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Spontaneous symmetry breaking in the O(N) model for large N†

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We study the O(N) generalization of the σ model in the limit of large N, for four, three, two, and one space-time dimensions. We compute the effective potential and some momentum-dependent Green's functions. In one and two dimensions, spontaneous symmetry breakdown is impossible; any asymmetric minimum inserted in the tree-approximation potential is immediately filled in by the effects of radiative corrections. This is in agreement with general theorems. In four dimensions, the model is inconsistent; it possesses a tachyon. In three dimensions, the model seems to be consistent, and offers an interesting example of some nonlinear effects associated with spontaneous symmetry breakdown that are not present in the usual (tree-approximation) models.

I. INTRODUCTION

The O(N) model is a theory of N real scalar fields, φ^a, with O(N)-symmetric quartic self-interactions. The Lagrangian density for the theory is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \varphi^a \partial^\mu \varphi^a - \tfrac{1}{2}\mu_0{}^2\varphi^a\varphi^a - \frac{\lambda_0}{8N}(\varphi^a\varphi^a)^2, \qquad (1.1)$$

where $\mu_0{}^2$ and λ_0 are real parameters (the bare squared mass and bare coupling constant), and the sum over repeated indices is implied. It has been known for some time[1] that it is possible to analyze this model in two different perturbative ways: One is ordinary perturbation theory in λ_0 for fixed N; the other is perturbation theory in $1/N$ for fixed λ_0. Computations in the second expansion are slightly more difficult than in the first; nevertheless, at least to leading order in $1/N$, it is possible to obtain formulas for many quantities of physical interest. (This situation is sometimes described by saying that the model is exactly soluble as N goes to infinity. This seems to us to be a slightly deceptive way of putting things, rather like describing the Born approximation by saying that quantum electrodynamics is exactly soluble as e goes to zero.) These formulas typically display richer structures than the corresponding leading-order expressions in ordinary perturbation theory. This is because the leading $1/N$ approximation preserves much more of the nonlinear structure of the exact theory than does ordinary lowest-order perturbation theory; for example, two-particle unitarity for the four-point function is exact.

Recently, several authors[2] have used the leading $1/N$ approximation to study spontaneous symmetry breakdown in the O(N) model. This paper is to be thought of as a comment upon this earlier work. We improve on previous investigations in three ways: (1) By introducing a different method of computation, we have been able to simplify considerably the detailed calculations. We hope that this will make this interesting and instructive model accessible to a wider audience than before. (2) We have carried out calculations for one, two, and three space-time dimensions as well as four. This would be a witless exercise were it not that strikingly different physics emerges in the four different cases. (3) Because of our improved computational method, we are able to calculate explicitly the propagators in the presence of spontaneous symmetry breakdown.

Before proceeding to detailed calculations, let us try to get a rough idea of what we expect to find. As a starting point, let us analyze (1.1) in the semiclassical (tree) approximation. Here, there is no renormalization, so we identify $\mu_0{}^2$ and λ_0 with μ^2 and λ, the renormalized parameters, and look for the minima of the energy density, for constant fields,

$$U = \tfrac{1}{2}\mu^2\varphi^a\varphi^a + \frac{\lambda}{8N}(\varphi^a\varphi^a)^2. \qquad (1.2)$$

We find the following:

(1) If λ is negative, U is unbounded below, and the theory has no ground state; therefore we restrict ourselves to positive λ.

(2) If μ^2 is positive, the state of lowest energy is $\varphi^a = 0$, and the symmetry is manifest; if μ^2 is negative, the state of lowest energy is any state for which

$$\varphi^a\varphi^a = -2\mu^2 N/\lambda \equiv \langle\varphi\rangle^2, \quad \langle\varphi\rangle > 0. \tag{1.3}$$

Which of these we choose as the ground state of the theory is irrelevant to the physics of the problem; whichever one we choose, the symmetry is spontaneously broken to O($N-1$). For convenience, we choose

$$\varphi^a = \delta^{aN}\langle\varphi\rangle\ . \tag{1.4}$$

(3) To investigate the properties of the broken-symmetry world, we define shifted fields,

$$\begin{aligned}\sigma &= \varphi^N - \langle\varphi\rangle\ ,\\ \pi^a &= \varphi^a \quad (a<N)\ .\end{aligned} \tag{1.5}$$

In terms of these fields

$$U = \frac{\lambda}{8N}[\pi^a\pi^a + \sigma^2 + 2\sigma\langle\varphi\rangle]^2 \tag{1.6}$$

plus an irrelevant constant. From this we see that the $N-1$ π fields are massless Goldstone bosons, while the mass of the σ field is given by

$$m_\sigma^2 = \lambda\langle\varphi\rangle^2/N = -2\mu^2\ . \tag{1.7}$$

How much of this structure do we expect to be preserved in the exact theory?

