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AN ORTHOGONAL PROPERTY OF THE HYPERGEOMETRIC POLYNOMIAL

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Communicated July 21, 1942

1. Mittag-Leffler's polynomial $g_n(z)$ has the orthogonal property

$$\begin{aligned}\int_{-\infty}^{\infty} g_m(-ix)g_n(ix)dx/xsh(x) &= 0 \qquad m \neq n \qquad m > 0 \\ &= 2/n \qquad m = n \qquad n > 0. \qquad (1.1)\end{aligned}$$

This is readily obtained by inverting the integral representation[1]

$$g_n(ix) = (1/\pi)\sin(\pi x)\int_{-\infty}^{\infty} e^{ixu}(\tanh {}^1/_2u)^n du/sh\, u \qquad n \geqq 1 \quad (1.2)$$

by means of Fourier's inversion formula. The resulting equation

$$\text{cosech}\, u\,(\tanh {}^1/_2u)^n = {}^1/_2\int_{-\infty}^{\infty} e^{-iux}\,\text{cosech}\,(\pi x)g_n(ix)dx \quad (1.3)$$

then gives the desired relation when sh u e^{-iux} is expanded in powers of $\tanh {}^1/_2u$. With the notation of the hypergeometric function the orthogonal relation may be written in the form

$$\begin{aligned}\int_{-\infty}^{\infty} F(1-m, 1+ix;\ 2;\ 2)F(1-n, 1-ix;\ 2;\ 2)\times \\ xdx/sh(\pi x) &= 0 \qquad m \neq n \\ &= {}^1/_2n, \qquad m = n > 0. \qquad (1.4)\end{aligned}$$

2. A more general relation may be obtained by writing Euler's integral in the form

$$\begin{aligned}F(-n, ix; c; z)B(ix, c-ix) = \int_{-\infty}^{\infty} e^{ixu - {}^1/_2cu}(1 - {}^1/_2z - {}^1/_2z\tanh {}^1/_2u)^n \\ du/(2\text{ch}\,{}^1/_2u)^c \quad (2.1)\end{aligned}$$

and treating it in much the same way as the integral used to represent $g_n(ix)$. The result is that if $f_n(x) = F(-n, ix; c; z)$ where z is real

$$\begin{aligned}\int_{-\infty}^{\infty} B(ix, c-ix)(z-1)^{ix}f_m(x)f_n(x)dx &= 0 \qquad m \neq n \\ &= (-)^n n!/(c, n)(z-1)^{c+n}z^{-c} \qquad m = n \qquad (2.2)\end{aligned}$$

where $B(p, q)$ is the Beta function. This includes the relation

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \operatorname{sech}\left(\tfrac{1}{2}\pi x\right) w_m(x) w_n(x) dx = 0 \qquad m \neq n$$
$$= 1 \qquad m = n \qquad (2.3)$$

satisfied by the polynomial $w_n(x)$ defined by means of the generating function

$$(1 + t^2)^{-1/2} \exp(-\, x \arctan t) = \sum_{n=0}^{\infty} w_n(x) t^n. \qquad (2.4)$$

A polynomial with an orthogonal property for the weight function sech (πx) was discovered in 1940 by G. H. Hardy.[2] My attention to the polynomial $w_n(x)$ was called by a letter from B. R. Wicker of Loyola University, Los Angeles. He defined the polynomial in the first place by means of a definite integral equivalent to

$$i^n w_n(x) = (1/\pi) \int_{-\infty}^{\infty} e^{-ixu} \operatorname{sech} u \tanh^n u \, du \qquad (2.5)$$

and by a contour integral. From these he obtained the generating function and a recurrence relation

$$(n + 1) w_{n+1}(x) + n w_{n-1}(x) + x w_n(x) = 0. \qquad (2.6)$$

He also noted the existence of an orthogonal relation but did not see the relation between his polynomial and that of Hardy. Originally Hardy used the notation $q_n(x)$ and Wicker the notation $Q_n(x)$ but as both of these notations are used for Legendre functions of the second kind the notation $w_n(x)$ is preferable.

3. The polynomial $w_n(x)$ may be expressed in terms of my polynomial[3] $F_n(z)$ by means of the equation

$$i^n w_n(x) = \sum_{m=0}^{n} a_{nm} F_m(ix)$$

where

$$t^n = \sum_{m=0}^{n} a_{n,\,m} P_m(t). \qquad (3.1)$$

If $F_n^m(z)$ denotes Pasternack's polynomial[4] which is such that when $R(m) > -1$ and $|R(z)| < 1 + R(m)$

$$2^m B\left(\tfrac{1}{2}m + \tfrac{1}{2} + \tfrac{1}{2}iz,\ \tfrac{1}{2}m + \tfrac{1}{2} - \tfrac{1}{2}iz\right) F_n^m(iz) =$$
$$\int_{-\infty}^{\infty} e^{-ixz} P_n (\tanh x) \frac{dx}{\operatorname{ch}^{m+1} x} \qquad (3.2)$$

we may deduce the orthogonal relation

$$^1/_{2\pi}\int_{-\infty}^{\infty}\frac{F_n^m(iz)F_{n'}^{-m}(-iz)dz}{\mathrm{ch}(\pi z)+\cos(m\pi)} = 0 \qquad n' \neq n$$
$$= 1/(2n+1) \qquad n' = n. \quad (3.3)$$

