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Viscosity, ion mobility, and the λ transition*

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A model is presented of the λ transition in superfluid helium in which fluctuations near the transition are approximated by distinct regions of normal fluid and superfluid. The macroscopic viscosity of such a medium is computed. The ion mobility is also computed, taking into account a region of normal fluid around the ion induced by electrostriction. The results are, for the viscosity, $\eta_\lambda - \eta \sim t^{0.67}$ and for the mobility $\mu - \mu_\lambda \sim t^{0.92}$, both in excellent agreement with recent experiments. The model suggests that the λ transition itself is the point at which superfluid regions become macroscopically connected.

The λ transition in liquid helium has been the subject of intense scrutiny in recent years because the very precise measurements possible in that medium have made it an important test case for predictions based on the scaling hypothesis. Scaling and related arguments have been successfully applied principally to static properties such as the heat capacity and the superfluid fraction ρ_s/ρ.[1] In this paper, an argument is presented which accounts for recently reported singular behavior in the shear viscosity and in the mobility of ions.

The observed ion mobilities μ on the superfluid side of the transition have been reported[2] to approach their values at the transition μ_λ according to the law

$$(\mu - \mu_\lambda)/\mu_\lambda = at^{\rho'}, \quad t = |(T_\lambda - T)/T_\lambda| , \tag{1}$$

with $\rho' = 0.94 \pm 0.02$. The coefficient $a \approx 12$ at saturated vapor pressure (SVP) and increases with pressure along the λ line. The viscosity η has been measured independently by two groups, one using an oscillating cylinder technique,[3] the other a vibrating wire.[4] The two groups report somewhat different values of the critical exponent, but a careful comparison of the published data show they are in excellent agreement where they overlap. The oscillating cylinder results extend closer to the transition, and so probably give the more correct asymptotic behavior. The results are reported[3] in the form

$$(\eta_\lambda - \eta)/\eta_\lambda = bt^{x'} , \tag{2}$$

where $x' = 0.65 \pm 0.03$. Both quantities μ and η remain finite at the transition, but pass through it with infinite slope.

In the general vicinity of T_λ, μ and η may be related to each other by way of Stokes law for the drag on a sphere in a viscous medium. Thus, the two measurements cited above may be taken to mean that in the asymptotic region the effective viscosity one measures depends on the size of the measuring probe. Moreover, the large-scale viscosity [Eq. (2)] evidently has a singular part which is proportional to the superfluid fraction ρ_s/ρ since ρ_s/ρ goes to zero with an exponent approximately equal to $\frac{2}{3}$.[1] These observations taken together suggest a novel interpretation of the behavior of helium at the λ transition.

Many years ago, Einstein showed that a fluid containing a suspension of hard spheres would have an effective viscosity that depended on the volume fraction occupied by the spheres. His result may be generalized to show that for small concentrations the viscosity of one fluid suspended in another will be close to the volume average of the two viscosities. Thus, the results in Eq. (2) suggest, crudely speaking, that the helium in the asymptotic region divides into separate superfluid and normal fluid parts. The difference between the viscosity and the mobility measurements may be accounted for if the division takes place on a scale that is small compared to the size of the viscosity measuring apparatus, but large compared to the ion. In this paper we would like to show that such a model does indeed lead to the observed results, not only qualitatively, but quantitatively as well.[5]

Critical phenomena, of which the λ transition is an example, are generally considered to be governed by local fluctuations which may be correlated over increasingly large distances as the transition is approached. In fact ξ, the correlation length diverges at the transition, depending on t according to

$$\xi = \xi_0 t^{-\nu} , \tag{3}$$

where, for helium, $\xi_0 \approx 1$ Å and $\nu \approx \frac{2}{3}$. It is important to remember, however, that ξ is the largest distance over which fluctuations are correlated at any t. Fluctuations also occur on all scales smaller than ξ, down to atomic dimensions.