(1) In the full theory, the vacuum state is found not by minimizing the classical potential energy density, U, but by minimizing the effective potential, V. [The effective potential is a function of a set of c-number variables called classical fields. There is one classical field for each quantum field in the theory. V has two equivalent definitions: V is the generating function for one-particle-irreducible (1PI) Green's functions, with all external momenta set equal to zero; V is the energy per unit volume in a state in which the expectation values of the quantum fields are constrained to be the values of the classical fields.[3]] Since the definition of the renormalized parameters has a large element of convention in it, we would not expect the exact equations obtained above to be preserved, except for a fortuitous definition of λ and μ^2. However, we would expect their qualitative features to be preserved: For some range of λ, the theory should be nonsense (no ground state), and for some range of λ it should make sense. For a sensible choice of λ, for some range of μ^2, the minimum of V should be at the origin (manifest symmetry), and for some other range the minimum should be away from the origin (symmetry breakdown).

(2) When there is symmetry breakdown, there should be $N-1$ massless pions, as a consequence of the Goldstone theorem. Since Eq. (1.6) contains a trilinear $\sigma\pi\pi$ coupling, the σ particle should be unstable; that is to say, the pole in the σ propagator should be displaced to the second sheet. However, as λ goes to zero, the pole should move onto the real axis, and its location should be given by Eq. (1.7).

(3) In the semiclassical approximation, Eq. (1.6), if we give the σ field a constant negative value, the pions acquire a negative squared mass. Thus, a situation of constant field lower than the minimum is unstable; we expect that, as a result of this, it should be impossible to define V in a physically sensible way to the left of the minimum. Such a situation already arises if we consider the one-loop corrections to V; here V develops an imaginary part to the left of the minimum, clear evidence of instability.[4]

(4) All of this is for three and four space-time dimensions. In one dimension, the system is simply a coupled set of N anharmonic oscillators. For such a system, if there is a ground state, it is unique; spontaneous symmetry breakdown is impossible. In two dimensions, spontaneous symmetry breakdown can occur, but only for discrete symmetries; the extreme infrared divergences that would be caused by the appearance of Goldstone bosons makes spontaneous breakdown of continuous symmetries impossible.[5] Thus, in both one and two dimensions, the minimum of V should always remain at the origin, no matter how we adjust μ^2.

The detailed computations we will present verify all of these expectations. In addition, there is one surprise: In the four-dimensional case, the theory possesses a tachyon, a pole in the σ propagator at spacelike momentum. We believe that this is merely a defect of the leading $1/N$ approximation, not of the exact theory. However, it does make suspect the use of the leading $1/N$ approximation to study the statistical mechanics of symmetry breakdown[6]; in this approximation, the system can never reach thermal equilibrium, since the energy spectrum contains (tachyon) states of arbitrarily large negative energy. Of course, since the purely thermodynamic computations display no internal inconsistencies, it remains possible that the leading $1/N$ approximation is valid for thermodynamic quantities but invalid for the detailed energy spectrum of the theory. Our computations do not go far enough nor our insight deep enough for us to do more than raise this possibility.

II. THE EFFECTIVE POTENTIAL

A. Introductory remarks and a combinatoric trick

Let us begin by recalling how certain Feynman diagrams come to be more important than others in the large-N limit. For simplicity, let us study the scattering of a meson of type a from one of type b, in the theory with unbroken symmetry.

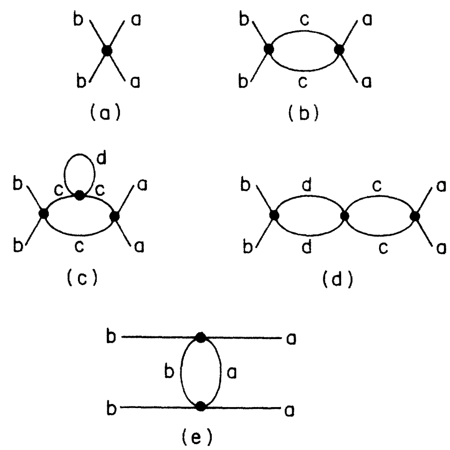

FIG. 1. Some diagrams for meson-meson scattering.

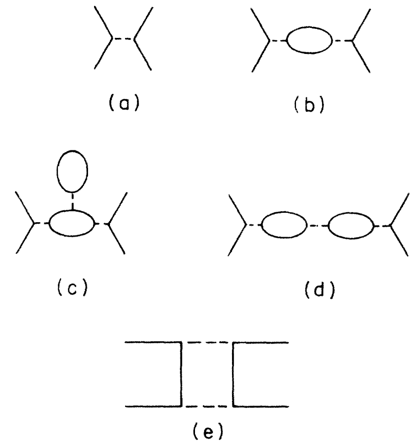

FIG. 2. The same diagrams as those of Fig. 1, in a different formalism.

Some diagrams that contribute to this process are shown in Fig. 1. (The summation over internal indices is implied.) The lowest-order diagram [Fig. 1(a)] is proportional to $1/N$, because of the explicit $1/N$ in the interaction, Eq. (1.1). The one-bubble and two-bubble diagrams [Figs. 1(b), 1(c), and 1(d)] are also proportional to $1/N$; the additional factors of $1/N$ introduced by the additional interactions are precisely canceled by the factors of N introduced by summing over the N internal mesons in each bubble. On the other hand, the diagram shown in Fig. 1(e) is proportional to $1/N^2$, since there is no internal summation.

