ON A THEOREM OF KOCH

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We give a short proof of a slightly stronger version of a
theorem of Koch: A complex quadratic field whose ideal class
group contains a subgroup of type (4,4,4) possesses an infinite
unramified Galois pro-2 extension.

1. Koch's Theorem.

If $K$ is a finite extension of $\mathbb{Q}$ and $p$ is a prime number, let $K^{(0)} = K$ and
for $n \geq 1$ define $K^{(n)}$ to be the maximal abelian unramified $p$-extension of
$K^{(n-1)}$. The smallest $n$ such that $K^{(n)} = K^{(n+1)}$ is called the length of the
$p$-class field tower of $K$; if no such integer $n$ exists, we say that $K$ has infinite
$p$-class field tower. By a group of type $(m_1, \ldots, m_t)$ we understand a group
isomorphic to $\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_t\mathbb{Z}$. The purpose of this note is to give a
short proof of (a slightly strengthened version of) a theorem of Koch [4]:

**Theorem 1.** If $K$ is a complex quadratic field whose ideal class group
contains a subgroup of type $(4,4,4)$, then the 2-class field tower of $K$ is
infinite.

Koch's proof proceeds by showing that in a minimal presentation of the
Galois group of the maximal unramified 2-extension of $K$ by a free pro-2
group $G$, the relations lie deep in the Zassenhaus filtration of $G$. We replace
this key ingredient of his proof, which can be thought of as the study of the
quadratic unramified extensions of the genus field of $K$ which are central over
$K$ [3, Satz 1], with a simple result from genus theory. Moreover, Koch's
proof requires a generalization of the Vinberg/Gashûtz sharpening of the
Golod-Shafarevich theorem on the structure of pro-$p$ groups [4, Satz 3]; for
our proof, the original Vinberg/Gashûtz inequality suffices (for an account
of these inequalities, see, e.g., Koch's book [5]). Indeed, we will need only
the following result (see Martinet [8]):

**Theorem 2.** Suppose $F$ is a totally real field of degree $n$, and $E$ is a totally
complex quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$
which ramify in $E$. The 2-rank of the ideal class group of $E$ is at least $t - 1$.
If

$$t \geq 3 + 2\sqrt{n + 1},$$

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then the 2-class field tower of \( E \) is infinite.

**Corollary 3.** Suppose \( F \) is a totally real degree 4 extension of \( \mathbb{Q} \). If two rational primes that split completely in \( F \) ramify in a complex quadratic field \( L \), then \( E = FL \) has an infinite 2-class field tower.

**Proof.** With notation as in the theorem, we have \( t \geq 8 \geq 3 + 2\sqrt{4+1} \).

**Proof of Theorem 1.** We know that at least four primes divide the discriminant \( D \) of \( K \). If six or more primes divide \( D \), then an application of Theorem 2 to \( K/\mathbb{Q} \) already yields the result. Assume first that exactly four primes divide \( D \). By the criterion of Rédei-Reichardt [9] on the 4-rank of the class group of \( K \), one knows that \( D = -p_1 \cdot p_2 \cdot p_3 \cdot p_4 \) where \( p_2, p_3, p_4 \) are odd primes satisfying \( \left( \frac{p_i}{p_j} \right) = +1 \) for \( i, j > 1, i \neq j \), and one of the following is satisfied:

1. \( p_1 = 8; p_j \equiv 1 \pmod{8}, j = 2, 3, 4 \).
2. \( p_1 = 8; p_2 \equiv 7 \pmod{8}; p_j \equiv 1 \pmod{8}, j = 3, 4 \).
3. \( p_1 \equiv 3 \pmod{4} \) is an odd prime, \( p_j \equiv 1 \pmod{4}, j = 2, 3, 4 \), and \( \left( \frac{p_i}{p_j} \right) = +1 \) for \( j = 2, 3, 4 \).

Incidentally, Koch’s theorem was originally stated for case (IV) only. Let \( F = \mathbb{Q}\left( \sqrt{p_3}, \sqrt{p_4} \right) \) and \( E = F\left( \sqrt{-p_1 \cdot p_2} \right) \). In all cases, \( (p_2) \) and the unique rational prime divisor of \( (p_1) \) split completely in \( F \). Hence, by Corollary 3, \( E \) has an infinite 2-class field tower. Since \( E/K \) is an unramified 2-extension, \( K \) has an infinite 2-class field tower as well. Now suppose exactly five primes \( p_1, \ldots, p_5 \) divide the discriminant of \( K \); using the Rédei-Reichardt criterion [9], or its equivalent form due to Rédei [10], it is straightforward to check that for some \( i, 1 \leq i \leq 5 \), we have

\[
p_i \equiv 1 \pmod{4}, \quad \left( \frac{p_i}{p_j} \right) = 1, j \neq i.
\]

Now let \( F = \mathbb{Q}\left( \sqrt{p_i} \right), E = K\left( \sqrt{p_i} \right); E/F \) is a CM-extension with 8 ramified primes. By Theorem 2, \( E \) has an infinite 2-class field tower, and so does \( K \).