We would like to suggest that in the case of helium the fluctuating quantities are the local values of the superfluid and normal fluid densities. In particular, we imagine that for $T < T_\lambda$, but close to the transition, the excitations which form the normal fluid component tend to agglomerate together into something like droplets, leaving behind regions rich in superfluid.[5] Within each agglomeration there is no quantum phase coherence; the fluid is simply normal. The largest agglomerates will have dimension $\sim\xi$. The existence of smaller fluctuations means not only that there are smaller patches of normal fluid, but also that within the normal regions there will be smaller inclusions of superfluid within those still smaller inclusions of normal, and so on, down to the dimensions of individual rotons. Viewed from $T > T_\lambda$, the situation is reversed; in a background of normal fluid, there are inclusions of superfluid on scales up to ξ. Notice that in this picture the λ transition approached from below is the point at which the superfluid regions lose their macroscopic connectivity. It is at that point that information about the quantum phase is no longer transmitted over large distances, and large scale superflow can therefore no longer take place. On the other hand, below T_λ, any experiment where the characteristic dimension is large compared to ξ will not detect the agglomerations of normal fluid, and hence will obey the conventional two fluid model in which the fluids are homogeneously mixed.

The explanations we wish to present for the behavior of η and μ rest upon detailed hydrodynamic calculations, i.e., solutions of the Navier-Stokes equations. These equations would be intractable for the complex inhomogeneous fluid we have described above. As we shall see, however, the leading order contributions to the singular parts of η and μ may be attributed to the influence of the largest-scale fluctations, those whose dimensions are of order ξ. This allows us to simplify the problem by presenting a heuristic model simple enough that calculations may be performed. Like many such models, we can expect it to become invalid sufficiently close to the transition, and we shall comment below on its range of validity.

The model is that, below T_λ, we have normal regions of dimension ξ embedded in a background of connected superfluid. Above T_λ, the situation is reversed, superfluid regions of dimension ξ embedded in a background of normal fluid. Each type of region is taken to be internally homogeneous, and to have a viscosity which is finite at the λ transition. The superfluid part has nonzero viscosity since it includes the average effects of smaller normal inclusions. If the two types of regions have viscosities η_i and occupy volume fractions x_i, the macroscopic viscosity of the medium will be given, aside from coefficients unimportant for our purposes, by

$$\eta = \sum_i x_i \eta_i \,. \tag{4}$$

Specific calculations that give essentially this form are discussed in Appendix A.

As the transition is approached from below, the normal regions grow, cutting off and isolating regions of superfluid, thus driving the volume fraction of connected superfluid to zero. The largest isolated inclusions of superfluid are, of course, always smaller than the normal regions of dimensions ξ within which they are included, and are therefore counted as part of the volume fraction occupied by the normal fluid. The remaining connected superfluid background, whose volume fraction we call x_s, is just the part of the medium that participates in large-scale superflow, and will thus be proportional to the measurable quantity ρ_s.

Passing through the transition, the correlation length having gone to infinity and retreated again, we find that those superfluid inclusions which were previously counted as part of the normal fraction are now the largest fluctuations, with dimensions of order ξ. We will call the volume fraction of isolated superfluid regions above the transition f. These superfluid fluctuations are included in the normal regions below T_λ, but they are not part of the normal fluid background above T_λ. For this reason, the viscosity attributed to the normal regions must be expected to have different values above and below the transition. Taking all of these considerations into account, we can rewrite Eq. (4) in a way that insures that the macroscopic viscosity η will be continuous at the transition:

$$\begin{aligned} &\eta = \eta_\lambda - x_s(\eta_\lambda - \eta_s), \quad T < T_\lambda \,, \\ &\eta = \eta_\lambda + (f_\lambda - f)(\eta_n - \eta_s), \quad T > T_\lambda \,, \end{aligned} \tag{5}$$

Here η_s is the viscosity of the super regions, η_n the viscosity of the normal regions above the transition (i.e., at $T > T_\lambda$), η_λ is the value of η at the transition and also its value in the normal regions belows the transition, and f_λ the value of f at the transition. Taking $x_s = \alpha\rho_s/\rho$, where α may depend on T, but is neither zero nor infinite at the transition, we have below the transition

$$(\eta_\lambda - \eta)/\eta_\lambda = \alpha[(\eta_\lambda - \eta_s)/\eta_\lambda]\rho_s/\rho \,. \tag{6}$$

Any missing coefficient in Eq. (4) may be absorbed into α. Equation (6) gives an excellent account of the experimental observations discussed above. Specifically [Eq. (2)] the experimental result may be written in the form

$$(\eta_\lambda - \eta)/\eta_\lambda = 0.53\,\rho_s/\rho \,.$$

Above the transition the viscosity is proportional to $f - f_\lambda$, the variation of the volume fraction of superfluid fluctuations, a quantity which does not seem to be measurable in any other way. The reported behavior [3,4] follows Eq. (2) with an exponent of 0.8.