Note that Figs. 1(b) and 1(e) have the same topological structure; thus we lose one of the great advantages of graphical analysis, the transformation of algebra into topology. This difficulty can be eliminated by changing the Lagrangian, adding to it a term involving a new field, χ,

$$\mathcal{L} \to \mathcal{L} + \frac{1}{2}\frac{N}{\lambda_0}\left(\chi - \frac{1}{2}\frac{\lambda_0}{N}\varphi^a\varphi^a - \mu_0{}^2\right)^2 . \qquad (2.1)$$

The added term has no effects on the dynamics of the theory. This is easy to see from the viewpoint of functional integration: The functional integral over χ is a trivial Gaussian integral; its only effect is to multiply the generating functional of the theory by an irrelevant constant. It is also easy to see from the viewpoint of canonical quantization: The Euler-Lagrange equation for χ is

$$\chi = \frac{1}{2}\frac{\lambda_0}{N}\varphi^a\varphi^a + \mu_0{}^2 . \qquad (2.2)$$

This involves no time derivatives; it is not a true equation of motion, but an equation of constraint, just like the Euler-Lagrange equation for the fourth component of a massive vector field. Therefore, before we construct the Hamiltonian, χ must be eliminated from the Lagrangian, using Eq. (2.2). Such an elimination simply cancels the added term.

However, although the dynamics defined by the new Lagrangian is the same as defined by the old, the Feynman rules are very different. By elementary algebra,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a + \frac{1}{2}\frac{N}{\lambda_0}\chi^2 - \tfrac{1}{2}\chi\varphi^a\varphi^a - \frac{N\mu_0{}^2}{\lambda_0}\chi , \qquad (2.3)$$

plus an irrelevant constant. Thus, in the new formalism, the only nontrivial interaction is a trilinear $\varphi\varphi\chi$ coupling. All factors of $1/N$ come from the χ propagator ($i\lambda_0/N$), and every closed φ loop gives a factor of N. Also, in this formalism, the mass of the φ is the combined result of the trilinear interaction and the χ tadpole, the last term in Eq. (2.3). Figure 2 shows some graphs in the new formalism that correspond to the graphs of Fig. 1. (The dashed lines represent χ propagators.) Notice that Figs. 2(b) and 2(e) are topologically distinct; the first contains an internal φ loop while the second does not.

B. The effective potential

We now turn to the computation of the effective potential, V. We remind the reader that this is a function of so-called classical fields, the Legendre conjugates of c-number sources coupled linearly to the quantum fields of the theory, in our case φ and χ. For notational simplicity, we will denote the classical fields by the same symbols as we use for the corresponding quantum fields; which we mean at any time will always be clear from the context.

From Eq. (2.3),

$$V(\varphi,\chi) = -\frac{1}{2}\frac{N}{\lambda_0}\chi^2 + \tfrac{1}{2}\chi\varphi^a\varphi^a + \frac{N\mu_0{}^2}{\lambda_0}\chi + \text{radiative corrections}. \qquad (2.4)$$

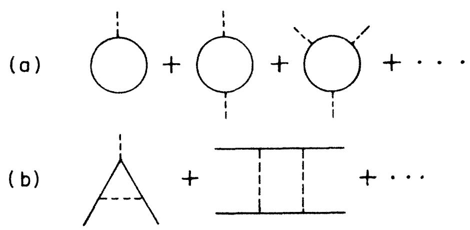

FIG. 3. (a) The diagrams that make the leading contribution to the effective potential in the large-N approximation. (b) Some diagrams that are unimportant compared to those in (a).

We are interested in V in the neighborhood of the tree-approximation minimum. In this region, φ is of the order of $N^{1/2}$, from Eq. (1.3), and χ is of the order of 1, from Eq. (2.2). Thus, all three terms explicitly displayed in Eq. (2.4) are of the same order of magnitude; they are all of order N. It is easy to see that the only radiative corrections of order N are those which arise from the graphs of Fig. 3(a); all other graphs, such as those of Fig. 3(b), are of order 1 or lower.

Graphs similar to those of Fig. 3(a) have been summed many times in the literature by many methods.[7] Rather than doing the computation in detail once more, we merely state the answer:

$$V=-\frac{1}{2}\frac{N}{\lambda_0}\chi^2+\tfrac{1}{2}\chi\varphi^2+\frac{N\mu_0{}^2}{\lambda_0}\chi + \tfrac{1}{2}N\int\frac{d^nk}{(2\pi)^n}\ln(k^2+\chi)\,, \tag{2.5}$$

where n is the number of space-time dimensions, φ^2 is $\varphi^a\varphi^a$, and the integral is over Euclidean momenta.

The ground state of the theory is determined by the stationary point of V, that is to say, by the equations

$$\frac{\partial V}{\partial\chi}=0\,, \tag{2.6a}$$

$$\frac{\partial V}{\partial\varphi^a}=0\,. \tag{2.6b}$$

We can also view these equations in a slightly different way: We can think of Eq. (2.6a) as an equation that eliminates χ, and enables us to write V as a function of φ alone. In this way we recreate the effective potential we would have constructed in our old formalism, before we introduced χ. We can then use Eq. (2.6b) to determine the ground state. [This is, of course, completely equivalent to applying Eqs. (2.6a) and (2.6b) simultaneously.] We shall adopt this procedure here because it enables us to check the conjectures about the effective potential (as a function of φ alone) made in Sec. I.

