Office of Naval Research
Department of the Navy
Contract Nonr-220(28)

WATER WAVES GENERATED BY THE TRANSLATORY AND OSCILLATORY SURFACE DISTURBANCE

T. Yao-tsu Wu

Engineering Division
California Institute of Technology
Pasadena, California

Report No. 85-3
July, 1957

Approved by
M. S. Plesset

WATER WAVES GENERATED BY THE TRANSLATORY AND OSCILLATORY SURFACE DISTURBANCE

T. Yao-tsu Wu

1. Introduction and Summary

The problem under consideration is that of two-dimensional gravity waves in water generated by a surface disturbance which oscillates with frequency $\Omega/2\pi$ and moves with constant rectilinear velocity U over the free water surface. The present treatment may be regarded as a generalization of a previous paper by De Prima and Wu (Ref. 1) who treated the surface waves due to a disturbance which has only the rectilinear motion. It was pointed out in Ref. 1 that the dispersive effect, not the viscous effect, plays the significant role in producing the final stationary wave configuration, and the detailed dispersion phenomenon clearly exhibits itself through the formulation of a corresponding initial value problem. Following this viewpoint, the present problem is again formulated first as an initial value problem in which the surface disturbance starts to act at a certain time instant and maintains the prescribed motion thereafter. If at any finite time instant the boundary condition is imposed that the resulting disturbance vanishes at infinite distance (because of the finite wave velocity), then the limiting solution, with the time oscillating term factored out, is mathematically determinate as the time tends to infinity and also automatically has the desired physical properties.

From the associated physical constants of this problem, namely Ω, U, and the gravity constant g, a nondimensional parameter of importance is found to be $\alpha = 4\Omega U/g$. The asymptotic solution for large time shows that the space distribution of the wave trains are different for $0 < \alpha < 1$ and $\alpha > 1$. For $0 < \alpha < 1$ and time large, the solution shows that there are three wave trains in the downstream and one wave in the upstream of the disturbance. For $\alpha > 1$, two of these waves are suppressed, leaving two waves in the downstream. At $\alpha = 1$, a kind of "resonance" phenomenon results in which the amplitude and the extent in space of one particular wave both increase with time at a rate proportional to $t^{1/2}$. Two other special cases: (1) $\Omega \to 0$ and $U > 0$, (2) $U = 0$, $\Omega > 0$ are also discussed; in these cases the solution reduces to known results.

2. Formulation of the Problem

We consider the propagation of surface gravity waves in an infinite ocean, initially at rest, due to a pressure disturbance which oscillates with frequency $\Omega/2\pi$ and moves with constant rectilinear velocity U over the free surface. The flow motion is taken to be two-dimensional in an xy-plane, with the y-axis taken vertically upward, $y = 0$ coinciding with the undisturbed free surface, and the origin being fixed with respect to the applied pressure so that the free stream has velocity U in the direction of x positive. The surface pressure may be described as

$$\begin{aligned} P &= 0 && \text{for time} \quad t \leq 0 , \\ &= p_o(x)\, e^{i\Omega t} && \text{for} \quad t > 0 , \end{aligned} \tag{1}$$

where $p_o(x)$ is an arbitrary function of x, assumed to be absolutely integrable with respect to x. The liquid medium is taken to be inviscid and incompressible of constant density ρ. Thus, there exists a perturbed velocity potential $\varphi(x, y, t)$ in the lower half plane where it satisfies

$$\varphi_{xx} + \varphi_{yy} = 0 \qquad \text{for} \quad y < 0, \quad t > 0. \tag{2}$$

(Here partial derivatives are denoted with subscripts.) The total flow velocity now has the components $(U + \varphi_x, \varphi_y)$. When surface tension is neglected, the linearized boundary conditions on the free surface (cf. Ref. 2) are

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta &= \varphi_y \qquad (3) \\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\varphi + g\zeta &= -\frac{1}{\rho}\, p_o(x)\, e^{i\Omega t} \qquad (4) \end{aligned}\right\} \text{ on } y = 0 \text{ for } t > 0,$$

where $\zeta(x, t)$ is the vertical displacement of the free surface, measured upward from $y = 0$, and g is the gravitational constant. At space infinity, we impose the condition that for any finite t,

$$\zeta, \; \zeta_x \longrightarrow 0 \qquad \text{as} \quad |x| \longrightarrow \infty, \tag{5}$$

and $$\varphi, \; |\mathrm{grad}\,\varphi| \longrightarrow 0 \qquad \text{as} \quad (x^2 + y^2) \longrightarrow \infty \quad \text{for} \quad y \leq 0; \tag{6}$$

in fact, they tend to zero in such a way that their Fourier transforms with respect to x exist. These conditions are suggested by the argument that the wave velocity of the waves having finite wave length is finite. At $t = 0$ we prescribe the zero initial conditions

$$\varphi(x, y, 0) = \zeta(x, 0) = 0 \qquad \text{for} \quad y \leq 0, \qquad (7)$$

which asserts that the flow is initially uniform with the free surface undisturbed. This completes the statement of the problem. In what follows both Ω and U are taken to be positive, and for physical problems one may interpret the result by taking its real or imaginary part.

It is not difficult to obtain an integral representation of the solution of this initial value problem. For example, the method of using the Fourier transform with respect to x and Laplace transform, with respect to t, as was used in Ref. 1, may readily be applied to this problem to yield

$$\varphi(x, y, t) = \frac{1}{\rho} e^{i\Omega t} \int_{-\infty}^{\infty} p_o(\xi) \, \Phi(x-\xi, y, t) \, d\xi , \qquad (8a)$$

$$\Phi(x, y, t) = -\frac{1}{4\pi^2} \int_\Gamma e^{st} \frac{ds}{s} \int_o^\infty e^{ky} \left\{ \frac{e^{-ikx}(Uk-\Omega+is)}{(Uk-\Omega+is)^2-gk} - \frac{e^{ikx}(Uk+\Omega-is)}{(Uk+\Omega-is)^2-gk} \right\} dk \qquad (8b)$$

for $y \leq 0$; and

$$\zeta(x, t) = \frac{1}{\rho} e^{i\Omega t} \int_{-\infty}^{\infty} p_o(\xi) \, H(x-\xi, t) \, d\xi , \qquad (9a)$$

$$H(x, t) = \frac{1}{4\pi^2 i} \int_\Gamma e^{st} \frac{ds}{s} \int_o^\infty \left\{ \frac{e^{-ikx}}{(Uk-\Omega+is)^2-gk} + \frac{e^{ikx}}{(Uk+\Omega-is)^2-gk} \right\} kdk. \qquad (9b)$$

In (8b) and (9b), the contour Γ is taken parallel to the imaginary s-axis, located to the right of all the singularities of the integrand in the complex s-plane. If the variable k is taken to be real, then the singularities of the integrand in the s-plane are five poles, all located on the imaginary s-axis at $s = 0$, $i\omega_1$, $i\omega_2$, $i\omega_3$, $i\omega_4$, where

$$\left.\begin{array}{c}\omega_1\\ \\ \omega_2\end{array}\right\} = Uk - \Omega \pm \sqrt{gk}\,, \qquad \left.\begin{array}{c}\omega_3\\ \\ \omega_4\end{array}\right\} = -\left[Uk + \Omega \mp \sqrt{gk}\right], \tag{10}$$

in which $\sqrt{k}$ is interpreted as its positive branch. Thus the contour Γ may be taken along $s = s_o + i\eta$ with $s_o = \text{const} > 0$ and η running from $-\infty$ to $+\infty$. It may be noted that this solution is also the unique solution of our problem.