We now turn to the ion mobility measurements, considering first the case $T < T_\lambda$. The mobilities of ions in the model result from the drag on a sphere in a viscous medium, but we must be careful to distinguish which of the viscosities we have introduced come into play. The ion in helium is an unshielded charge which has long-range electrostrictive effects, setting its own characteristic scales. In particular, as the charge is approached from far away, the local pressure P_0 rises above the applied pressure P according to[6]

$$P_0 - P = c_0/r^4 \ , \tag{7}$$

where r is the distance from the ion and c_0 depends on the polarizability of helium. At some distance R_0, P_0 becomes sufficiently large (in the case of positive ions) to cause the helium to freeze, so that the positive ion is basically a solid sphere ($R_0 \approx 6$ Å). Below T_λ, there is another length

$$R_\lambda = [c_0/(P_\lambda - P)]^{1/4} = ct^{-1/4} \ , \tag{8}$$

where $P_\lambda(T)$ is the λ pressure at the bath temperature, and c relates $P_\lambda - P$ to $T_\lambda - T$ by way of the slope of the λ line. At distances smaller than R_λ, the local pressure and temperature in the fluid always correspond to bulk helium above the λ transition. Although R_λ diverges as the transition is approached, it is always smaller than ξ. Using Eqs. (3) and (8), we have $(R_\lambda/\xi) \sim t^{0.42}$. At $t = 10^{-5}$, $(R_\lambda/\xi) \approx 3\times 10^{-2}$.[7]

When the ion is in an already normal region, the effect of electrostriction is to suppress whatever superfluid inclusions might be present. Thus an ion in a normal region, even below T_λ, will sense an effective viscosity η_n, and the mobility is given by the formula for Stokes drag,

$$\mu_n = e/6\pi\eta_n R_0 \ . \tag{9}$$

If, instead, the ion is in a part of the connected superfluid background, we have a rather more complicated hydrodynamic problem to solve. A hard sphere of radius R_0 is surrounded out to radius R_λ by fluid of viscosity η_n, and beyond R_λ by fluid of the viscosity η, the macroscopic average value. The Navier-Stokes equations can be solved analytically for this situation, using no slip boundary conditions at R_0 and requiring continuity of velocity and stress at R_λ. Details of the calculation are given in Appendix B. The result is a drag coefficient given by

$$\zeta = 8\pi\eta_n A \ , \tag{10}$$

where

$$A^{-1} = \frac{4}{3}\,\frac{R+q-1}{R} - \frac{10}{3}\,\frac{(R^2-1)^2(1-q)}{R^5(3q+2)+2(q-1)} \ , \tag{11}$$

with $q = \eta_n/\eta$ and $R = R_\lambda/R_0$. Noting that R_λ diverges at the transition we find near T_λ

$$\zeta = 6\pi\eta_n[1 - \varphi R_0/R_\lambda + O((R_0/R_\lambda)^3)] \ , \tag{12}$$

where

$$\varphi = \tfrac{3}{2}(q-1)(2q+3)/(3q+2) \ . \tag{13}$$

Then

$$\mu_s = e/\zeta R_0 = \mu_n(1 + \varphi R_0/R_\lambda) \ , \tag{14}$$

where μ_s is the mobility of an ion in the connected superfluid background.