Inserting Eq. (2.5) in Eq. (2.6a), we obtain

$$\varphi^2=\frac{2N}{\lambda_0}\chi-\frac{2N\mu_0{}^2}{\lambda_0}-N\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2+\chi}\,. \tag{2.7}$$

Also, trivially, from Eqs. (2.5) and (2.6a),

$$\frac{dV}{d\varphi^2}=\frac{\partial V}{\partial\varphi^2}+\frac{\partial V}{\partial\chi}\frac{\partial\chi}{\partial\varphi^2}=\tfrac{1}{2}\chi. \tag{2.8}$$

It is these two equations we shall use below. Note that Eq. (2.8) implies that if there is a broken-symmetry ground state, it occurs at $\chi=0$.

From this point onward, it is necessary to consider the cases of four, three, two, and one dimensions separately.

C. Four dimensions

Equation (2.7) contains a divergent integral. Since this is a renormalizable theory, the divergences can be absorbed into mass and coupling-constant renormalizations. In particular, if we define

$$\frac{\mu^2}{\lambda}\equiv\frac{\mu_0{}^2}{\lambda_0}+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\,, \tag{2.9}$$

and also define

$$\frac{1}{\lambda}\equiv\frac{1}{\lambda_0}+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{k^2+M^2} \tag{2.10}$$

where M is an arbitrary parameter with dimensions of mass, we obtain

$$\varphi^2=\frac{2N}{\lambda}\chi-\frac{2N\mu^2}{\lambda}-\frac{N}{16\pi^2}\chi\ln(\chi/M^2)\,. \tag{2.11}$$

We emphasize that M is completely arbitrary. Any nonzero choice of M is as good as any other; a change in M merely redefines λ.

Let us now discuss the qualitative behavior of V as a function of φ. We begin by assuming that V possesses an asymmetric minimum, $\chi=0$. By Eq. (2.11), at this minimum

$$\varphi^2=-2N\mu^2/\lambda\,. \tag{2.12}$$

[This is the same as the tree-approximation result, Eq. (1.3), but this is a fortuity of no physical significance; it would not be true had we defined our renormalized parameters otherwise.] In the neighborhood of the minimum, as χ becomes positive, φ^2 increases. Thus V is a monotonically increasing function of φ^2 to the right of the minimum. This situation does not persist indefinitely; as we continue increasing χ, φ^2 passes through a maximum and begins to decrease. This maximum occurs at

$$\ln(\chi/M^2)=-1+\frac{32\pi^2}{\lambda}\,, \tag{2.13a}$$

$$\varphi^2=-\frac{2N\mu^2}{\lambda}+\frac{N\chi}{16\pi^2}\equiv\varphi_{\max}{}^2\,. \tag{2.13b}$$

In order to obtain a greater value of φ^2 than this, we must make χ complex. By Eq. (2.8), this means that V has an imaginary part for φ^2 larger than $\varphi_{\max}{}^2$. This is a complete surprise; we did not anticipate such behavior in our introductory discussion. To the left of the minimum, a similar phenomenon occurs. Real negative χ implies imaginary φ^2; to keep φ^2 real we must make χ, and therefore V, complex. This is a sign of the unphysical nature of the regime to the left of the minimum, anticipated in the introduction.

To summarize: The condition for an asymmetric minimum and spontaneous symmetry breakdown is that μ^2/λ be negative. If this condition is fulfilled, to the left of the minimum the effective potential is complex, as expected. To the right of the minimum, the effective potential is real, monotonically increasing, and convex, but only for a while; at a certain point it becomes complex again. At this stage in our investigation, we have no explanation of this second region of complexity. (We will make some speculations later.) Nowhere in any of this is there any constraint on the sign or magnitude of λ.[8]

D. Three dimensions

The computations for the three-dimensional case are almost identical with those for the four-dimensional one, although the consequences are somewhat different. Once again, we must absorb divergences into unrenormalized parameters. Therefore, we define

$$\frac{\mu^2}{\lambda} \equiv \frac{\mu_0{}^2}{\lambda_0} + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2}. \tag{2.14}$$

There is no need for further subtractions, so we simply define the renormalized coupling constant to be the same as the bare one,

$$\lambda \equiv \lambda_0 . \tag{2.15}$$

In this way we obtain

$$\varphi^2 = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} + \frac{N}{4\pi}\sqrt{\chi} . \tag{2.16}$$

In this case, since the bare coupling constant is a well-defined quantity, we certainly want to choose λ to be always positive, so the Hamiltonian will be bounded below and the theory will possess a ground state. Then the right-hand side of Eq. (2.16) is a monotonically increasing function of χ for positive χ, and has an imaginary part for negative χ.