By carrying out the s-integral, some alternative forms of the solution are obtained:

$$\Phi(x, y, t) = -\frac{1}{\pi}\int_o^\infty e^{ky}\, dk \int_o^t e^{-i\Omega\tau} \cos k(x - U\tau) \cos \sqrt{gk}\,\tau\, d\tau\,, \quad y \leq 0\,; \tag{11a}$$

or

$$\Phi = \frac{1}{4\pi i}\int_o^\infty e^{ky}\left\{e^{-ikx}\left[\frac{1-e^{i\omega_1 t}}{\omega_1} + \frac{1-e^{i\omega_2 t}}{\omega_2}\right] + e^{ikx}\left[\frac{1-e^{i\omega_3 t}}{\omega_3} + \frac{1-e^{i\omega_4 t}}{\omega_4}\right]\right\} dk \tag{11b}$$

where $\omega_1(k) \ldots \omega_4(k)$ are given by (10). Also,

$$H(x, t) = -\frac{1}{\pi}\int_o^\infty \sqrt{\frac{k}{g}}\, dk \int_o^t e^{-i\Omega\tau} \cos k(x - U\tau) \sin \sqrt{gk}\,\tau\, d\tau\,, \tag{12a}$$

or

$$H(x,t) = \frac{1}{4\pi}\int_0^\infty \sqrt{\frac{k}{g}} \left\{ e^{-ikx}\left[\frac{1-e^{i\omega_2 t}}{\omega_2} - \frac{1-e^{i\omega_1 t}}{\omega_1}\right] + e^{ikx}\left[\frac{1-e^{i\omega_4 t}}{\omega_4} - \frac{1-e^{i\omega_3 t}}{\omega_3}\right]\right\} dk. \tag{12b}$$

It will be seen that the zeros of $\omega_1(k) \ldots \omega_4(k)$ play an important role in the subsequent analysis. They are given by the relation

$$\Lambda_1(k) \equiv \omega_1(k)\,\omega_2(k) = (Uk-\Omega)^2 - gk = U^2(k-\kappa_1)\,(k-\kappa_2) \tag{13}$$

where

$$\left.\begin{matrix}\kappa_1\\ \kappa_2\end{matrix}\right\} = \kappa_o\left[(1+a)^{1/2} \mp 1\right]^2, \qquad \kappa_o = \frac{g}{4U^2}, \qquad a = \frac{4\Omega U}{g}. \tag{14}$$

The nondimensional parameter a actually represents a reduced frequency of the motion which provides an estimate of the combined effect of the oscillatory and translational motion relative to the gravitational effect. If $\sqrt{k}$ is taken as the positive branch in the complex k-plane with a branch cut introduced along the negative axis, then from the definition of ω_1 and ω_2 in (10) it is readily seen that

$$\omega_1(\kappa_1) = 0, \qquad \omega_2(\kappa_2) = 0 \tag{15}$$

and that they have no other zeros in the cut plane. On the other hand,

$$\Lambda_2(k) \equiv \omega_3(k)\,\omega_4(k) = (Uk+\Omega)^2 - gk = U^2(k-\kappa_3)(k-\kappa_4) \tag{16}$$

where

$$\left.\begin{matrix}\kappa_3\\ \kappa_4\end{matrix}\right\} = \kappa_o\left[1 \mp (1-a)^{1/2}\right]^2. \tag{17}$$

Hence the two zeros of $\Lambda_2(k)$ are (i) real and unequal $(\kappa_3 < \kappa_o < \kappa_4)$ for $\alpha < 1$ (or $4 < \cancel{U\Omega g}$); (ii) real and equal $(\kappa_3 = \kappa_4 = \kappa_o)$ for $\alpha = 1$ and (iii) complex conjugate $(\kappa_3 = \overline{\kappa}_4)$ for $\alpha > 1$. These different regions of α will be seen to correspond to different types of flows. The curve $\alpha = 1$ is a hyperbola in the ΩU-plane, dividing the regions $\alpha < 1$ and $\alpha > 1$, as shown in Fig. 1. Again, from (10) and (16) it can be seen that in the complex k-plane cut along the negative axis,

$$\omega_3(\kappa_3) = \omega_3(\kappa_4) = 0 \qquad \text{for} \quad \alpha > 0 \tag{18}$$

and ω_4 has no zero in the cut plane.

Thus, the singularities of the integrand in (11) or (12) are clearly all removable. Consequently the integrand of (11) or (12) is an analytic function of the complex variable k, regular everywhere in the cut plane. It can also be seen that the integral in (12) integrated along the real k converges uniformly in any finite closed interval of x and t excluding the point $x = 0$; the same statement holds true for the integral in (11) (in fact it converges also absolutely for $y < 0$). In view of this analytical behavior of the integrand, the path of integration for these integrals may be deformed into a new contour in the complex k-plane as long as the integral converges. Needless to say, when a term of the integrand which has a non-removable singularity is integrated separately along the real k-axis, its integral may be interpreted as its Cauchy principal value.

3. The Limiting Solution as $t \to \infty$

In the above integral representation of the solution, Φ and H are actually the fundamental solution of the problem, only with the time oscillating term $\exp(i\Omega t)$ factored out. Therefore, on physical grounds, one would expect that at least under certain general conditions, $\Phi(x, y, t)$ and $H(x, t)$ will tend to a limit which is independent of t as $t \to +\infty$. Their respective limit as such will first be calculated by applying the Tauberian theorem. For simplicity, the analysis will be presented only for H; the result for Φ, however, will be given in the last section. Since the two terms in the integrand of (9b) have different behavior, they are considered separately. Rewrite (9b) as

$$H(x, t) = H_1(x, t) + H_2(x, t)\ , \tag{19a}$$

$$\left.\begin{matrix} H_1(x, t) \\ \\ H_2(x, t) \end{matrix}\right\} = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{st} \left\{\begin{matrix} G_1(s, x) \\ \\ G_2(s, x) \end{matrix}\right\} \frac{ds}{s}\ , \quad (b>0) \tag{19b}$$

with

$$G_1(s, x) = \frac{1}{2\pi}\int_0^\infty \frac{e^{-ikx}\, kdk}{(Uk-\Omega+is)^2-gk}\ , \quad G_2(s, x) = \frac{1}{2\pi}\int_0^\infty \frac{e^{ikx}\, kdk}{(Uk+\Omega-is)^2-gk}\ . \tag{19c}$$