The measured mobility of ions depends on the time τ required by an ion to traverse a path of length L (the size of the experimental cell) through the helium. When $L \gg \xi$, the ion will encounter a large number of normal regions randomly placed along its path. Since all paths through the fluid must on the average be equivalent, the probability that any given line segment is to be found in a normal region must be proportional to $1 - x_s$. Thus a portion of the path $(1-x_s)L$ will be spent in normal regions corresponding to a time $(1-x_s)L/v_n$, where v_n is the mean velocity of the ion in a normal region. Applying the same argument to the super-regions as well, we thus have

$$1/v \equiv \tau/L = (1-x_s)/v_n + x_s/v_s \ . \tag{15}$$

In the limit of small electric field E, the mobility is given by $v = \mu E$. Defining separate mobilities for the two regions by $\mu_i = v_i/E$, where $i = s, n$, we have

$$1/\mu = (1-x_s)/\mu_n + x_s/\mu_s \tag{16}$$

(Ref. 8), or, to leading order in x_s,

$$\mu = \mu_n[1 + x_s(\mu_s - \mu_n)/\mu_s] \ . \tag{17}$$

Defining μ_λ to be the value of μ at the transition, taking once again $x_s = \alpha\rho_s/\rho$, substituting Eqs. (9) and (14) into (17), we have to leading order in singular terms

$$(\mu - \mu_\lambda)/\mu_\lambda = \alpha(\rho_s/\rho)\varphi R_0/R_\lambda \ . \tag{18}$$

The singular factors on the right-hand side are $\rho_s/\rho \sim t^{2/3}$ (approximately) and $R_\lambda^{-1} \sim t^{1/4}$. Thus the predicted exponent in Eq. (1) is

$$\rho' = \tfrac{2}{3} + \tfrac{1}{4} \simeq 0.92 \ ,$$

in excellent agreement with the observed value of

0.94 ± 0.02. The parameters in this model, α, η_λ, η_n, and η_s can be chosen within reasonable limits to give the observed coefficients in Eqs. (1) and (2). This result for ρ' and Eq. (6) are the principal results of the model.

As pointed out earlier, this model is expected to break down sufficiently close to the transition, at least insofar as its application to ion mobilities is concerned. The reason is that we have had to make a distinction between the behavior of the ions in isolated superfluid regions and the behavior in connected superfluid regions. That distinction must become invalid when ξ becomes sufficiently large. We have assumed that electrostriction entirely suppresses isolated superfluid patches below T_λ, where their size is necessarily small compared to ξ, whereas in the connected superfluid background the effect of electrostriction is to induce a normal region limited to radius R_λ. When ξ becomes very large, it will be possible to have inclusions of superfluid which, still smaller than ξ, are nevertheless large compared to R_λ. Then in the isolated super patches, the mobility will be larger than μ_n owing to the presence of unsuppressed superfluid far from the ion (farther than R_λ, but less than the size of the inclusion, which in turn is small compared to ξ). The same will be true qualitatively of the ions just above the transition, although no characteristic scale R_λ then exists in terms of which the effect can be discussed.

The magnitude of the effect that this phenomenon has on μ is difficult to determine, since fluctuations smaller than ξ have no other characteristic scale. However, it is possible to make a rough estimate of the temperature at which μ should depart from the prediction, Eq. (18). It is necessary to have a substantial probability of superfluid inclusions of dimension d, which satisfies $R_\lambda \ll d \ll \xi$. Thus we might expect the prediction to start to break down when R_λ is, say, roughly two orders of magnitude smaller than ξ. As we have seen earlier, that requires t smaller than 10^{-5}. The result reported in Eq. (1) is based on data for $t \gtrsim 10^{-4}$, where the present model should be valid.

To conclude then: recent measurements have indicated that the asymptotic behavior of flow dissipation near the superfluid phase transition depends on whether the measuring probe is microscopic or macroscopic. We have shown that the observed results could be accounted for by means of a model in which attention is directed to long ranged fluctuations in the normal and superfluid densities. The success of the model suggests that the thermodynamic λ transition is a transition in the connectivity of the superfluid regions.

