Organized structures, memory, and the decay of turbulence

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The rapid increase in computational power has led to an unprecedented enhancement of our ability to study the behavior of complex systems in the physical, biological, and social sciences. However, there are still many systems that are too complex to tackle. A turbulent fluid is the archetypal example of such a complex system. Its complexity is manifested as the appearance of organized structures across all of the scales available to a turbulent fluid. Thus, the task that a numerical analyst working on turbulence faces is to reduce the complexity of the problem into something manageable, which at the same time preserves the essential features of the problem. Although much knowledge about the Euler and Navier–Stokes equations has accumulated over the years (1–8), it has proven very difficult to incorporate this knowledge in the construction of effective models. The work of Hald and Stinis (9) in this issue of PNAS is an attempt toward the construction of an effective model that utilizes qualitative information about the structure of a turbulent flow. The work in ref. 9 rests on the idea that the organization of a fluid flow in vortices leads to "long memory" effects, i.e., the motion of a vortex at one scale is influenced by the past history of the motion of vortices in other scales. This line of thought first appeared in the work of Alder and Wainwright (ref. 10; see also ref. 11 for a recent review on memory and problem reduction).

The incorporation of a long memory in the construction of an effective model leads to a simplified form of the equations for the model if the assumption of long memory is taken to the extreme, i.e., no separation of time scales between the scales resolved and those left unresolved is assumed. Of course, there is no proof yet that such an extreme assumption is valid, but the surprising accuracy of the model's predictions, presented in ref. 9, about the inviscid Burgers (see also ref. 12) and 2D Euler equations, where one knows what to expect of the solution, gives hope that the assumption is not misguided. The model, called the t-model by the authors (9), first appeared in previous work of Chorin *et al.* (13, 14) concerning the application of the Mori–Zwanzig formalism of irreversible statistical mechanics (15, 16) in the construction of dimensionally reduced models for complex systems of equations.

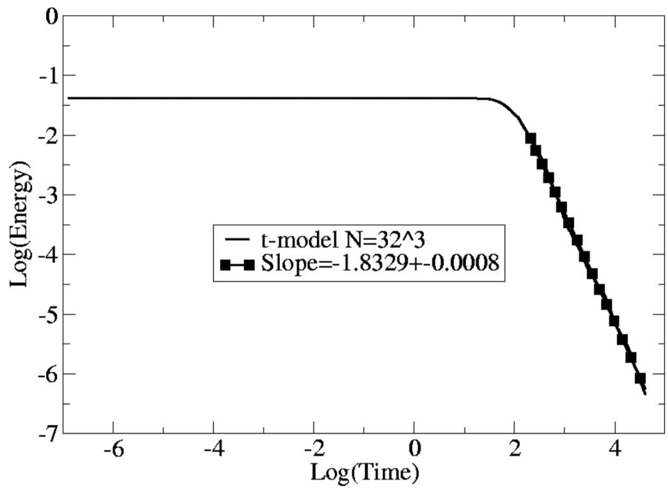

Fig. 1. Energy evolution of the t-model with $n = 32^3$ modes for the 3D Euler equations.

In ref. 9, the t-model is analyzed as a numerical method, and its convergence, in the case when the solutions of the original equations remain regular, is proven. A regular solution conserves its energy. The convergence theorem for the case of regular solutions means that, if one increases the resolution of the reduced model, it should, after enough variables are kept, predict a solution with a conserved energy. For the 3D Euler equations, it is not known whether the solutions behave badly enough to cause a finite-time blowup of the vorticity and even of the fluid velocity itself; in other words, it is not known whether the solution remains regular for all times. For these equations, the t-model predicts a solution whose energy starts out a constant but after a finite time starts to decay, as a power law of the elapsed time (see Fig. 1). This power law decay cannot be considered irrefutable evidence that the solutions of the Euler equations lose smoothness in finite time. It may well be that the solutions predicted by the t-model with a higher resolution will conserve energy. Even if this is the case, the t-model may prove to be a good candidate for the construction of simple effective models involving a small number of variables. In this case, one needs to be able to model the drain of energy out of the resolved range of scales. For the inviscid Burgers and 2D Euler equations, the model appears to give the correct rate of energy drain to the unresolved scales. On the other hand, the numerical solution of the 3D Euler equations is a much more challenging problem because of the large number of active scales that emerge, even if one starts with a smooth initial condition. Currently, there is no broad consensus as to the rate at which energy cascades from the large to the small scales (this cascade concept is based on the intuitive picture of large vortices breaking up into smaller vortices). Thus, at this time there is no way to test the accuracy of the energy-drain rate predicted by the t-model.

It is important to put the t-model in perspective with regard to the considerable number of effective models that have appeared in the literature. The models appearing in the literature can be divided in two categories. In the first category are the models that involve introducing some terms that model the behavior of the unresolved subgrid scales and then proving the convergence of the model's solution to the solution of the original equations, in the limit of

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an infinite number of resolved modes (17–24). The second category of models involves some kind of perturbative argument, where the perturbation parameter is the strength of the nonlinear term appearing in the equations (see ref. 25 and references therein). The models in the first category inevitably carry some adjustable parameters that need to be calibrated for the model to give accurate results. But for a problem like the 3D Euler equations, where one does not know *a priori* what the solution will do, such an adjustment is problematic. The models in the second category fail when the strength of the nonlinear term is large due to the breakdown of the perturbative expansion. An important aspect of the t-model is that it comes directly from the Euler equations, and it is not based on a perturbative expansion in the strength of the nonlinear term. The model is based on the assumption of the absence of separation of time scales in a turbulent flow because of the appearance of organized structures.

In conclusion, the work of Hald and Stinis (9) presents a fresh look at the problem of the construction of effective models for systems that exhibit no separation of time scales, with an application to model the notoriously difficult 3D Euler equations. The precise evaluation of the accuracy of the proposed model (through the application of the model with a larger number of resolved variables), as well as its applicability to other systems exhibiting a wide range of active scales, remains to be seen.

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