The Tauberian theorem (see Ref. 3, p. 187), when applied to this problem, may be stated as follows. Omitting the subscript for the moment, if

$$\lim_{s \to 0+} G(s, x) = F(x) \qquad \text{exists,} \tag{20a}$$

then

$$\lim_{t \to +\infty} H(x,t) = F(x) \qquad \text{for fixed } x \tag{20b}$$

if, and only if

$$\frac{1}{t}\int_0^t \tau \frac{\partial H}{\partial \tau}\, d\tau \longrightarrow 0 \qquad \text{as } t \to +\infty \text{ with } x \text{ fixed.} \tag{20c}$$

To evaluate the limit of $G_1(s,x)$ as $s \to 0+$, one may neglect higher powers of s other than the terms necessary to keep the integral convergent. Thus,

$$\lim_{s \to 0+} G_1(s,x) = \frac{1}{2\pi} \lim_{s \to 0+} \int_0^\infty \frac{\Lambda_1 - 2iUs(k - a\kappa_o)}{\Lambda_1^2 + \left[2Us(k - a\kappa_o)\right]^2} e^{-ikx}\, kdk$$

$$= \frac{1}{2\pi} P\int_0^\infty \frac{e^{-ikx}}{\Lambda_1} kdk - \frac{i}{2\pi} \lim_{s \to 0+} \int_0^\infty \frac{2Us(k - a\kappa_o)\, e^{-ikx}}{\Lambda_1^2 + \left[2Us(k - a\kappa_o)\right]^2} kdk \tag{21}$$

where P denotes the Cauchy principal value of the integral, and Λ_1, a, κ_o are given by (13), (14). Since $\Lambda_1(k)$ has only two simple zeros at κ_1 and κ_2, with $0 < \kappa_1 < a\kappa_o < \kappa_2$ (see Eq. 13), the above principal value exists for all $a > 0$. Now

$$P\int_0^\infty \frac{e^{-ikx}}{\Lambda_1(k)} kdk = \frac{1}{U^2(\kappa_2 - \kappa_1)} P\int_0^\infty \left(\frac{\kappa_2}{k - \kappa_2} - \frac{\kappa_1}{k - \kappa_1}\right) e^{-ikx} dk.$$

The integral of the form

$$P\int_0^\infty \frac{e^{-ikx}}{k-\kappa}\,dk\,, \qquad \kappa>0\,,$$

can be evaluated by means of a complex integral. We construct in the complex k-plane a closed contour consisting of the original path along the positive real axis which is broken at κ (as required by its principal value), a semicircle of small radius ϵ indented below for $x>0$ (or indented above κ for $x<0$), a large semicircle of radius $|k| = R$ below (or above) the real axis for $x>0$ (or $x<0$) and a return to $k=0$ by the negative real axis. By Cauchy's theorem, the integral along the closed contour is zero. Upon passing to the limit $R\rightarrow\infty$, $\epsilon\rightarrow 0$, the contribution from the ϵ-circle accounts for half the residue. Finally, one obtains

$$P\int_0^\infty \frac{e^{-ikx}}{k-\kappa}\,dk = -\pi i\ \mathrm{sgn}\,x\ e^{-i\kappa x} + e^{-i\kappa x}\int_\kappa^\infty \frac{e^{i\eta x}}{\eta}\,d\eta$$

where $\mathrm{sgn}\,x = 1$ for $x>0$ and -1 for $x<0$. Hence

$$P\int_0^\infty \frac{e^{-ikx}}{\Lambda_1(k)}\,kdk = \frac{\pi i\ \mathrm{sgn}\,x}{U^2(\kappa_2-\kappa_1)}\left[\kappa_1 e^{-i\kappa_1 x} - \kappa_2 e^{-i\kappa_2 x}\right] + L(x;\kappa_1,\kappa_2) \tag{22a}$$

$$L(x;\kappa_1,\kappa_2) = \frac{1}{U^2(\kappa_2-\kappa_1)}\left\{\kappa_2 e^{-i\kappa_2 x}\int_{\kappa_2}^\infty \frac{e^{i\eta x}}{\eta}\,d\eta - \right.$$

$$\left. - \kappa_1 e^{-i\kappa_1 x}\int_{\kappa_1}^\infty \frac{e^{i\eta x}}{\eta}\,d\eta\right\}\,. \tag{22b}$$

This function L(x) represents the local effect which levels off rather fast from the origin. For, by integrating (22b) by parts, one obtains

$$L(x; \kappa_1, \kappa_2) \cong \frac{1}{U^2(\kappa_2 - \kappa_1)} \sum_{n=1}^{N} \frac{n!}{(\kappa_2^n - \kappa_1^n)} (ix)^{-(n+1)}$$

$$\text{as } |x| \to \infty \qquad (22c)$$

in which the proper choice of N depends on the value of x. L(x) thus behaves like x^{-2} for x large. However, L(x) has a logarithmic singularity at $x = 0$; this is because first, H(x, t) is a singular solution and second, the surface tension of the interface is being neglected.

To evaluate the limit of the second integral in (21), the range of integration may be reduced to two short stretches: from $\kappa_1 - \epsilon$ to $\kappa_1 + \epsilon$ and from $\kappa_2 - \epsilon$ to $\kappa_2 + \epsilon$, for outside these two short stretches the integrand tends to zero uniformly. Within these stretches one may approximate $\exp(-ikx)$ by $\exp(-i\kappa_1 x)$ or $\exp(-i\kappa_2 x)$, and $(k - a\kappa_o)$ by $(\kappa_1 - a\kappa_o) = -2(\kappa_o \kappa_1)^{1/2}$ or $(\kappa_2 - a\kappa_o) = 2(\kappa_o \kappa_2)^{1/2}$. Denoting the positive quantity $(\kappa_o \kappa_1)^{1/2}$ or $(\kappa_o \kappa_2)^{1/2}$ by B, then

the limit of the second integral in (21)

$$= 2(\kappa_2 e^{-i\kappa_2 x} - \kappa_1 e^{-i\kappa_1 x}) \lim_{s \to 0+} \int_o^{\epsilon/(4UBs)} \frac{d\xi}{[U^2(\kappa_2 - \kappa_1)\xi]^2 + 1}$$

$$= \frac{\pi}{U^2(\kappa_2 - \kappa_1)} \left[\kappa_2 e^{-i\kappa_2 x} - \kappa_1 e^{-i\kappa_1 x} \right]. \qquad (23)$$

Substituting (22) and (23) into (21), one obtains

$$F_1(x) = \lim_{s \to 0+} G_1(s, x) = \frac{i(1 + \operatorname{sgn} x)}{2U^2(\kappa_2 - \kappa_1)} \left[\kappa_1 e^{-i\kappa_1 x} - \kappa_2 e^{-i\kappa_2 x} \right] +$$

$$+ \frac{1}{2\pi} L(x; \kappa_1, \kappa_2) . \tag{24}$$

The above limit is valid for all $\alpha > 0$ since $\kappa_1 \neq \kappa_2$ (actually $\kappa_2 - \kappa_1 \geq 4\kappa_o\sqrt{1+\alpha}$). Aside from the local effect $L(x)$, $F_1(x)$ represents two waves, having wave length $\lambda_1 = 2\pi/\kappa_1$ and $\lambda_2 = 2\pi/\kappa_2$, both propagating on the downstream side.