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APPENDIX A

We wish to find the effective large-scale viscosity of a fluid which is in fact inhomogeneous, having a background viscosity η_0, but including a small volume fraction x_1 of regions where the viscosity is η_1. The equations of motion for steady flow anywhere in the fluid (assumed uniform density and incompressible) are[9]

$$\nabla \cdot \vec{v} = 0 , \tag{A1}$$

$$-\nabla P + \nabla \cdot (\eta \overleftrightarrow{E}) = 0 , \tag{A2}$$

where η is the viscosity, $\vec{v}$ is the velocity field, P the pressure, and $\overleftrightarrow{E}$ is a second-rank tensor whose components are

$$E_{ik} = \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} , \tag{A3}$$

where the x_i are Cartesian coordinates and the v_i are corresponding components of the velocity. For the case we are interested in, the fluid is inhomogeneous (i.e., η is not uniform) on a small scale, but on a larger scale we expect to recover Eqs. (A2) and (A3) with P, η, and E_{ik} replaced by their volume averages, e.g.,

$$\overline{E}_{ik} = \frac{1}{V} \int E_{ik} dV , \tag{A4}$$

the volume V being large compared to the scale of the inhomogeneities. In particular, the stress tensor is

$$\sigma_{ik} = -P\delta_{ik} + \eta E_{ik} . \tag{A5}$$

Then when averaged over a sufficiently large volume, we should find

$$\overline{\sigma}_{ik} = -\overline{P}\delta_{ik} + \eta_{\text{eff}} \overline{E}_{ik} , \tag{A6}$$

where η_{eff}, the quantity we seek, is a constant defined by (A6), and the large-scale equations of motion replacing (A2) may be written

$$\frac{\partial \overline{\sigma}_{ik}}{\partial x_k} = 0 \tag{A7}$$

(the summation convention for repeated indices is observed throughout).

Clearly, $\overline{\sigma}_{ik}$ may be written

$$\overline{\sigma}_{ik} = -\overline{P}\delta_{ik} + \eta_0 \overline{E}_{ik} + \frac{1}{V} \int (\sigma_{ik} - \eta_0 E_{ik} + P\delta_{ik})\, dV . \tag{A8}$$

The integrand is zero in the background region, so the last term in Eq. (A8) gives the contribution of that small fraction of the fluid whose local viscosity is η_1.

We now consider a velocity field

$$v_l = \alpha_{lk} x_k \ , \tag{A9}$$

where α_{lk} is a constant, symmetric tensor. Substitution in Eq. (A1) gives $\alpha_{ii}=0$, and in (A2) gives the corresponding pressure $P_0=\text{const}$ in any region of uniform viscosity. We will take this to be the unperturbed flow field. If we imagine the flow field remains unperturbed by the presence of regions of viscosity η_1, we find immediately on substitution into (A8)

$$\bar{\sigma}_{ik} = -\bar{P}\delta_{ik} + \eta_{\text{eff}}\bar{E}_{ik} = -P_0\delta_{ik} + 2\eta_{\text{eff}}\alpha_{ik} \ , \tag{A10}$$

with

$$\eta_{\text{eff}} = \eta_0 + (\eta_1 - \eta_0)x_1 \ , \tag{A11}$$

where $x_1 = V_1/V$ and V_1 is the volume occupied by regions of viscosity η_1. Equation (A11) is equivalent to Eq. (4) of the text. We wish, however, to investigate whether the perturbation of the flow owing to the inhomogeneities changes the essential result, that $\eta_{\text{eff}} - \eta_0$ is proportional to x_1.

Since x_1 is small, we can find the leading-order effect by assuming that each region of viscosity η_1 acts independently, later multiplying by the concentration of such regions. Assume an "inner" region, about the origin, of viscosity η_1, surrounded to infinity by a fluid of viscosity η_0. We can take the inner region to be spherical for convenience, arguing that we are interested in the time-average behavior of an ensemble of randomly shaped fluctuations, but that is not essential to the argument. The velocity field in the inner region will be

$$u_l = v_l + u_{1l} \quad \text{(inner region)} \ , \tag{A12}$$

and in the outer region

$$u_l = v_l + u_{0l} \quad \text{(outer region)} \ . \tag{A13}$$

Here u_{0l} must vanish at infinity, u_{1l} must remain finite at the origin, and both must obey Eqs. (A1)–(A3) in their own regions, each depending parametrically on the tensor α_{ik}. It is easy to verify by direct substitution that the required solutions are

$$u_{1l} = 48\,dr^3\alpha_{ik}n_i n_k n_l - (24cr + 120\,dr^3)\alpha_{lk}n_k \ ,$$

$$P = P_0 - \eta_1 504\,dr^2\alpha_{ik}n_i n_k \quad \text{(inner region)} \tag{A14}$$

and

$$u_{0l} = (3a/r^2 - 15b/r^4)\alpha_{ik}n_i n_k n_l + 6(b/r^4)\alpha_{lk}n_k \ ,$$

$$P = P_0 + \eta_0(6a/r^3)\alpha_{ik}n_i n_k \quad \text{(outer region)} . \tag{A15}$$