Thus, if μ^2 is positive, χ is positive at $\varphi^2=0$ and grows larger as φ^2 increases. The minimum of V is at the origin, and away from the origin V is real, monotonically increasing, and convex. If μ^2 is negative, the minimum is away from the origin and given by Eq. (2.12). To the right of the minimum, V is real, monotonically increasing, and convex; in contrast to the four-dimensional case, there is no unphysical region of imaginary V on the far right. Something unexpected does happen, though, to the left of the minimum. Although this region is unphysical, as anticipated, nevertheless V remains real.

The most direct way to see this is from the explicit solution to Eq. (2.16),

$$\sqrt{\chi} = -\frac{\lambda}{4\pi} + \left(\frac{\lambda^2}{16\pi^2} + \mu^2 + \frac{\lambda\varphi^2}{2N}\right)^{1/2} . \tag{2.17}$$

(We have chosen the solution of the quadratic equation that gives a physically sensible answer to the right of the minimum.) From this we see that as we move to the left of the minimum, all that happens is that $\sqrt{\chi}$ moves from positive to negative values; χ remains real and positive, and therefore V remains real. We emphasize that this region is as surely unphysical as one in which V has an imaginary part: We are not free to choose a negative sign for the square root in Eq. (2.16); the $\sqrt{\chi}$ in this expression comes from the integral in Eq. (2.7), and for real positive χ the integral unambiguously gives the positive square root.[9]

We have no simple physical picture of this curious persistence of reality in the unphysical region. We suspect that an imaginary part may develop in higher orders of $1/N$, but we have not carried out the computations necessary to verify this suspicion.

E. Two dimensions

In two dimensions, we still require mass renormalization. Therefore, we define

$$\frac{\mu^2}{\lambda} = \frac{\mu_0{}^2}{\lambda_0} + \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+M^2}, \tag{2.18}$$

where M is an arbitrary parameter with the dimensions of mass, as in the four-dimensional case. As before, no coupling-constant renormalization is needed, so we identify λ with λ_0, and only consider the case of positive λ.

We thus obtain

$$\varphi^2 = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} + \frac{N}{4\pi}\ln(\chi/M^2). \tag{2.19}$$

The qualitative behavior of this equation is completely different from that in the preceding two cases; the right-hand side increases monotonically from $-\infty$ at $\chi=0$ to $+\infty$ at $\chi=+\infty$.

Thus, no matter how we adjust μ^2, there is no way to make χ vanish for positive φ^2, and therefore no way to produce an asymmetric minimum in V. The minimum in V is always at the origin; *spontaneous symmetry breakdown is impossible.*

Of course, this is just what we anticipated in our introductory discussion on the basis of general principles. Nevertheless, it is always pleasant to see general principles verified in an explicit example.

F. One dimension

In one dimension, there are no divergences, so we identify μ^2 and λ with $\mu_0{}^2$ and λ_0. As before, we only consider the case of positive λ.

We thus obtain

$$\varphi^2 = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} - \frac{N}{2\sqrt{\chi}}. \tag{2.20}$$

As expected, we find the same qualitative features as in two dimensions; whatever the value of μ^2, the right-hand side of this equation is a monotonically increasing function of χ, going from $-\infty$ to $+\infty$ as χ traverses the positive real axis. Thus, as before, spontaneous symmetry breakdown is impossible.

III. SOME GREEN'S FUNCTIONS

A. General formulas

Let us recall the general prescription for computing Green's functions in a theory with spontaneous symmetry breakdown: (1) First, one computes the effective action, the generating functional for one-particle irreducible (1PI) Green's functions. This is a functional of the same classical fields that appear in the effective potential; the difference is that in the effective action the classical fields are not constants, but space-time-dependent functions. When the classical fields are constants, the effective action reduces to minus the space-time integral of the effective potential. (2) Then, one determines the ground state of the theory by minimizing the effective potential, and defines shifted classical fields which vanish at the minimum, like the fields of Eq. (1.5). (3) Finally, one expands the effective action in a functional Taylor series in these shifted fields. The coefficients in this expansion are the 1PI Green's functions of the theory with spontaneous symmetry breakdown.

We will now carry through these steps, to leading order in $1/N$, for those cases for which we have found spontaneous symmetry breakdown (i.e., the four- and three-dimensional models, with negative μ^2/λ). In order to avoid worrying about $i\epsilon$'s, we will do all our computations in Euclidean space. Thus, k^2 will always denote the (positive) Euclidean squared length of a vector, and $\Box^2$ will denote the Euclidean n-dimensional Laplace operator (not the Minkowskian wave operator). When necessary, we will return to Minkowski space by analytic continuation.

By the same reasoning as in the previous section, the relevant terms are those that come from the original Lagrangian, Eq. (2.3), and the radiative corrections from the diagrams of Fig. 3(a). Of course, since the classical χ field is now no longer a constant, the external lines on these diagrams now carry nonzero momentum. However, at least formally, they may still be summed into a closed form. Thus we find for the effective action

$$\Gamma = \int d^n\chi\left(\tfrac{1}{2}\varphi^a\Box^2\varphi^a + \frac{1}{2}\frac{N}{\lambda_0}\chi^2 - \tfrac{1}{2}\chi\varphi^a\varphi^a - \frac{N\mu_0{}^2}{\lambda_0}\chi\right) - \tfrac{1}{2}N\,\mathrm{tr}[\ln(-\Box^2+\chi)], \tag{3.1}$$

where tr denotes the trace of the operator in square-brackets, considered as an integral operator in Euclidean n-space. Note that if the fields are constants, this reduces to minus the integral of V, as it should. This completes the first step of the prescription.