Next we show that the necessary and sufficient condition (20c) is satisfied by $H_1(x, t)$. From (19) and (12) one can derive

$$\frac{\partial H_1}{\partial t} = \frac{i}{4\pi} \lim_{A \to +\infty} \int_o^A e^{-ikx} \left[e^{i\omega_1 t} - e^{i\omega_2 t} \right] \sqrt{\frac{k}{g}}\, dk .$$

By using $\sqrt{k}$ as a new variable, the integration can be carried out to yield*

$$\frac{\partial H_1}{\partial t} = \frac{e^{-i\Omega t}}{2\pi\sqrt{g}} |Ut - x|^{-3/2} \left\{ \sqrt{\frac{\pi}{2}} (1 - 2i\beta^2) e^{-i\beta^2} \cdot \right.$$

$$\left. \cdot \left[\pm C\left(\sqrt{\frac{2}{\pi}}\beta\right) + i\, S\left(\sqrt{\frac{2}{\pi}}\beta\right) \right] + \beta \right\}$$

where

$$\beta = \frac{t}{2} (g/|Ut - x|)^{1/2} ,$$

C and S denote the Fresnel integrals defined as usual (Ref. 4) and

---

* It should be remarked that $\partial H_1/\partial t$ and $\partial H_2/\partial t$ are bounded near the point $x = Ut$ if the surface tension is also taken into account. In either case, however, H is always bounded near $x = Ut$.

the + sign is for $x < Ut$, the - sign, for $x > Ut$. For x fixed and t large, $\beta^2 \cong gt/4U$, then the asymptotic expansions of C and S can be used to give

$$\frac{\partial H_1}{\partial t} \cong \frac{1}{8U^2}\left(\frac{g}{4Ut}\right)^{1/2} \exp\left[-i\Omega\left(1+\frac{1}{\alpha}\right)t - i\frac{\pi}{4}\right]\left\{1+0(t^{-1})\right\}$$

$$\text{as } t \to \infty . \tag{25}$$

Hence, to evaluate the limit (20c), the interval of integration can be divided into two parts, from $\tau = 0$ to T in which H_1 itself is bounded, and from $\tau = T$ to t in which the above asymptotic representation is valid. Upon integration by parts in both intervals, one obtains

$$\frac{1}{t}\int_0^t \tau \frac{\partial H_1}{\partial \tau}\, d\tau = 0(t^{-1/2}) \qquad \text{as } t \to \infty, \quad x \text{ fixed.} \tag{26}$$

Therefore, it follows from the Tauberian theorem that as $t \to \infty$, $H_1(x, t) \to F_1(x)$.

Next, the expression for $H_2(x, t)$ will be treated separately for the cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$.

3.1 The case $\alpha < 1$.

After neglecting the small terms with s^2 in $G_2(s, x)$ of (19),

$$\lim_{s \to 0+} G_2(s, x) = \frac{1}{2\pi} P\int_0^\infty \frac{e^{ikx}}{\Lambda_2(k)}\, kdk + \frac{i}{2\pi} \lim_{s \to 0+}\int_0^\infty \frac{2Us(k+\alpha\kappa_0)\, e^{ikx}k}{\Lambda_2^2 + \left[2Us(k+\alpha\kappa_0)\right]^2}\, dk \tag{27}$$

In this case $(\alpha < 1)$ $\Lambda_2(k)$ has two simple zeros at κ_3 and κ_4 on the real axis, these integrals can be calculated in a manner similar to the above analysis. The following results are readily verified.

$$P\int_0^\infty \frac{e^{ikx}}{\Lambda_2}\, k\,dk = \frac{\pi i \operatorname{sgn} x}{U^2(\kappa_4 - \kappa_3)}\left[\kappa_4 e^{i\kappa_4 x} - \kappa_3 e^{i\kappa_3 x}\right] + \bar{L}(x; \kappa_3, \kappa_4), \tag{28a}$$

$$\text{the last term of (27)} = \frac{i}{2U^2(\kappa_4 - \kappa_3)}\left[\kappa_4 e^{i\kappa_4 x} + \kappa_3 e^{i\kappa_3 x}\right], \tag{28b}$$

where $\bar{L}(x)$ denotes the complex conjugate of $L(x)$ given by (22b). Hence

$$F_2(x) = \lim_{s \to 0+} G_2(s, x) = \frac{i}{2U^2(\kappa_4 - \kappa_3)}\Big[(1 + \operatorname{sgn} x)\kappa_4 e^{i\kappa_4 x} +$$

$$+ (1 - \operatorname{sgn} x)\kappa_3 e^{i\kappa_3 x}\Big] + \frac{1}{2\pi}\bar{L}(x; \kappa_3, \kappa_4). \tag{29}$$

It can also be proven that $H_2(x, t)$ satisfies condition (20c) for $\alpha < 1$. Therefore for $\alpha < 1$ and $t \to \infty$, $H(x, t) \to H(x) = F_1(x) + F_2(x)$ so that

$$H(x) = \frac{i}{g\sqrt{1+\alpha}}\left[\kappa_1 e^{-i\kappa_1 x} - \kappa_2 e^{-i\kappa_2 x}\right] + \frac{i\kappa_4}{g\sqrt{1-\alpha}}\, e^{i\kappa_4 x} \qquad \text{for } x > 0,$$

$$= \frac{i\kappa_3}{g\sqrt{1-\alpha}}\, e^{i\kappa_3 x} \qquad \text{for } x < 0, \tag{30}$$

In the above expression the local effect $\left[L(x; \kappa_1, \kappa_2) + \bar{L}(x; \kappa_3, \kappa_4)\right]/2\pi$ are omitted for both $x > 0$ and $x < 0$.

