_1, \omega_2\})y_i(\omega_1) + P_1(\omega_2 | \{\omega_1, \omega_2\})y_i(\omega_2) \) is the expected value of \( y_i \) conditional on \( \{\omega_1, \omega_2\} \). Since the conditional probabilities \( P_1(\omega_1 | \{\omega_1, \omega_2\}) \) and \( P_1(\omega_2 | \{\omega_1, \omega_2\}) \) are the same for both investors, \( \overline{y}_1, \overline{y}_2 \) is an allocation. Moreover, the strict concavity of \( v_i \) implies

\[
u_i(\overline{y}_1) - u_i(y_i) = P_1(\{\omega_1, \omega_2\})v_i(z_i) - [P_1(\omega_1)v_i(y_i(\omega_1)) + P_1(\omega_2)v_i(y_i(\omega_2))] > 0
\]

for each \( i = 1, 2 \), which contradicts the Pareto optimality of the allocation \((y_1, y_2)\).
Next we claim that all Pareto optimal allocations (with the exception of "corner" allocations where one investor has zero consumption) are state dependent across $A$ and $A^c$. This is obvious since the marginal rate of substitution between consumption in any two states $\omega \in A$ and $\omega' \in A^c$ at a state independent consumption plan equals $\frac{P_i(\omega)}{P_i(\omega')}$, and is different for $i = 1$ and for $i = 2$. Equality of marginal rates of substitution is a necessary condition for Pareto optimality.

Since the asset span $M$ is the space of all eventually constant sequences, Pareto optimal consumption plans do not belong to the asset span $M$. However, they can be approximated by consumption plans in $M$ so that the difference in utility is arbitrarily small. More precisely, let $(y_1, y_2)$ be a Pareto optimal allocation. Consider the sequence of allocations $\{(y^n_1, y^n_2)\}$ such that $y^n_i(\omega) = y_i(\omega)$ for $\omega \leq n$, and $y^n_i(\omega) = \frac{R(\hat{\theta}(\omega))}{2}$ for $\omega > n$, $i = 1, 2$. We have that $y^n_i \in M$ and $\{y^n_i\}$ converges in the weak topology $\sigma(\ell_{\infty}, \ell_1)$ to $y^i$. Since the expected utility function $u_i$ is weakly continuous on the order interval $[0, R(\hat{\theta})]$ of $\ell_{\infty}$, it follows that $u_i(y^n_i)$ converges to $u_i(y_i)$. So, the utility set $U$ is not closed.\[\triangle\]

6 Existence of a Portfolio Equilibrium

Our analysis of the existence of a portfolio equilibrium follows the approach of A. Mas-Colell [21]; see also [5, Theorem 3.5.12, p. 161].

Suppose that the cone of positive payoff portfolios has a Yudin basis—which is, of course, equivalent to saying that the cone $M^+ = M \cap X^+$ of the asset span $M$ has a Yudin basis. Then, by Theorem 14, the utility possibility set of security markets is closed. Thus, there exist optimal portfolio allocations—a necessary condition for the existence of an equilibrium. The remaining issue is the existence of the supporting prices, i.e., security prices that would make an optimal portfolio allocation a portfolio quasiequilibrium for a suitable allocation of initial portfolios. That issue—which is specific to infinite dimensional equilibrium theory—is handled by restricting the class of investors’ utility functions to those that are also uniformly proper on $\phi^+_R$.

Recall that a portfolio utility function $\hat{u}_i$ is $\theta$-uniformly $\xi$-proper on $\phi^+_R$, if there is a neighborhood (in the inductive limit topology) $V$ of zero such that $\hat{u}_i(\theta - \alpha \hat{\theta} + \gamma) \geq \hat{u}_i(\theta)$ implies $\gamma \notin \alpha V$ for every $\alpha > 0$ and $\theta \in \phi^+_R$ with $\theta - \alpha \hat{\theta} + \gamma \in \phi^+_R$. Observe, that if $\hat{u}_i$ is $\hat{\theta}$-uniformly $\tau$-proper for a Hausdorff locally-convex topology $\tau$ on $\phi$, then (in view of $\tau \subseteq \xi$) $\hat{u}_i$ is automatically $\hat{\theta}$-uniformly $\xi$-proper.

We can now state our existence of portfolio equilibrium result as follows.

