

Letters to the Editor

LETTERS to the Editor include two sections: *Research Notes and Comments and Errata*. Communications published in these sections should not exceed 1000 words in length including the space allowed for figures and tables. *Research Notes* include important research results of a preliminary nature which are of special interest to physics of fluids and new research contributions modifying or improving results published earlier in the scientific literature. *Comments and Errata* refer to papers published in The Physics of Fluids. The Board of Editors will not hold itself responsible for the opinions expressed in the Letters to the Editor.

Research Notes

Nearly Free Molecular Flow Through an Orifice

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THE problem of the flow through an orifice is a very interesting one in fluid mechanics, as it promises to be one of the few configurations which can be investigated over virtually the whole range of possible motions. For this reason Liepmann¹ has recently made measurements of the mass flow through an orifice at what are practically infinite pressure ratios, through a range of Knudsen numbers covering the transition from continuum to free molecule flow. The mass flow rate per unit area in the Knudsen limit (i.e., at high $K = \lambda_1/R$ where λ_1 is the mean free path at upstream infinity and R is the radius of the hole) is well known from kinetic theory to be $\dot{m} = \frac{1}{4}\rho_1\bar{c}_1$ where ρ_1 is the density and \bar{c}_1 the mean molecular speed at upstream infinity. The purpose of this note is to estimate the effect on \dot{m} of a Knudsen number K that is not so large.

In steady flow, and in the absence of any external forces, Boltzmann's equation for the distribution function f is

$$\begin{aligned} \mathbf{v} \cdot \nabla f &= \int f(\mathbf{v}')f(\mathbf{w}')gI \, d\Omega \, D\mathbf{w} \\ &- f(\mathbf{v}) \int f(\mathbf{w})gI \, d\Omega \, D\mathbf{w} \\ &= G(f) - fL(f), \end{aligned} \quad (1)$$

where \mathbf{v} is the velocity of the molecules, and G and L are the 'gain' and 'loss' operators on f . As the mass flow through the orifice (Fig. 1) is given by

$$\dot{m} = m \int_{z=0} (-v_z)f(\mathbf{v}) \, D\mathbf{v}, \quad (2)$$

where m is the mass of each molecule, our aim is to calculate the perturbation on the free-molecule distribution f^0 when $\epsilon = K^{-1} = R/\lambda_1$ is a small quantity but not zero. To tackle Eq. (1) directly seems a rather hopeless task, but we can write it along the center-line of the orifice as an ordinary differential equation

$$v_z(df/dz) = G(f) - fL(f). \quad (3)$$

We obtain an approximate solution f^1 of this equation by an iteration procedure suggested by Willis.² We replace $G(f)$ and $L(f)$ by $G(f^0)$ and $L(f^0)$, and solve Eq. (3) for f^1 at $z = 0$. This gives us, writing $v_a = -v_z$,

$$f'(z = 0) = \int_0^\infty \frac{G(f^0)}{v_a} \left\{ \exp - \int_0^{z'} \frac{L(f^0)}{v_a} dz'' \right\} dz'. \quad (4)$$

This integral is still too difficult to evaluate exactly, so we use the approximations for $G(f^0)$ and $L(f^0)$ suggested by Krook's model³:

$$\begin{aligned} L(f^0) &= An^0, \\ G(f^0) &= A(n^0)^2(\beta^0/\pi)^{\frac{3}{2}} \exp - \beta^0(\mathbf{v} - \mathbf{u})^2, \end{aligned} \quad (5)$$

where (\mathcal{R} being the gas constant)

$$\begin{aligned} n^0 &= \int f^0(\mathbf{v}) \, D\mathbf{v}, \quad \mathbf{u}^0 = \frac{1}{n^0} \int f^0(\mathbf{v})\mathbf{v} \, D\mathbf{v}, \\ T^0 &= \frac{1}{2\mathcal{R}\beta^0} = \frac{1}{3\mathcal{R}n^0} \int f^0(\mathbf{v})(\mathbf{v} - \mathbf{u})^2 \, D\mathbf{v}, \end{aligned} \quad (6)$$

are, respectively, the number density, the mean velocity, and the temperature in free molecule flow, and are known.

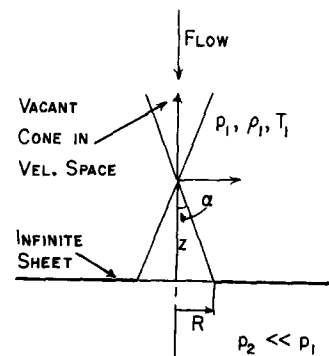


FIG. 1.