3.2 The case $a > 1$.

In this case $\Lambda_2(k)$ has two simple zeros which are complex conjugate. Hence to obtain the limit of $G_2(s,x)$ as $s \to 0+$, s may be neglected altogether, for the resulting integral still converges. Write

$$\kappa = \kappa_o\left[1 + i(a-1)^{1/2}\right]^2 = \kappa_o\left[2 - a + 2i(a-1)^{1/2}\right] \quad \text{for} \quad a > 1. \tag{31}$$

Here $R\ell\,\kappa > 0$, $= 0$ or < 0 for $1 < a < 2$, $a = 2$ or $a > 2$. Now, from (19c)

$$F_2(x) = \lim_{s \to 0+} G_2(s,x) = \frac{1}{2\pi U^2}\int_o^\infty \frac{e^{ikx}\,k\,dk}{(k-\kappa)(k-\bar{\kappa})}.$$

If one chooses a closed contour bounded by the first quadrant for $x > 0$, or the fourth quadrant for $x < 0$, and applies the theorem of residues, one readily obtains

$$\begin{aligned} F_2(x) &= \frac{i\,m\kappa}{U^2(\kappa-\bar{\kappa})}\, e^{i\kappa x} + \frac{1}{2\pi U^2}\int_o^\infty \frac{e^{-x\eta}\,\eta\,d\eta}{(\eta+i\kappa)(\eta+i\bar{\kappa})} \qquad \text{for} \quad x > 0, \\ &= \frac{i\,m\bar{\kappa}}{U^2(\kappa-\bar{\kappa})}\, e^{i\bar{\kappa} x} + \frac{1}{2\pi U^2}\int_o^\infty \frac{e^{x\eta}\,\eta\,d\eta}{(\eta-i\kappa)(\eta-i\bar{\kappa})} \qquad \text{for} \quad x < 0, \end{aligned} \tag{32}$$

where $m = 1$ for $1 < a < 2$, $m = 1/2$ for $a = 2$ and $m = 0$ for $a > 2$; also the above integrals are interpreted as their Cauchy principal value when $a = 2$. In any case, however, the first term is proportional to $\exp(-2\sqrt{a-1}\ \kappa_o|x|)$ and hence falls off exponentially away from $x = 0$. The second term again has a logarithmic singularity at $x = 0$.

For $|x|$ large, however, Watson's lemma (Ref. 5) can be applied to these integrals to yield

$$F_2(x) \cong -(2\pi U^2 a^2)^{-1} \left\{ (\kappa_o x)^{-2} + 4i(2-a)a^{-2}(\kappa_o x)^{-3} + 0(\kappa_o x)^{-4} \right\}. \tag{32a}$$

Furthermore, it can also be shown that $H_2(x,t)$ satisfies (20c) in this case. Therefore for $a > 1$ and $t \to \infty$, $H(x,t) \to H(x)$ where

$$\begin{aligned} H(x) &\cong \frac{i}{g\sqrt{1+a}} \left[\kappa_1 e^{-i\kappa_1 x} - \kappa_2 e^{-i\kappa_2 x} \right] + 0(x^{-2}) \quad \text{for positive large } x; \\ &\cong 0(x^{-2}) \quad \text{for negative large } x. \end{aligned} \tag{33}$$

3.3 The case $a = 1$.

As $a \to 1$, both (30) and (32) become meaningless. This result would imply that no steady state limit of $H_2(x,t)$ really exists when $a = 1$ $(4\Omega U = g)$. To see this, we put

$$a = 1 \qquad \text{so that} \qquad \Lambda_2(k) = U^2(k - \kappa_o)^2$$

and further introduce the nondimensional quantities

$$u = k/\kappa_o, \qquad \xi = \kappa_o x, \qquad \tau = U\kappa_o t, \qquad z = s/U\kappa_o,$$

then from (19), after neglecting the terms in s^2,

$$\begin{aligned} H_2(x,t) &\cong \frac{1}{(2\pi U)^2 i} \int_{b-i\infty}^{b+i\infty} e^{z\tau} \frac{dz}{z} \int_o^\infty \frac{e^{iu\xi}\, u\, du}{(u-1)^2 - 2iz(u+1)} \\ &\cong \frac{1}{2\pi U^2} \int_o^\infty e^{iu\xi} \left[1 - \exp\left(-\frac{i(u-1)^2}{2(u+1)}\tau\right) \right] \frac{u\, du}{(u-1)^2}. \end{aligned} \tag{34}$$

For τ large, the most significant contribution to this integral is seen to come from a neighborhood of $u = 1$. Thus, one may divide the interval into two parts: from $u = 0$ to 2 and from 2 to ∞; the second integral can be estimated, upon integration by parts, to be of $0(\tau^{-1/2})$ for $|\xi| > 0$. For the first integral, one may expand the integrand about the point $u = 1$, hence for $|\xi| > 0$ and τ large with $(u-1) = 2v/\sqrt{\tau}$,

$$H_2(x,t) \cong \frac{\sqrt{\tau}\, e^{i\xi}}{4\pi U^2} \int_{-\sqrt{\tau}/2}^{\sqrt{\tau}/2} e^{i\frac{2\xi}{\sqrt{\tau}}v} (1-e^{iv^2}) \frac{dv}{v^2} \left[1 + 0(\tau^{-1/2})\right].$$

For general values of $\xi/\sqrt{\tau}$, this integral can be expressed in terms of the Fresnel integrals. However, in the region $|\xi| << \sqrt{\tau}$, $\exp(2i\xi\, v/\sqrt{\tau})$ may be neglected, then

$$H_2(x,t) \cong \frac{\sqrt{\tau}\, e^{i\xi}}{4\pi U^2} \int_{-\infty}^{\infty} (1-e^{iv^2}) \frac{dv}{v^2} \left[1 + 0(\tau^{-1/2})\right]$$

$$\cong \frac{1}{4U^2} \left(\frac{gt}{\pi U}\right)^{1/2} \exp\left\{ i \left(\frac{gx}{4U^2} + \frac{\pi}{4}\right)\right\} \left[1 + 0(\tau^{-1/2})\right], \quad (35a)$$

valid for $0 < |x| << 2U(Ut/g)^{1/2}$, so that in this region H_2 grows beyond all bounds as $t \to \infty$. It can also be shown that for $|x| >> 2U(Ut/g)^{1/2}$

$$H_2(x,t) \cong -\left(\frac{Ut}{\pi g}\right)^{1/2} \frac{1}{Ux} \exp\left\{ i \left(\frac{gx}{4U^2} + \frac{gx^2}{4U^3 t} - \frac{\pi}{4}\right)\right\} \left[1 + 0(\tau^{-1/2})\right]. \quad (35b)$$