**Theorem 17** Assume that the cone of positive payoff portfolios $\phi^+_R$ has a Yudin basis, and that each portfolio utility function $\hat{u}_i$ is also $\theta$-uniformly $\xi$-proper on $\phi^+_R$. Then there exists a portfolio quasiequilibrium.
If \( \hat{u}_i \) is the indirect utility function \( u_i(R(\cdot)) \) obtained from the consumption utility function \( u_i: X^+ \to \mathbb{R} \) which is \( \bar{\varepsilon} \)-uniformly \( \tau \)-proper for a Hausdorff locally convex topology \( \tau \) on \( X \) (where \( \bar{\varepsilon} = R(\bar{\theta}) \)), then \( \hat{u}_i \) is \( \bar{\theta} \)-uniformly \( \xi \)-proper on \( \phi_R^+ \). Properness of consumption utility functions is a standard assumption in infinite dimensional general equilibrium theory; see, for instance [5] or [22].

**Corollary 18** Assume that the positive cone \( M^+ \) of the asset span has a Yudin basis, and that each consumption utility function \( u_i \) is continuous and \( \bar{\varepsilon} \)-uniformly proper for a Hausdorff locally convex topology on \( X \). Then there exists a portfolio quasiequilibrium.

### 7 Equivalent Market Structures

A sequence \( x = \{x_n\} \) of security payoffs is said to define a market structure if the set \( \{x_1, x_2, \ldots, x_n\} \subseteq X \) is simply a linearly independent set.

Associated with a market structure \( x = \{x_n\} \) is an asset span space \( M_x \), the payoff operator \( R_x: \phi \to M_x \) defined by \( R_x(\theta) = \sum_{n=1}^{\infty} \theta_n x_n \), a portfolio dominance order \( \geq_{R_x} \), and a cone of positive payoff portfolios \( \phi^+_R \). For simplicity, we shall denote the portfolio dominance order \( \geq_{R_x} \) by \( \geq_x \) and the cone of positive payoff portfolios \( \phi^+_R \) by \( \phi^+_x \).

Two market structures that give rise to the same asset span provide investors with the same opportunities of insuring against the consumption risk and will be referred to as equivalent market structures.

**Definition 19** Two market structures \( x = \{x_n\} \) and \( z = \{z_n\} \) are equivalent if they have the same asset span, i.e., if \( M_x = M_z \).

Now assume that two market structures \( x = \{x_n\} \) and \( z = \{z_n\} \) are equivalent. Let \( M = M_x = M_z \) and put \( M^+ = M \cap X^+ \). Clearly, \( \phi^+_x = R_z^{-1}(M^+) \) and \( \phi^+_z = R_x^{-1}(M^+) \). Then a linear operator \( \Phi: \phi \to \phi \) is naturally defined via the formula

\[
\Phi(\theta) = R_x^{-1}(R_x(\theta)).
\]

The operator \( \Phi: (\phi, \phi^+_x) \to (\phi, \phi^+_z) \) is an order isomorphism between these two partially ordered portfolio spaces. By Lemma 2, the portfolio dominance \( \geq_x \) is a lattice order if and only if \( \geq_z \) is likewise a lattice order—and this is equivalent to assuming that \( M^+ \) is a lattice cone of \( M \) (or that \( M \) is a lattice-subspace of \( X \)).

Thus the property of portfolio dominance being a lattice order is independent of the market structure as long as market structures are equivalent and \( M \) is a lattice-subspace of \( X \). Furthermore, the cone of positive payoff portfolios \( \phi^+_x \) has a Yudin basis if and only if the cone \( \phi^+_z \) has a Yudin basis—and this is, of course, equivalent to requiring that \( M^+ \) has a Yudin basis.
The operator \( \Phi: (\phi, \phi^+_x, \xi) \rightarrow (\phi, \phi^+_x, \xi) \) is also a topological order isomorphism. The adjoint operator \( \Phi' \) is therefore a well-defined positive operator. It maps the space of security prices \( \mathbb{R}^\infty \) into itself and is given via the duality identity

\[
\Phi'(q) \cdot \theta = q \cdot \Phi(\theta)
\]

for \( \theta \in \phi \) and \( q \in \mathbb{R}^\infty \).