We choose $A = \bar{c}_1/(n_1\lambda_1)$ so that the number of collisions when the distribution is Maxwellian agrees with the number suggested by Krook's model. The distribution f^0 at any point is just a Maxwellian everywhere in velocity space except in the 'vacant cone' formed by the orifice at the point, so from Eq. (6)

$$n^0 = n_1 \cos^2(\alpha/2), \quad u_a^0 = -u_z^0 = \bar{c}_1 \sin^2(\alpha/2),$$

$$\beta^0 = \beta_1 \left[1 - \frac{2 \sin^4 \alpha}{3\pi(1 + \cos \alpha)^2} \right]^{-1}, \quad (7)$$

where $\tan \alpha = (R/z)$ (Fig. 1). Substituting these results into Eq. (4) we get after a little reduction

$$f^1(0) = \frac{\bar{c}_1}{\lambda_1} n_1 \left(\frac{\beta_1}{\pi} \right)^{\frac{3}{2}} \exp(-\beta_1 v^2)$$

$$\cdot \int_0^\infty g_1(z') g_2(z', \mathbf{v}) H(z', \mathbf{v}) dz'/v_a, \quad (8)$$

where

$$g_1(z') = \left(\frac{n^0}{n_1} \right)^2 \left(\frac{\beta^0}{\beta_1} \right)^{\frac{3}{2}} \exp(-\beta^0 u^2),$$

$$g_2(z', \mathbf{v}) = \exp \{ 2\beta^0 v_a u_a + (\beta_1 - \beta^0) v^2 \}, \quad (9)$$

$$H(z', \mathbf{v}) = \exp - \int_0^{z'} \frac{\bar{c}_1 n^0 dz''}{v_a n_1 \lambda_1}.$$

The functions g_1, g_2 rapidly approach unity as (z/R) increases, but H , the mean free path term, has a characteristic distance λ_1 . Hence we can write Eq. (8) correct to $O(\epsilon)$ as

$$f^1 = f_1(\bar{c}_1/\lambda_1 v_a) \left[R \int_0^\infty (g_1 g_2 - 1) d(z'/R) \right. \\ \left. + \lambda_1 \int_0^\infty H d(z'/\lambda_1) \right],$$

where f_1 is the (Maxwellian) distribution at infinity. The second integral above, to $O(\epsilon)$, can be shown to be

$$\int_0^\infty H d(z'/\lambda_1) = (v_a/\bar{c}_1) + \frac{1}{2}(R/\lambda_1),$$

and so we get

$$f^1 = f_1 \left[1 + \epsilon \left(\frac{\bar{c}_1}{v_a} \right) \left\{ \frac{1}{2} + \int_0^\infty (g_1 g_2 - 1) d\left(\frac{z'}{R}\right) \right\} \right]. \quad (10)$$

The correction to f is thus of the first order in ϵ , and there is a contribution from the variation of λ near the orifice which cannot be neglected. The integral in Eq. (10) has been numerically evaluated and gives for f^1 :

$$f^1 \simeq f_1 [1 + (8\epsilon/\pi V_a) \{ -0.490 + 0.398 V_a \\ + 0.032 V_a^2 + 0.004 V_a^3 + 0.00044 V_a^4 + \dots \}],$$

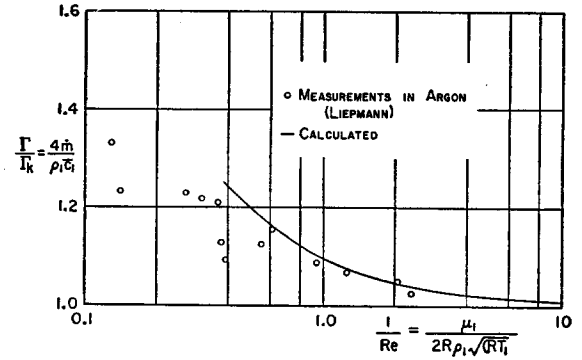


FIG. 2. Comparison of experimental data with the calculation, $Re \simeq 2.5 (R/\lambda_1)$.

$$V_a = 8v_a/\pi\bar{c}_1.$$

One may get a very crude estimate of the mass flow now by using Eq. (2), assuming that the distribution function, on other rays, has the same value as above. One should expect this result to be a little too high, as the radial component of the mean velocity is smaller at other positions than on the center-line. Carrying out the computation for what it is worth, we get

$$\dot{m} = (\rho_1 \bar{c}_1/4)(1 + 0.26R/\lambda_1).$$

Figure 2 shows this relation plotted with the experimental data of Liepmann.¹ The agreement is reasonable, but the accuracy of theory or measurements is not great enough for more refined comparisons.

Investigations into the problems of orifice flow are being continued, and it is hoped that a fuller account will be published in the near future.

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¹ H. W. Liepmann, "Gaskinetics and gasdynamics of orifice flow," First All-Union Congress on Theoretical and Applied Mechanics, Moscow, 1960.

² D. R. Willis, Princeton University Rept. 442.

³ P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).

Viscous Dissipation of Shallow Water Waves

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FOR a free surface gravity wave in a nonviscous incompressible fluid, the total energy is conserved and is proportional to the square of the wave amplitude a . In the case of a wave in shallow water, i.e., one whose wavelength λ is very much greater than the water depth h ,