4. An Alternative Method of Obtaining the Limiting Solution

An alternative method to calculate the limit of $H(x, t)$ as $t \to +\infty$ is by deforming the original path of integration (taken along the real k-axis) into an appropriate contour in the complex k-plane along which the transient waves that depend on time die out as $t \to +\infty$. Considering the behavior of ω_1, ω_2 and ω_3 near their zeros, one can derive from (10) and (13)-(18) that

$$\omega_1(k) = a_1(k-\kappa_1)-b_1(k-\kappa_1)^2+\ldots, \quad a_1 = U\left[(1+\alpha)\frac{\kappa_o}{\kappa_1}\right]^{1/2}, \quad b_1 = \frac{U^3}{g}\left(\frac{\kappa_o}{\kappa_1}\right)^{3/2};$$

$$\omega_2(k) = a_2(k-\kappa_2)-b_2(k-\kappa_2)^2+\ldots, \quad a_2 = U\left[(1+\alpha)\frac{\kappa_o}{\kappa_2}\right]^{1/2}, \quad b_2 = \frac{U^3}{g}\left(\frac{\kappa_o}{\kappa_2}\right)^{3/2};$$

$$\omega_3(k) = a_3(k-\kappa_3)-b_3(k-\kappa_3)^2+\ldots, \quad a_3 = U\left[(1-\alpha)\frac{\kappa_o}{\kappa_3}\right]^{1/2}, \quad b_3 = \frac{U^3}{g}\left(\frac{\kappa_o}{\kappa_3}\right)^{3/2};$$

$$\omega_{\cancel{3}\,4}(k) = a_4(k-\kappa_4)-b_4(k-\kappa_4)^2+\ldots, \quad a_4 = -U\left[(1-\alpha)\frac{\kappa_o}{\kappa_4}\right]^{1/2}, \quad b_4 = \frac{U^3}{g}\left(\frac{\kappa_o}{\kappa_4}\right)^{3/2}. \tag{36}$$

Thus, a_1 and a_2 are positive for $\alpha > 0$; but $a_3 > 0$, $a_4 < 0$ for $0 < \alpha < 1$, they vanish at $\alpha = 1$ and become complex conjugate for $\alpha > 1$. Consider now the terms depending on t in (12b). In a neighborhood of κ_1, $|\exp(i\omega_1 t)| \cong \exp\left[-a_1 t\,\mathrm{Im}(k-\kappa_1)\right]$ which diminishes exponentially as $t \to +\infty$ only for $\mathrm{Im}(k-\kappa_1) > 0$. Also, near $k = \kappa_2$, $|\exp(i\omega_2 t)| = \exp\left[-a_2 t\,\mathrm{Im}(k-\kappa_2)\right] \to 0$ as $t \to +\infty$ for $\mathrm{Im}(k-\kappa_2) > 0$. As $t \to +\infty$ with $0<\alpha<1$, however, $|\exp(i\omega_3 t)| \to 0$ near κ_3 for $\mathrm{Im}(k-\kappa_3) > 0$ and near κ_4 for $\mathrm{Im}(k-\kappa_4) < 0$. Consequently, the integration path for the first two terms in (12b) may be deformed into a contour C_{12} which consists of the positive real k-axis indented above at κ_1 and κ_2; the path for the last two terms in (12b) may be deformed into C_{34} which consists of the positive real k-axis indented above at κ_3 and indented below at κ_4 (see Fig. 2). It can be shown that the terms containing t integrated along the aforementioned contour tend to zero as $t \to +\infty$ (the details of the proof, similar to those described in Ref. 1, will be omitted here). Finally one obtains for $0<a<1$, $H(x,t) \to H(x)$ as $t \to +\infty$, where $H(x)$ is defined by the contour integral

$$H(x) = \frac{1}{2\pi U^2}\int_{C_{12}} \frac{e^{-ikx}\,kdk}{(k-\kappa_1)(k-\kappa_2)} + \frac{1}{2\pi U^2}\int_{C_{34}} \frac{e^{ikx}\,kdk}{(k-\kappa_3)(k-\kappa_4)} \; . \tag{37}$$

These contour integrals can be evaluated by applying the theorem of residues as follows. Construct for the first integral a closed contour Γ_1 which consists of C_{12} and a large semicircle $|k| = R$ in the lower half plane for $x>0$ (or in the upper half plane for $x<0$, as required by the behavior of $\exp(-ikx)$) and back to the origin along the negative real axis. For the second integral a similar closed contour Γ_2 is constructed, only here the large semicircle is in the upper half plane for $x>0$, or in the lower half plane for $x<0$. By passing to the limit $R \to \infty$, one obtains again the solution (30).

For $a>1$, κ_3 and κ_4 become complex conjugate, hence C_{34} may be deformed back to the real axis. In this case one also obtains the solution (33).

It is noted that when $\alpha = 1$, this method also indicates that no steady state limit of the integral containing ω_3 exists because in this case the expansion of ω_3 near κ_3 and κ_4 starts with the quadratic term $(k-\kappa_3)^2$, $(k-\kappa_4)^2$ so that $\exp(i\omega_3 t)$ does not tend to zero everywhere on the indentations as $t \to +\infty$. Its time behavior then should be calculated separately, such as was done in the previous section.

5. Asymptotic Behavior of the Solution for Large Time

Consideration will be given here to the range of validity of the asymptotic representation of the solution for large time. The analysis presented below will be a first approximation. For convenience, rewrite (12b) as

$$H(x,t) = h_o(x) + \sum_{n=1}^{4} h_n(x,t) , \tag{38a}$$

$$h_o(x) = \frac{1}{2\pi} \; P\int_o^\infty \left\{ \frac{e^{-ikx}}{\Lambda_1(k)} + \frac{e^{ikx}}{\Lambda_2(k)} \right\} kdk , \tag{38b}$$

$$\begin{aligned} h_n(x,t) &= \frac{(-1)^{n+1}}{4\pi} \; P\int_o^\infty \sqrt{\frac{k}{g}} \; e^{-ikx+i\omega_n t} \frac{dk}{\omega_n} \qquad n=1,2, \\ &= \frac{(-1)^{n+1}}{4\pi} \; P\int_o^\infty \sqrt{\frac{k}{g}} \; e^{ikx+i\omega_n t} \frac{dk}{\omega_n} \qquad n=3,4. \end{aligned} \tag{38c}$$

(In fact the operation indicated by P is not needed for n=4.)