**2. Further Remarks.**

Koch and Venkov [6] have proved that a complex quadratic field whose ideal class group has a subgroup of type \((p,p,p)\) for some odd prime \( p \) has an
infinite $p$-class field tower. Therefore, a complex quadratic field possesses an infinite Hilbert class field tower whenever its ideal class group contains a subgroup of type $(m, m, m)$ with $m \geq 3$. On the other hand, the field $\mathbb{Q}\left(\sqrt{-105}\right)$, whose ideal class group is of type $(2, 2, 2)$, has a finite class field tower, since its root discriminant is just below the Odlyzko bound (see e.g. [8]). I am indebted to the referee for the above remark.

Note that the proof of Koch's theorem we have given relies only on the existence of two primes that split completely in a real biquadratic field. For instance, the primes 31, 89 split completely in $\mathbb{Q}\left(\sqrt{2}, \sqrt{5}\right)$, hence $\mathbb{Q}\left(\sqrt{-2 \cdot 5 \cdot 31 \cdot 89}\right)$ has an infinite 2-class field tower; its 2-ideal class group is of type $(4, 2, 2)$.

Taussky-Todd [12] proved that a number field with 2-ideal class group of type $(2, 2)$ has a finite 2-class field tower of length at most 2. It is natural to ask whether there are number fields with infinite 2-class field tower whose 2-class group is of type $(4, 2)$ or $(2, 2, 2)$ (simplest non-cyclic 2-groups after type $(2, 2)$). Using a minor variation on an idea first introduced by Schoof [11], we now show that there are complex quadratic fields with these properties. Consider, for example, $K = \mathbb{Q}\left(\sqrt{-5 \cdot 7 \cdot 41 \cdot 61}\right)$, which has 2-ideal class group of type $(2, 2, 2)$. To show that this field has infinite 2-tower, let $H_0$ be the Hilbert class field of $K_0 = \mathbb{Q}\left(\sqrt{5 \cdot 41 \cdot 61}\right)$, a real quadratic field with class number 16. Since 7 is inert in $K_0$, it splits into 16 prime ideals in $H_0$, all of which ramify in the CM extension $L = H_0\left(\sqrt{-7}\right)$. Theorem 2 shows that $L$, an unramified 2-extension of $K$, has an infinite 2-class field tower, proving the claim. In fact, for any prime $q$ satisfying $q \equiv 7 \pmod{5 \cdot 41 \cdot 61}$ (there are infinitely many such primes by Dirichlet's theorem), the same argument shows that $K_q = \mathbb{Q}\left(\sqrt{-5 \cdot 41 \cdot 61 \cdot q}\right)$ has infinite 2-class field tower; furthermore, by Rédei-Reichardt, $K_q$ has 2-class group of type $(2, 2, 2)$.

For the second example, let $K = \mathbb{Q}\left(\sqrt{-5 \cdot 11 \cdot 461}\right)$; this field has 2-ideal class group of type $(4, 2)$. Observe that the rational prime ideal $(11)$ splits into 16 prime ideals in $H_0$, the Hilbert class field of the real quadratic field $K_0 = \mathbb{Q}\left(\sqrt{5 \cdot 461}\right)$ with class number 16. Therefore, by the same argument as above, $L = H_0\left(\sqrt{-11}\right)$, and thereby $K$, have infinite 2-class field tower. Let $H$ be the 2-Hilbert class field of $K$. Benjamin [1] has shown that the 2-class field tower of a complex quadratic field $E$ with 2-class group of type $(4, 2)$ has length at most 2 if the 2-Hilbert class field of $E$ has elementary 2-class group ($E = \mathbb{Q}\left(\sqrt{-5 \cdot 13}\right)$ is an example). Since $K$ has infinite 2-tower, we conclude that $H$ does not have elementary abelian 2-class group.
Finally, note that the 2-rank of the ideal class group of $L$ is at least 15. Using Louboutin [7], we compute the 2-rank of the ideal class group of the biquadratic field $E = \mathbb{Q}\left(\sqrt{-11}, \sqrt{5 \cdot 461}\right)$ to be 2. The arguments of [2] then show that the 2-rank of the ideal class group of $L$ is 15, 16 or 17.

References


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