Here r is the distance from the origin, n_i is the ith component of a unit vector directed along $\vec{r}$, and a, b, c, and d are constants to be evaluated by means of the boundary conditions at the interface between the two regions.

If the inner region has an irregular shape, the interface may not be stationary in time, the inner region conserving only its volume. However, we can see that the shape of the region is not important for our problem by the following argument. Using the equation of motion, $\partial\sigma_{ik}/\partial x_k = 0$ and the consequent identity $\sigma_{ik} = \partial(\sigma_{il}x_k)/\partial x_l$, the integral in Eq. (A8) may be converted to a surface integral to be evaluated at very large r (compared to the dimension of the inner region). Performing the integral, only the term proportional to a/r^2 from u_{0l} in Eq. (A15) will survive. The result is that the correction to $\bar{\sigma}_{ik}$ in Eq. (A8) is proportional to a/V. The constant a has the dimensions of a volume, and must therefore be of order of the only volume in the problem, the volume of the inner region. It follows then that the correction $\eta_{\text{eff}} - \eta_0$ will be proportional to $x_1 = V_1/V$.

To illustrate the point, let us complete the problem explicitly for the case where the inner region is a sphere of radius R. The boundary conditions at $r=R$ are that u_l and $n_i\sigma_{ik}$ be continuous (conditions which allow flow across the boundary). The resulting constants are

$$a = -[5(q-1)/3(3+2q)]R^3 \ ,$$

$$b = -[(q-1)/3(3+2q)]R^5 \ , \tag{A16}$$

$$c = [(q-1)/12(3+2q)] \ , \quad d = 0 \ ,$$

where we have written $q = \eta_1/\eta_0$. Performing the integral in Eq. (A8) using these results, we find, in accordance with Eq. (A10),

$$\bar{\sigma}_{ik} = -P_0\delta_{ik} + 2\eta_{\text{eff}}\alpha_{ik} \ ,$$

where

$$\eta_{\text{eff}} = \eta_0 + [5/(3+2q)](\eta_1 - \eta_0)x_1 \tag{A17}$$

(we have multiplied the correction term by the concentration of η_1 regions). As promised, this result differs from the equation used in the text only in the coefficient of x_1. Notice that if we let $\eta_1 \to \infty$ (so that $q \to \infty$) we recover Einstein's result for the effective viscosity of a suspension of hard spheres $\eta_{\text{eff}} = \eta_0(1 + \frac{5}{2}x_1)$.

APPENDIX B

In this case, we wish to compute the drag on a solid sphere of radius R_0 moving with velocity $\vec{v}$ in a fluid whose viscosity is $\eta = \eta_1$ for $R_0 < r < R_\lambda$ and $\eta = \eta_0$ for $R_\lambda < r$, flow being allowed across the boundary at R_λ. For steady incompressible flow the equations of motion are once again Eqs. (A1)–(A3). We consider a frame whose origin lies at the center of the sphere, and take the velocity at infinity to be $\vec{U}$ in the z direction. We look for solutions of the form

$$\vec{v} = \vec{a}_r \cos\theta v_a(r) + \vec{a}_\theta \sin\theta v_b(r) , \tag{B1}$$

$$P = p(r)\cos\theta , \tag{B2}$$

where r and θ are spherical polar coordinates and $\vec{a}_r$ and $\vec{a}_\theta$ the respective unit vectors. The problem is symmetric with respect to the azimuthal angle φ. For the components of E_{ik} we find[10]

$$E_{rr} = S_1(r)\cos\theta, \quad E_{\theta\theta} = E_{\varphi\varphi} = -\tfrac{1}{2}E_{rr} , \quad E_{r\theta} = S_2(r)\sin\theta , \tag{B3}$$

where

$$S_1(r) = 2\frac{\partial v_a}{\partial r} , \quad S_2(r) = \frac{\partial v_b}{\partial r} - \frac{v_a + v_b}{r} . \tag{B4}$$