The integrals of (38b) have been evaluated in Section 3 (see Eqs. 22a, 28a); hence, with the local elevation near the origin again neglected, the result is

$$h_o(x) = \frac{i\,\mathrm{sgn}\,x}{2g}\left\{\frac{\kappa_1 e^{-i\kappa_1 x} - \kappa_2 e^{-i\kappa_2 x}}{\sqrt{1+a}} + \frac{\kappa_4 e^{i\kappa_4 x} - \kappa_3 e^{i\kappa_3 x}}{\sqrt{1-a}}\right\} \qquad \text{for} \quad 0<a<1,$$

$$= \frac{i\,\mathrm{sgn}\,x}{2g\sqrt{1+a}}\left\{\kappa_1 e^{-i\kappa_1 x} - \kappa_2 e^{-i\kappa_2 x}\right\} \qquad \text{for} \quad a>1. \tag{39}$$

In order to calculate the asymptotic value of $h_1(x, t)$, we further decompose it as

$$h_1(x,t) = \left[\int_0^{\kappa_1-\epsilon} + P\int_{\kappa_1-\epsilon}^{\kappa_1+\epsilon} + \int_{\kappa_1+\epsilon}^{\infty}\right]\sqrt{\frac{k}{g}}\;\frac{\exp i\left[\omega_1(k)\,t-kx\right]}{4\pi\,\omega_1(k)}\,dk \tag{40}$$

$$= h_{11} + h_{12} + h_{13}, \qquad \text{according to the order,}$$

where ϵ is a small positive quantity, chosen to be less than κ_1. Substituting the expansion (36) into the expression for h_{12}, one obtains

$$h_{12} \cong \frac{\kappa_1 e^{-i\kappa_1 x}}{2\pi g\sqrt{1+a}}\; P\int_{-\epsilon}^{\epsilon} e^{i(a_1 t-x)z-ib_1 t z^2}\left[1+0(z)\right]\frac{dz}{z}$$

$$\cong -\frac{i\kappa_1 e^{-i\kappa_1 x}}{2g\sqrt{1+a}}\int_0^{\infty} e^{-iu^2}\sin 2\gamma u\,\frac{du}{u} + 0(t^{-1/2})$$

$$\cong -\frac{i\kappa_1 e^{-i\kappa_1 x}}{2g\sqrt{1+\alpha}}\left\{(1-i)\left[C\left(\sqrt{\frac{2}{\pi}}\,\gamma\right)+i\,S\left(\sqrt{\frac{2}{\pi}}\,\gamma\right)\right]+0(t^{-1/2})\right\} \tag{41a}$$

where

$$\gamma = (x-a_1 t)/(4b_1 t)^{1/2} \quad , \tag{41b}$$

with a_1, b_1 given by (36), and C, S again stand for the Fresnel integrals. As $t \to \infty$, γ may be large (such as at $x=0$) or small (such as at $x=a_1 t$). However, γ is large in the region $|x-a_1 t| \gg (4b_1 t)^{1/2}$, where h_{12} assumes the asymptotic value

$$h_{12}(x,t) \cong -\frac{i\,\mathrm{sgn}(x-a_1 t)}{2g\sqrt{1+\alpha}}\,\kappa_1 e^{-i\kappa_1 x} + 0(t^{-1/2}) \quad \text{for} \quad |x-a_1 t| \gg (4b_1 t)^{1/2}, \tag{42}$$

the sgn term appears since C and S are odd functions of their argument. In the region $|x-a_1 t| \ll (4b_1 t)^{1/2}$ where γ is small, h_{12} is of the order γ. On the other hand, the asymptotic value of h_{11} and h_{13} for large t can be calculated by applying the method of stationary phase, as was carried out in detail in Section 5 of Ref. 1. The final result in the present case is

$$h_{11} \cong 0(t^{-1/2}) \qquad \text{for} \qquad x > a_1 t + (4b_1 t)^{1/2},$$

$$h_{13} \cong 0(t^{-1/2}) \qquad \text{for} \qquad Ut < x < a_1 t - (4b_1 t)^{1/2},$$

and they are otherwise of the order t^{-1}. Therefore, for t large, $h_1(x,t)$ has the same asymptotic value as h_{12} of (42).

In a similar manner it can be shown that for t large,

$$h_2(x,t) \cong \frac{i\,\mathrm{sgn}\,(x-a_2t)}{2g\sqrt{1+\alpha}}\,\kappa_2 e^{-i\kappa_2 x} + 0(t^{-1/2}) \quad \text{for } |x-a_2t| \gg (4\,b_2t)^{1/2}, \tag{43}$$

where a_2, b_2 are given by (36). The integral for h_3 nevertheless has to be treated separately for $0<\alpha<1$ and $\alpha>1$. The final result for $0<\alpha<1$ can be shown to be

$$h_3(x,t) \cong \frac{i\,\mathrm{sgn}\,(x+a_3t)}{2g\sqrt{1-\alpha}}\,\kappa_3 e^{i\kappa_3 x} + 0(t^{-1/2}) \quad \text{in} \quad |x+a_3t| \gg (4\,b_3t)^{1/2}, \tag{44a}$$

$$-\frac{i\,\mathrm{sgn}\,(x+a_4t)}{2g\sqrt{1-\alpha}}\,\kappa_4\, e^{i\kappa_4 x} + 0(t^{-1/2}) \quad \text{in} \quad |x+a_4t| \gg (4\,b_4t)^{1/2};$$

whereas for $\alpha>1$,

$$h_3(x,t) \cong 0\,(t^{-1/2}) \quad \text{for} \quad x<Ut, \quad \cong 0\,(t^{-1}) \quad \text{for} \quad x>Ut. \tag{44b}$$

On the other hand, for all positive α,

$$h_4(x,t) \cong 0\,(t^{-1/2}) \quad \text{for} \quad x>Ut, \quad \cong 0\,(t^{-1}) \quad \text{for} \quad x<Ut. \tag{45}$$

In these equations, a_3, a_4, b_3, b_4 are again given by (36). The asymptotic expressions for h_2 and h_3 in the regions not given above are omitted here because they are of less importance.

In summary, by combining the above results, one obtains finally that for $0<\alpha<1$ and t large, $H(x,t)$ has the following component waves:

$$
\begin{aligned}
H(x,t) \cong\ & \frac{i\kappa_1}{g\sqrt{1+\alpha}}\, e^{-i\kappa_1 x} + 0(t^{-1/2}) \quad \text{in} \quad 0<x<a_1 t-(4b_1 t)^{1/2}, \\
& - \frac{i\kappa_2}{g\sqrt{1+\alpha}}\, e^{-i\kappa_2 x} + 0(t^{-1/2}) \quad \text{in} \quad 0<x<a_2 t-(4b_2 t)^{1/2}, \\
& + \frac{i\kappa_4}{g\sqrt{1-\alpha}}\, e^{i\kappa_4 x} + 0(t^{-1/2}) \quad \text{in} \quad 0<x<-a_4 t-(4b_4 t)^{1/2}, \\
& + \frac{i\kappa_3}{g\sqrt{1-\alpha}}\, e^{i\kappa_3 x} + 0(t^{-1/2}) \quad \text{in} \quad -a_3 t+(4b_3 t)^{1/2}<x<0,
\end{aligned} \tag{46}
$$

and $H(x,t)$ is of the order $t^{-1/2}$ outside these regions. However, for $\alpha > 1$ and t large, the last two components then drop out and only the first two components remain.

As $t \to +\infty$, the regions in which these waves propagate extend to infinity and all transient waves whose amplitude depends on t diminish. It is in this manner that the limiting solution $H(x)$ for $\alpha \neq 1$, as was obtained previously, is approached.