Now for the second step. We introduce shifted fields, precisely as in our introductory discussion:

$$\sigma = \varphi^N - \langle\varphi\rangle,$$

and

$$\pi^a = \varphi^a \quad (a<N), \tag{3.2}$$

where

$$\langle\varphi\rangle^2 = -2\mu^2 N/\lambda. \tag{3.3}$$

Note that there is no need to shift χ, because χ always vanishes at the asymmetric minimum.

Now for the third step. In terms of these shifted fields,

$$\Gamma = \int d^n\chi\left[\tfrac{1}{2}\pi^a\Box^2\pi^a + \tfrac{1}{2}\sigma\Box^2\sigma + \frac{1}{2}\frac{N}{\lambda_0}\chi^2 - \chi\sigma\langle\varphi\rangle - \tfrac{1}{2}\chi\sigma^2 - \tfrac{1}{2}\chi\pi^a\pi^a - \left(\frac{N\mu_0{}^2}{\lambda} + \tfrac{1}{2}\langle\varphi\rangle^2\right)\chi\right] - \tfrac{1}{2}N\,\mathrm{tr}[\ln(-\Box^2+\chi)]. \tag{3.4}$$

Of course, for consistency, the terms linear in χ must cancel the first-order term in the expansion of the logarithm. It is easy to check that this indeed happens, i.e., that

$$-\frac{N\mu_0{}^2}{\lambda_0} + \frac{N\mu^2}{\lambda} - \frac{N}{2}\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2} = 0. \tag{3.5}$$

This is simply the definition of μ^2/λ, Eqs. (2.9) and (2.14).

Equation (3.4) is the complete formal solution to our problem; from this expression, all of the 1PI Green's functions can be obtained by functional differentiation. We shall now perform the differentiations to obtain explicit forms for the 1PI two-point functions, the inverse propagators.

The simplest propagator to study is that of the π fields. Only the first term in Eq. (3.4) is quadratic in π. Thus, the π propagator is that of a free massless field,

$$D_{ab}(p^2)=\delta_{ab}/p^2\,. \tag{3.6}$$

(Remember, we are working in Euclidean space, so the propagator of a free scalar field is real and positive.) That the π particle is massless is, of course, a consequence of Goldstone's theorem; that its propagator is free is, of course, only an artifact of our approximation; standard unitarity arguments applied to Eq. (1.6) show that the imaginary part of the π propagator is proportional to $1/N$.

We now turn to the σ-χ system. The fourth term in Eq. (3.4) introduces mixing between these fields; thus, we will have to consider a matrix inverse propagator. We find

$$D^{-1}(p^2)=\begin{pmatrix} p^2 & \langle\varphi\rangle \\ \langle\varphi\rangle & -\dfrac{N}{\lambda_0}-NB(p^2)\end{pmatrix}\,, \tag{3.7}$$

where the first entry is σ, the second is χ. $B(p^2)$ is the contribution of the second functional derivative of the last term in Eq. (3.4); it is shown graphically in Fig. 4, and its explicit form is

$$B(p^2)=\frac{1}{2}\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2}\frac{1}{(k+p)^2}\,. \tag{3.8}$$

Of course, the propagators themselves are found by inverting Eq. (3.7). The diagonal entries in $D(p^2)$, the σ and χ propagators, are of special interest. The σ propagator is of interest because it is independent of our computational formalism; the σ propagator is, as always, simply the Fourier transform of the vacuum expectation value of the time-ordered product of two σ fields. We would have obtained the same expression, albeit with considerably more labor, even if we had never introduced the χ field. (The same cannot be said of the 1PI σ propagator; this does depend on the formalism.) The χ propagator is of interest because it is, in principle, a directly measurable quantity. Figure 5 shows the only graphs that contribute to on-mass-shell $\pi\pi$ scattering, in our approximation. These are all proportional to the χ propagator.

To go further, we must consider three and four dimensions separately.

B. Three dimensions

In three dimensions, Eq. (3.7) is free of divergences; λ is λ_0, and B is a convergent integral,

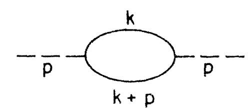

FIG. 4. The function $B(p^2)$. Note that because of the absence of χ tadpoles, the internal lines are massless.

$$B=1/16p\,, \tag{3.9}$$

where p is the positive square root of p^2. Thus we find

$$D_{\sigma\sigma}=\frac{p+\lambda/16}{p(p^2+\lambda p/16-2\mu^2)}\,. \tag{3.10}$$

Also,

$$D_{\chi\chi}\equiv\frac{\lambda p^2}{N(2\mu^2-\lambda p/16-p^2)}\,. \tag{3.11}$$

Note that the σ propagator blows up at $p=0$, but this is not the result of a pole in p^2 (a massless σ) but of a pole in p, that is to say, a cut in p^2. This cut is obviously a result of the two-π intermediate states; by unitarity and elementary dimensional analysis, these massless states contribute a term proportional to $1/p$ to the σ propagator.