Equation (A1) reduces to

$$\frac{\partial v_a}{\partial r} + \frac{2}{r}(v_a + v_b) = 0 . \tag{B5}$$

From the radial component of Eq. (A2) we have

$$-\frac{\partial p}{\partial r} + \frac{\partial(\eta S_1)}{\partial r} + \frac{3}{r}\eta S_1 + \frac{2}{r}\eta S_2 = 0 , \tag{B6}$$

and from the θ component,

$$\frac{p}{r} + \frac{\partial(\eta S_2)}{\partial r} + \frac{3}{r}\eta S_2 + \frac{\eta S_1}{2r} = 0 . \tag{B7}$$

The general solutions of Eqs. (7)–(10) are of the form

$$v_a = \alpha_1 + \alpha_2 r^{-1} + \alpha_3 r^{-3} + \alpha_4 r^2 ,$$

$$v_b = -\alpha_1 - \tfrac{1}{2}\alpha_2 r^{-1} + \tfrac{1}{2}\alpha_3 r^{-3} - 2\alpha_4 r^2 ,$$

where the α's are constants. Since for $r > R_\lambda$ the solutions must have the property that $v_a = U$ and $v_b = -U$ at $r = \infty$, we have[11]

$$v_a = U(1 - 2a/r + 2b/r^3) , \quad v_b = U(-1 + a/r + b/r^3), \quad r > R_\lambda . \tag{B8}$$

In the same region this yields

$$p = -2\eta_0(a/r^2)U, \quad S_1 = U(4a/r^2 - 12b/r^4), \quad S_2 = -(6b/r^4)U, \quad r > R_\lambda . \tag{B9}$$

For the inner solutions we have

$$v_a = (\Gamma - 2A/r + 2B/r^3 + Cr^2)U , \quad v_b = (-\Gamma + A/r + B/r^3 - 2Cr^2)U, \quad R_0 < r < R_\lambda , \tag{B10}$$

which yields

$$p = (-2\eta_1 A/r^2 + 10\eta_1 Cr)U , \quad S_1 = (4A/r^2 - 12B/r^4 + 4Cr)U , \quad S_2 = (-6B/r^4 - 3Cr)U, \quad R_0 < r < R_\lambda . \tag{B11}$$

We have in Eqs. (B8)–(B11) six unknowns to be determined by the boundary conditions at $r = R_0$ and at $r = R_\lambda$. Since there are no unbalanced forces in the fluid, the force on the sphere can be written in terms of the constants in Eq. (B8) alone[12]:

$$F = 8\pi a\eta_0 U . \tag{B12}$$

The problem thus reduces to choosing boundary conditions and eliminating the other constants in favor of a.

The boundary conditions at $r = R_\lambda$ are

$$\Delta v_a = 0, \quad \Delta v_b = 0 , \quad \Delta(-p + \eta S_1) = 0, \quad \Delta(\eta S_2) = 0, \quad r = R_\lambda . \tag{B13}$$

Of these, the first two conserve mass, while the last two assure, respectively, that the radial and tangential components of the force (i.e., the rr and $r\theta$ components of the stress tensor) are continuous. At $r = R_0$, assuming the sphere to be solid, we apply no-slip conditions

$$v_a = 0, \quad v_b = 0, \quad r = R_0, \quad \text{no-slip condition.} \tag{B14}$$

Equations (B13) and (B14) suffice to determine all of the constants. Writing $R = R_0/R_\lambda$ and $q = \eta_1/\eta_0$, we find, after tedious but straightforward algebra,

$$a = qAR_0 , \tag{B15}$$

with A as given in Eq. (11) of the text.