When $m = 0$ we have $F_n^m(z) = F_n(z)$ and the foregoing relation gives the known orthogonal property of $F_n(z)$. When $m = 1$ the function $F_n^m(z)$ reduces to the function $E_n(z)$ for which an orthogonal relation has not yet been found. This polynomial $E_n(z)$ was defined by the operational equation

$$E_n(d/du)\,\mathrm{sech}^2 u = \mathrm{sech}^2 u P_n(\tanh u). \quad (3.4)$$

When $m = -{}^1/_2$ the orthogonal relation satisfied by $F_n(iz)$ is of the same type as that satisfied by $w_n(z)$ and, indeed, we have the relation

$$F_n^{-1/2}({}^1/_2 ix) = i^n w_n(x). \quad (3.5)$$

4. In the case of Rice's polynomial[5] $H_n(x, p, v)$ which is represented by the integral

$$H_n(x, p, v)B(x, p - x) = \int_{-\infty}^{\infty} e^{xt}(1 + e^t)^{-p} P_n\left[1 - \frac{2ve^t}{1 + e^t}\right]dt \quad (4.1)$$

the orthogonal relation seems to be

$$\int_{-\infty}^{\infty} H_n(iz, p, v)B(iz, p - iz)A_m(-iz, p, v)dz = 0 \qquad m \neq n$$
$$= 2\pi \qquad m = n \quad (4.2)$$

where $A_n(x, p, v)$ is defined by means of the expansion

$$\left(\frac{1-u}{u+2v-1}\right)^x\left(\frac{2v}{u+2v-1}\right)^p = \sum_{n=0}^{\infty} A_n(x, p, v)P_n(u). \quad (4.3)$$

It is readily found that

$$A_n(x, p, v) = (2n + 1)\left[\frac{1}{x+1}F(x + p, x + 1;\ x + 2;\ v^{-1}) - \right.$$
$$\{n(n + 1)/1!1!\}\frac{1}{x+2}F(x + p, x + 2;\ x + 3;\ v^{-1}) +$$
$$\{(n - 1)n(n + 1)(n + 2)/2!2!\}\frac{1}{x+3}F(x + p, x + 3;\ x + 4;$$
$$\left. v^{-1}) + \ldots\right]. \quad (4.4)$$

5. Hardy[6] has shown that the functions $W_n(x) = \dfrac{\sin \pi(x - n)}{(x - n)}$ form an orthogonal system for the range $- \infty : \infty$, and it is natural to ask whether the functions $b(x) = \operatorname{sech} x\, F_n\left(\dfrac{2ix}{\pi}\right) i^{-n}$ can be expressed as linear combinations of the functions $W(x)$ by means of the formula of interpolation

$$b_m(x) = \sum_{n=-\infty}^{\infty} W_n(x) b_m(n). \tag{5.1}$$

This formula has been tested numerically for the case $n = 0$, $x = {}^1/_2$ when it becomes

$$\pi . \operatorname{sech} x = \sin \pi x \left[\frac{1}{x} - \frac{2x}{x^2 - 1} \operatorname{sech} 1 + \frac{2x}{x^2 - 4} \operatorname{sech} 2 - \frac{2x}{x^2 - 9} \operatorname{sech} 3 + \frac{2x}{x^2 - 16} \operatorname{sech} 4 - \ldots\right] \tag{5.2}$$

Using the values
$$\begin{aligned} \operatorname{sech} 1 &= .64805\ 42734\ 04663\ 6 \\ \operatorname{sech} 2 &= .27111\ 82739\ 42365\ 7 \\ \operatorname{sech} 3 &= .09132\ 79274\ 19433\ 4 \\ \operatorname{sech} 4 &= .03661\ 89934\ 73686\ 5 \\ \operatorname{sech} 5 &= .01334\ 05293\ 99091\ 5 \end{aligned}$$

$$\operatorname{ch} {}^1/_2 = 1.12762\ 59652\ 06381, \quad \pi = 3.14159\ 26535\ 89793$$

derived from numbers in the British Association Tables, vol. 1, we find that $\pi \operatorname{sech} {}^1/_2 = 2.786\ldots$ while the right-hand side exceeds 2.791774 when only the first three terms are taken into consideration. The complete value is greater than this and so this numerical test indicates that the supposition (5.1) is false.

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