Both propagators are meromorphic functions of p, that is to say, double-valued functions of p^2. The first sheet in p^2 is the right half p plane, the unitarity cut is the imaginary p axis, and the physical region for timelike p^2 is the negative imaginary p axis. The propagators are free of poles on the first sheet (the σ is unstable), but they have two poles on the second sheet, with locations given by

$$p=-\tfrac{1}{32}\lambda\pm[2\mu^2+(\tfrac{1}{32}\lambda)^2]^{1/2}\,. \tag{3.12}$$

(Remember, μ^2 is negative.) As λ goes to zero, these poles approach the physical region, heading for the point $p^2=2\mu^2$, the tree-approximation prediction for the σ mass. It is easy to check that both propagators have positive imaginary parts in the physical region, as they should.

To summarize: In three dimensions, everything

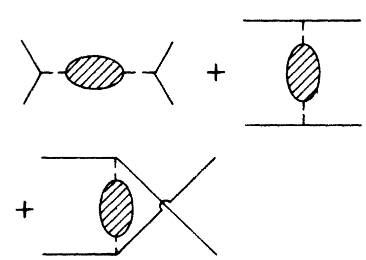

FIG. 5. The diagrams that contribute to $\pi\pi$ scattering, in leading order. The shaded blob is the χ propagator, to leading order.

behaves as it should. There are no pathologies, and no surprises.

C. Four dimensions: the tachyon disaster

In four dimensions, both λ_0 and B are divergent. Of course, these divergences cancel:

$$\frac{1}{\lambda_0}+B=\frac{1}{\lambda}+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(\frac{1}{(p+k)^2}-\frac{1}{k^2+M^2}\right)$$
$$=\frac{1}{\lambda}+\frac{1}{32\pi^2}\left[1-\ln(p^2/M^2)\right] . \tag{3.13}$$

Thus we find

$$D_{\sigma\sigma}=\frac{1/\lambda+(1/32\pi^2)[1-\ln(p^2/M^2)]}{-2\mu^2/\lambda+p^2\{1/\lambda+(1/32\pi^2)[1-\ln(p^2/M^2)]\}} . \tag{3.14}$$

Also

$$D_{\chi\chi}=-\frac{p^2/N}{-2\mu^2/\lambda+p^2\{1/\lambda+(1/32\pi^2)[1-\ln(p^2/M^2)]\}} . \tag{3.15}$$

In many ways, these expressions are similar to their three-dimensional analogs. The σ propagator blows up at $p^2=0$, but this is a consequence of the Goldstone-boson cut, not of a vanishing σ mass.[10] The cut is of logarithmic rather than square-root type, but this is just the difference between four-dimensional and three-dimensional kinematics. For small coupling constant, the σ pole is hiding on the second sheet, approaching $p^2=2\mu^2$ as λ goes to zero, etc.

However, these similarities are overshadowed by a disastrous difference; these propagators possess a tachyon, a pole at Euclidean p^2. This is easy to see: The denominators of these expressions are positive at $p^2=0$ (remember, μ^2/λ is negative) and go continuously to $-\infty$ at $p^2=\infty$. Thus, they must have a zero somewhere in between. Note that the tachyon appears whatever the sign of λ.

It is easy to understand the origin of the tachyon if we imagine cutting off the charge-renormalization integral, Eq. (2.10), at some large momentum, Λ,

$$\frac{1}{\lambda}=\frac{1}{\lambda_0}+\frac{1}{32\pi^2}\ln(\Lambda^2/M^2) . \tag{3.16}$$

This gives us the privilege of discussing energetics in terms of unrenormalized quantities, rather than in terms of renormalized ones as was done previously. For any fixed finite λ, as Λ goes to infinity, λ_0 goes to zero, through *negative* values. Negative λ_0 means that the energy is unbounded below; thus the theory contains states of arbitrarily large negative energy, and such are the tachyon states we have found.[11]

Until now, we have spoken only of that of which we were certain, but now we must begin to speculate, for we wish to inquire whether the pathology we have discovered is a genuine disease of the O(N) model in four dimensions, for sufficiently large N, or an artifact of the leading $1/N$ approximation. Artifactual pathologies are not unknown in field theory; for example, the lowest-order renormalization-group approximation to the photon propagator in quantum electrodynamics has just such a tachyon pole[12] as we have found here. However, in this case, there is an easy out; the pole occurs at a value of p^2 where the invariant charge is very large, and where, therefore, the lowest-order approximation is manifest nonsense. It is important to realize that *there is no such easy out here.* Our expansion parameter is $1/N$, but the location of the tachyon is *independent* of N. Therefore, in complete contrast to the electrodynamic case, there is no reason to believe that the terms we have neglected should be large compared to the terms we have retained at the tachyon pole.