In case \( M \) has a Yudin basis, then the adjoint operator \( \Phi': (\mathbb{R}^\infty, (\phi^+_x)) \rightarrow (\mathbb{R}^\infty, (\phi^+_x)) \) is a (surjective) lattice isomorphism. In particular, it maps the cone of weakly arbitrage-free prices under \( z \) onto the cone of weakly arbitrage-free prices under \( x \). Thus, security prices \( q \) are weakly arbitrage-free under the market structure \( z \), i.e., \( q \cdot \theta \geq 0 \) for every \( \theta \geq 0 \), if and only if security prices \( \Phi'(q) \) are weakly arbitrage-free under the market structure \( x \). The same holds true for arbitrage-free prices.

Suppose that each investor’s portfolio utility functions \( \hat{u}^x_i: \phi^+_x \rightarrow \mathbb{R} \) and \( \hat{u}^z_i: \phi^+_z \rightarrow \mathbb{R} \) are indirect utilities of a consumption utility function \( u_i \) given by \( \hat{u}^x_i(\theta) = u_i(R_x(\theta)) \) and \( \hat{u}^z_i(\theta) = u_i(R_z(\theta)) \). Furthermore, let the initial portfolios \( \vec{\theta}^i_x \) and \( \vec{\theta}^i_z \) be such that \( R_x(\vec{\theta}^i_x) = R_z(\vec{\theta}^i_z) \), i.e., they have the same payoff. The duality properties of the operator \( \Phi \) allow us to state an interesting invariance result.

**Theorem 20** For two equivalent market structures \( x = \{x_n\} \) and \( z = \{z_n\} \) and portfolios \( \theta_1, \ldots, \theta_m \in \phi \) we have the following invariance results:

1. \((\theta_1, \ldots, \theta_m)\) is an optimal portfolio allocation with respect to utility functions \( \hat{u}^x_i \) if and only if \((\Phi(\theta_1), \ldots, \Phi(\theta_m))\) is an optimal portfolio allocation with respect to utility functions \( \hat{u}^z_i \).

2. \((\theta_1, \ldots, \theta_m)\) is a portfolio equilibrium with respect to a price \( q \in \mathbb{R}^\infty \) relative to the market structure \( x \) if and only if \((\Phi(\theta_1), \ldots, \Phi(\theta_m))\) is a portfolio equilibrium relative to the market structure \( z \) with respect to the price \( (\Phi')^{-1}(q) \).

Proof: (1) This is straightforward.

(2) Let \((\theta_1, \ldots, \theta_m)\) be a portfolio equilibrium with respect to the price \( q \in \mathbb{R}^\infty \) relative to the market structure \( x \). We have \( \theta_i \in \phi^+_x \), \( q \cdot \theta_i \leq q \cdot \vec{\theta}^i_x \) for each \( i \), and

\[
\hat{u}^x_i(\theta_i) = \max\{\hat{u}^x_i(\theta): \theta \in \phi^+_x \text{ and } q \cdot \theta \leq q \cdot \vec{\theta}^i_x\}.
\]

From the duality identity \( q \cdot \theta = (\Phi')^{-1}(q) \cdot \Phi(\theta) \) and the facts that \( \Phi \) is onto and \( q \neq 0 \), we see that \((\Phi')^{-1}(q) \neq 0 \).

Clearly \( \Phi(\theta_i) \in \phi^+_x \). The market clearing \( \sum_{i=1}^{m} \theta^i = \sum_{i=1}^{m} \vec{\theta}^i_x \) under the market structure \( x \) implies that \( \sum_{i=1}^{m} \Phi(\theta^i) = \sum_{i=1}^{m} \Phi(\vec{\theta}^i_x) = \sum_{i=1}^{m} \vec{\theta}^i_z \) which guarantees that \((\Phi(\theta^1), \ldots, \Phi(\theta^m))\) is a portfolio allocation under the market structure \( z \). Now if \((\Phi')^{-1}(q) \cdot \eta \leq (\Phi')^{-1}(q) \cdot \Phi(\theta^i) \) for \( \eta \in \phi^+_z \), then \( q \cdot (\Phi')^{-1}(\eta) \leq q \cdot \theta^i = q \cdot \vec{\theta}^i_x \). Therefore,
\[ \hat{u}_x^\varepsilon(\eta) = \hat{u}_x^\varepsilon((\Phi^{-1}(\eta)) \leq \hat{u}_x^\varepsilon(\theta^i) = \Phi(\theta^i), \text{ and } \Phi(\theta^i) \text{ maximizes the indirect utility } \hat{u}_x^\varepsilon \text{ in the portfolio set of the market structure } z. \text{ Thus } (\Phi(\theta_1), \ldots, \Phi(\theta_m)) \text{ is a portfolio equilibrium relative to the market structure } z \text{ with respect to the price } (\Phi^i)^{-1}(q). \]

The proof of the converse implication is analogous.