In the case of the negative ion in liquid helium, which is thought to be an electron bubble, one might wish to apply pure slip boundary conditions at $r = R_0$ instead of Eqs. (B14). In this case we have

$$v_a = 0, \quad S_2 = 0, \quad r = R_0, \quad \text{pure-slip condition.} \tag{B16}$$

The result that replaces Eq. (11) of the text is

$$A^{-1} = 2 + \frac{2(q-1)[(2q+3)R^4 - 2(q-1)/R]}{(3q+2)R^5 - 3(q-1)} . \tag{B17}$$

If this result is to be applied to the model in the text, the factors 6π in Eqs. (9) and (12) are replaced by 4π and the factor $\frac{3}{2}$ in Eq. (13) is replaced by 1. The argument is otherwise unchanged.

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[1]For review of experiments in liquid helium, see G. Ahlers, in *The Physics of Liquid and Solid Helium*, edited by K. H. Bennemann and J. P. Ketterson (Wiley, New York, 1976), Chap. VIII. More recent experimental results appear in K H. Mueller, G. Ahlers,

and F. Pobell, Phys. Rev. B 14, 2096 (1976). For scaling theory applied to helium, see M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A 8, 1111 (1973). The scaling hypothesis in general is discussed in H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University, New York, 1971); and D. L. Goodstein, *States of Matter* (Prentice-Hall, Englewood Cliffs, N. J., 1975). The latter book also discusses the application of scaling to the superfluid transition.

[2]D. Goodstein, A. Savoia, and F. Scaramuzzi, Phys. Rev. A 9, 2151 (1974).

[3]R. Biskeborn and R. W. Guernsey, Jr., Phys. Rev. Lett. 34, 455 (1975).

[4]L. Bruschi, G. Mazzi, M. Santini, and G. Torzo, J. Low Temp. Phys. 18, 487 (1975).

[5]A similar idea has recently been applied to calculate shear viscosity at gas-liquid and other critical points. See D. W. Oxtoby and H. Metiu, Phys. Rev. Lett. 36, 1092 (1976). Ideas along the same lines have also been applied to binary mixtures. See, D. Jasnow and E. Gerjuoy, Phys. Rev. A 11, 340 (1975).

[6]K. R. Atkins, Phys. Rev. 116, 1339 (1959).

[7]In an earlier (unsuccessful) attempt to predict the behavior of μ near T_λ, based on a modified Ginzburg-Pitaevskii (GP) theory, the effect of electrostriction was specifically excluded because, as noted here, $R_\lambda/\xi \ll 1$. See A. A. Sobyanin, Zh. Eksp. Teor. Fiz. 63, 1780 (1972) [Sov. Phys.-JETP 36, 941 (1973)]. However the GP theory fails near the transition precisely because it cannot account for changes in the order parameter on a scale smaller than ξ. The present model is similar to GP in that attention is focussed on the longest-ranged fluctuations, but we do not exclude the shorter-range variations produced by electrostriction.

[8]The averaging of conductivities for the case of binary metallic mixtures has been discussed by R. Landauer [J. Appl. Phys. 23, 779 (1952)]. In the present case (which differs from metals) the arguments we have given assume the time scale for changes in the fluctuations be large compared to the diffusion time of an ion acorss a fluctuation of dimension ξ. Jasnow and Gerjouy (Ref. 5) estimate the ratio of those times from mode-mode coupling theory. In our application, the ratio of times is ξ/R_λ, which is indeed always large compared to one.

[9]The treatment here follows the general outline of similar arguments given in L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Mass., 1959), see especially p. 76 et sec. Attention has been paid in the fluid dynamics literature to problems similar to this one, including such details as the deformation of droplets with finite surface tension, and the influence of droplets on each other at higher concentrations. However, the present case in which the droplets are essentially a different phase of the same matter as the background does not seem to have been considered. See S. J. Choi and W. R. Schowalter, Phys. Fluids 18, 420 (1975). The results are of sufficient complexity that a simple derivation specific to the problem at hand seemed justified here.

[10]For the equations of motion in spherical polar coordinates, see Landau and Lifshitz, Ref. 9, p. 52; or I. S. Sokolnikoff, *Mathematical Theory of Elasticity* (McGraw-Hill, New York, 1956), p. 184.

[11]See, for example, Landau and Lifshitz, Ref. 9, p. 65, Eq. (20.7).

[12]Landau and Lifshitz, Ref. 9, p. 66.