However, all is not lost. For we are studying not just a certain range of p^2, but also a certain range of φ (φ^2 on the order of N). Therefore, we must consider the possibility that the minimum of the effective potential we have found does not represent the true vacuum state of the theory. After all, if we had been so foolish as to take the stationary point of the potential at $\varphi=0$ for the ground state of the theory, we would also have found tachyons (this time in the tree approximation) and this would have been the sign of our error. This example is somewhat artificial, since the origin is a maximum of V, not a minimum. However, we can easily construct instances where there are relative minima of V that are nevertheless false vacuums. (For example, consider the theory of a single scalar field with both quartic and cubic interactions. Here the potential may possess two relative minima, only one of which is an absolute minimum. In simple approximations, it looks like either of these will do for the ground state; nevertheless, it is easy to construct energetic arguments that show that only the absolute minimum is a satisfactory vacuum.)

Therefore, we conjecture that there is a minimum of the effective potential at very large φ^2 that is in fact lower than the minimum that we have found, and which represents the true vacuum of the theory. Such a minimum must be at very large φ^2, because it is only in this case that the diagrams we have neglected can make a contribution to V comparable to the diagrams we have retained. (Remember that our counting of powers of N in Sec. II depended not only on the powers of $1/N$ coming from the diagrams themselves, but also on the factors of N coming from the classical fields on the external lines.) With our current methods, we cannot verify this conjecture. However, there is one small piece of evidence that it is true. This is the imaginary part we found in V for sufficiently large φ^2. We suspect that this is the beginning of a long stretch of imaginary V that will terminate at the true minimum, just as there is a stretch of unphysical V extending from the origin that terminates at the true minimum in the nonpathological three-dimensional case.

IV. CONCLUSIONS

In four dimensions, the leading $1/N$ approximation to the O(N) model leads to inconsistencies. We have given tentative arguments that these inconsistencies represent a sickness of the approximation rather than of the theory, but we have not been able to definitely resolve the matter. In any case, the theory in the leading $1/N$ approximation is sick, and furthermore sick in a uniform way; the situation is not the familiar one where an approximation is accurate in some kinematic region but leads to pathologies in a different kinematic region where it is invalid.

In three, two, and one dimensions, in contrast, the leading $1/N$ approximation is free of inconsistencies and is an interesting and instructive example. In three dimensions, we can explicitly work out many of the nonlinear effects of Goldstone bosons, such as the instability of the σ meson. In one and two dimensions, we can see how the effects of radiative corrections prevent the occurrence of spontaneous symmetry breakdown.

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[1]The O(N) model is closely related to the spherical model of statistical mechanics. See H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford Univ. Press, New York, 1971); K. Wilson, Phys. Rev. D 7, 2911 (1973).

[2]L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974). H. J. Schnitzer, Phys. Rev. D 10, 1800 (1974); 10, 2042 (1974). After this work was completed, we received a paper by D. Gross and A. Neveu [Phys. Rev. D (to be published)] which analyzes O(N) models with four-Fermi interactions in two dimensions, using the same combinatoric tricks we employ.

[3]These parenthetical remarks are a highly condensed summary of what is, by now, standard lore. For a detailed exposition see, for example, S. Coleman, in Proceedings of the 1973 International Summer School of Physics "Ettore Majorana" (to be published).

[4]At first it might seem surprising that there are regions of classical-field space in which it is impossible to define V in a physically sensible way, but in fact this is a common phenomenon. Let us recall some details of the construction of the effective potential. The desired expectation value of the quantum field is obtained by adding a source term to the Lagrange density:

$$\mathcal{L} \to \mathcal{L} + J\varphi,$$

where J is a c-number external field. From the vacuum-to-vacuum transition matrix element in the presence of J, V is constructed by a Legendre transformation. Now, it is quite possible that there are some expectation values of the quantum field that are impossible to attain, no matter what the value of J; if so, the corresponding regions of classical-field space are unphysical. For example, in quantum electrodynamics, it is impossible to distribute external c-number charges so that a constant electric field is maintained over an arbitrarily large space-time region; the constant electric field has a nonzero probability for producing real electron-positron pairs from the vacuum, and these will eventually shield the external charges and destroy the constant field. That is to say, a configuration of constant electric field is unstable. In many cases, it is possible to continue V analytically into unphysical regions. The continuation typically possesses an imaginary part, which can be identified with the decay probability of the unstable field configuration per unit space-time volume. (For more details, see Ref. 3.)

[5]S. Coleman, Commun. Math. Phys. 31, 259 (1973).

[6]Dolan and Jackiw, Ref. 2.

[7]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973); R. Jackiw, *ibid.* 9, 1686 (1974).

[8]This disagrees with Schnitzer, Ref. 2, first paper. The contradiction is only apparent; it results from Schnitzer's renormalization procedure, which differs from ours. According to private conversations with Schnitzer, all "physical" conclusions are in agreement.

[9]If we were so foolish as to accept the negative square root, and therefore to extend V to the left of the minimum, we would find that the minimum was not a minimum at all, but an inflection point.

[10]This resolves the puzzle of apparently vanishing σ mass raised by Schnitzer (Ref. 2, first paper).

[11]This is an ancient pathology of truncated field theories. For example, almost identical problems arise in the pair model of G. Wentzel [Helv. Phys. Acta 15, 111 (1942)].

[12]L. D. Landau, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (McGraw-Hill, New York, 1955).