Theorem 20(1) implies that the utility set is independent of the market structure.

8 Mutual Funds and Short Sales Restrictions

Let the security payoffs \( \{x_n\} \subseteq X \) be such that the asset span \( M \) is a lattice-subspace of \( X \), and that the positive cone \( M^+ \) has a Yudin basis. By Theorem 17 these properties guarantee the existence of optimal portfolio allocations and the existence of portfolio equilibria.

Let \( \{f_n\} \) be the Yudin basis of \( M^+ \). Each payoff \( f_n \) is the payoff of some portfolio \( \eta^n \in \phi \), i.e., \( f_n = R(\eta^n) \). Note that portfolio \( \eta^n \) may involve short (i.e., negative) position in some securities. The portfolios \( \eta^n \) can be thought of as being mutual funds. Since the market structure \( \{f_n\} \) is equivalent to the market structure \( \{x_n\} \), trading in mutual funds provides the same spanning opportunities as trading in original securities. The invariance results of Section 7 do hold for these two market structures. In particular, there exists a portfolio equilibrium if and only if there exists an equilibrium in the mutual funds' markets.

Let \( \geq_f \) be the portfolio dominance order associated with the market structure \( \{f_n\} \), where, of course

\[ \lambda \geq_f \lambda' \text{ if } \sum_{n=1}^{\infty} \lambda_n f_n \geq \sum_{n=1}^{\infty} \lambda'_n f_n. \]

By the definition of Yudin basis, the portfolio dominance order \( \geq_f \) coincides with the standard pointwise order \( \geq \) so that

\[ \lambda \geq_f \lambda' \iff \lambda_n \geq \lambda'_n \text{ for each } n. \]

Consequently, the cone \( \phi^+_f \) of portfolios of mutual funds with positive payoffs equals the standard positive cone \( \phi^+ \). A portfolio of mutual funds has a positive payoff if and only if the share-holding of each fund is positive. The restriction of positive wealth is therefore equivalent to the restriction of no short sales of mutual funds. For an investor who plans to have positive consumption, the restriction of no short sales of mutual funds is nonbinding.

Finally, the condition of the existence of a Yudin basis of the positive cone \( M^+ \) which played a crucial role in our analysis in Sections 5 and 6 can be given the following simple interpretation: There exist mutual funds such that the restriction of no short sales of mutual funds is nonbinding.
9 The Portfolio-Consumption Duality

Our analysis of security markets thus far has been focused on portfolio allocations and security prices. Portfolio equilibria and optimal portfolio allocations have their counterparts in the payoff space. There is a simple duality between portfolios and consumption plans, and between security prices and consumption prices. This duality is the subject of this section.

The payoff operator $R$ maps the portfolio space $\phi$ into the payoff space $X$, and its image $R(\phi)$ is the asset span $M \subseteq X$. The asset span $M$ is a vector space with positive cone $M^+ = M \cap X^+$. We equip the space $M$ with its inductive limit topology. Every linear functional on $M$ is continuous in the inductive limit topology, i.e., the topological dual of $M$ coincides with the algebraic dual. Let $M'$ be the space of all linear functionals on $M$. Each functional $p$ in $M'$ is a consumption price system. The value of consumption $y \in M$ under the price $p$ is $p \cdot y$.

The dual system $(M, M')$ is the consumption-price duality. Between the portfolio-price dual system and the consumption-price dual system we have the payoff operator that maps portfolios into constrained consumptions, and the adjoint payoff operator that maps consumption prices into security prices. The payoff operator is continuous in the inductive limit topologies of $\phi$ and $M$, and therefore the adjoint payoff operator $R'$ is well-defined.

A **constrained consumption allocation** is any vector $(y_1, \ldots, y_m)$ with $y_i \in M^+$ for each $i$, and $\sum_{i=1}^m y_i = \bar{x} = R(\bar{\theta})$.

**Definition 21** A constrained consumption allocation $(y_1, \ldots, y_m)$ is said to be a **constrained consumption equilibrium**, if there exists a non-zero price system $p \in M'$ such that each $y_i$ maximizes $u_i(x)$ subject to $x \in M^+$ and $p \cdot x \leq p \cdot R(\bar{\theta})$.

The portfolio equilibria are in duality with the constrained consumption equilibria.

**Proposition 22** We have the following:

1. If $(y_1, \ldots, y_m; p)$ is a constrained consumption equilibrium, then
   $$(R^{-1}(y_1), \ldots, R^{-1}(y_m); R'(p))$$
   is a portfolio equilibrium.

2. If $(\theta_1, \ldots, \theta_m; R'(p))$ is a portfolio equilibrium, then $(R(\theta_1), \ldots, R(\theta_m); p)$ is a consumption equilibrium.

**Proof:** (1) Let $(y_1, \ldots, y_m)$ be a constrained consumption equilibrium supported by an equilibrium price $p \in M'$. So, we have $y_i \in M^+$, $p \cdot y_i \leq p \cdot R(\bar{\theta})$ for each $i$, and

$$u_i(y_i) = \max\{u_i(x) : x \in M^+ \text{ and } p \cdot x \leq p \cdot R(\bar{\theta})\}.$$
Clearly, $\theta^i = R^{-1}(y_i) \in \phi$ is a feasible portfolio for each investor $i$. Let $q = R'(p) \in \mathbb{R}^\infty$. From the duality identity $R'(p) \cdot \theta = p \cdot R(\theta)$ and the facts that $R$ is onto and $p \neq 0$, we see that $q = R'(p) \neq 0$.

The identity $\sum_{i=1}^m \theta^i = \sum_{i=1}^m R^{-1}(y_i) = R^{-1}(\sum_{i=1}^m y_i) = R^{-1}(\bar{x}) = \bar{\theta}$ guarantees that $(\theta^1, \ldots, \theta^m)$ is a portfolio allocation. Now if $q \cdot \theta \leq q \cdot \dot{\theta}^i$, then $p \cdot R(\theta) \leq p \cdot R(\dot{\theta}^i) = p \cdot y^i$. Therefore, $\dot{u}_i(\theta) = u_i(R(\theta)) \leq u_i(y^i) = \dot{u}_i(\dot{\theta}^i)$, and $\dot{\theta}^i$ maximizes the indirect utility $\dot{u}^i$ in the portfolio budet set $B_i(q)$.

(2) Assume that $((\theta^1, \ldots, \theta^m); R'(p))$ is a portfolio equilibrium, where $p \in \mathbb{R}^\infty$. Let $y_i = R(\theta^i)$. Clearly, $(y_1, \ldots, y_m)$ is a constrained consumption allocation. If $p \cdot x \leq p \cdot y_i$ for some $x \in M^+$, then $q \cdot R^{-1}(x) \leq q \cdot \theta^i$. Therefore, $u_i(x) = \dot{u}_i(R^{-1}(x)) \leq \dot{u}_i(R^{-1}(y_i)) = u_i(\theta^i)$, and $y_i$ maximizes utility $u_i$ in the consumption budget set.

A constrained optimal consumption allocation is defined as follows.

**Definition 23** A constrained consumption allocation $(y_1, \ldots, y_m)$ is said to be **constrained optimal**, if there is no other constrained consumption allocation $(z_1, \ldots, z_m)$ satisfying $u_i(z_i) \geq u_i(y_i)$ for every $i$ and $u_i(z_i) > u_i(y_i)$ for at least one $i$.

Of course, a constrained optimal allocation need not be Pareto optimal among all (unrestricted) consumption allocations. A Pareto optimal consumption allocation is defined, as usual, as follows.

**Definition 24** A consumption allocation $(y_1, \ldots, y_m)$ is **Pareto optimal**, if there is no other consumption allocation $(z_1, \ldots, z_m)$ satisfying $u_i(z_i) \geq u_i(y_i)$ for every $i$ and $u_i(z_i) > u_i(y_i)$ for at least one $i$.

The term “consumption allocation” in Definition 24 means any vector $(y_1, \ldots, y_m)$ satisfying $y_i \in X^+$ for each $i$ and $\sum_{i=1}^m y_i = x = R(\bar{\theta})$.

It is easily seen that optimal the portfolio allocations are in duality with the constrained consumption allocations.

**Proposition 25** A constrained consumption allocation $(y_1, \ldots, y_m)$ is constrained optimal if and only if the portfolio allocation $(\theta^1, \ldots, \theta^m)$, where $R(\theta^i) = y_i$ for every $i$, is an optimal portfolio allocation.

**References**