The asymptotic behavior of the solution when $\alpha = 1$ is already discussed in Sec. 3. 3.

6. Discussion of the Result

Consider the fundamental case: if

$$
p_o(x) = \rho g B\ \delta(x), \qquad \delta(x) = \text{Dirac delta function,} \tag{47}
$$

(here B has the dimension of area), then

$$\varphi(x,y,t) = gB\, e^{i\Omega t}\ \Phi(x,y,t), \qquad \zeta(x,t) = gB\, e^{i\Omega t}\, H(x,t). \qquad (48)$$

Moreover, we are interested at present in the behavior of the solution as $t \to +\infty$. In what follows the local effect which is important only near the origin will be omitted. The following cases will be discussed.

(1) $0<\alpha<1$ and $t \to +\infty$.

In this case the solution (48) becomes

$$\zeta \cong \frac{iB}{\sqrt{1+\alpha}}\left[\kappa_1 e^{-i(\kappa_1 x-\Omega t)} - \kappa_2 e^{-i(\kappa_2 x-\Omega t)}\right] + \frac{iB\kappa_4}{\sqrt{1-\alpha}}\, e^{i(\kappa_4 x+\Omega t)}$$

$$\text{for } x>0\,, \qquad (49)$$

$$\cong \frac{iB\kappa_3}{\sqrt{1-\alpha}}\, e^{i(\kappa_3 x+\Omega t)} \qquad \text{for } x<0\,.$$

The corresponding solution of φ is

$$\varphi \cong -\frac{B}{\sqrt{1+\alpha}}\left[\sqrt{g\kappa_1}\, e^{\kappa_1 y-i(\kappa_1 x-\Omega t)} + \sqrt{g\kappa_2}\, e^{\kappa_2 y-i(\kappa_2 x-\Omega t)}\right]$$

$$-B\sqrt{\frac{g\kappa_4}{1-\alpha}}\, e^{\kappa_4 y+i(\kappa_4 x+\Omega t)} \qquad \text{for } x>0\,, \qquad (50)$$

$$= -B\sqrt{\frac{g\kappa_3}{1-\alpha}}\, e^{\kappa_3 y+i(\kappa_3 x+\Omega t)} \qquad \text{for } x<0\,.$$

The solution above may be interpreted physically by taking its real or imaginary part. In this case there are three wave trains on the downstream side, but only one wave propagates upstream. The wave with $\exp(i\Omega t - i\kappa_1 x)$ has wave length $\lambda_1 = 2\pi/\kappa_1$, amplitude

$A_1 = B\kappa_1/\sqrt{1+a}$; its phase velocity is $c_1 = \Omega/\kappa_1 = Ua(\sqrt{1+a}-1)^{-2}$, and its group velocity is $c_{g1} = (d\omega_1/dk)_{k=\kappa_1} = a_1$ (with a_1 given by Eq. 36), both velocities being referred to the fixed applied surface force. Similar computations can be made for the other waves. These results may be collected as follows:

$$\lambda_1 = \frac{2\pi}{\kappa_1}, \qquad A_1 = \frac{B\kappa_1}{\sqrt{1+a}}, \qquad c_1 = U\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}, \qquad c_{g1} = a_1 = \frac{U\sqrt{1+a}}{\sqrt{1+a}-1}; \tag{51a}$$

$$\lambda_2 = \frac{2\pi}{\kappa_2}, \qquad A_2 = \frac{B\kappa_2}{\sqrt{1+a}}, \qquad c_2 = U^2/c_1, \qquad c_{g2} = a_2 = \frac{U\sqrt{1+a}}{\sqrt{1+a}+1}; \tag{51b}$$

$$\lambda_3 = \frac{2\pi}{\kappa_3}, \qquad A_3 = \frac{B\kappa_3}{\sqrt{1-a}}, \qquad c_3 = -U\frac{1+\sqrt{1-a}}{1-\sqrt{1-a}}, \qquad c_{g3} = -a_3 = -\frac{U\sqrt{1-a}}{1-\sqrt{1-a}}; \tag{51c}$$

$$\lambda_4 = \frac{2\pi}{\kappa_4}, \qquad A_4 = \frac{B\kappa_4}{\sqrt{1-a}}, \qquad c_4 = U^2/c_3, \qquad c_{g4} = -a_4 = \frac{U\sqrt{1-a}}{1+\sqrt{1-a}}. \tag{51d}$$

In the above, c_{g3} and c_{g4} take the value of $(-d\omega_3/dk)$ at κ_3 and κ_4 respectively (as can be seen from Eq. 12b). Positive values of c indicate the wave propagation in the direction of positive x (or downstream side), otherwise the propagation is upstream. Thus, waves 1 and 2 propagate toward downstream on the downstream side, wave 4 propagates toward upstream on the downstream side, while wave 3 propagates toward upstream on the upstream side. On the other hand, the group velocity, arising from the dispersion phenomenon, has the important physical significance that the wave energy is transmitted at the rate c_g. The above values of c_g for the different waves are certainly consistent with the previous result of the wave extent in space for large time (see Eq. 46). It may be noted, however, that c_4 and c_{g4} are opposite in sign; this is due to the special coordinate system to which they are being referred. With respect to the fluid at rest, for example, c and c_g of these waves all have the same sign. More precisely, one has the relation:

$$c_{gn} - U = \frac{1}{2}(c_n - U)\,, \qquad n = 1, 2, 3, 4 \qquad (52)$$

which means that the group velocity is half the phase velocity when both are referred to the fluid at rest, a well-known result for gravity waves (Ref. 2).

As to the relative magnitudes of the above quantities, one may note that

$$\kappa_1 < \kappa_3 < \alpha\kappa_o < \kappa_4 < \kappa_2 \qquad \text{for} \quad 0<\alpha<1\,. \qquad (53a)$$

Hence,

$$\lambda_1 > \lambda_3 > 2\pi U/\Omega > \lambda_4 > \lambda_2\,, \qquad (53b)$$

$$A_1 < \alpha\,(4U^2\sqrt{1+\alpha})^{-1} < A_2\,, \qquad A_3 < \alpha\,(4U^2\sqrt{1-\alpha})^{-1} < A_4, \qquad (53c)$$

$$c_1 > |c_3| > U > |c_4| > c_2\,, \qquad (53d)$$

$$c_{g1} > U\left[(1+\alpha)/\alpha\right]^{1/2} > c_{g2}\,, \quad |c_{g3}| > U\left[(1-\alpha)/\alpha\right]^{1/2} > c_{g4}\,. \qquad (53e)$$

From the viewpoint of energy conservation (see also Ref. 2, p. 415), the work done per unit time (the power) by the external disturbance in order to maintain this motion must be equal to the energy transmitted, at the rate c_g, to these waves. In other words, since the extent of these waves enlarges at the rate $|c_{gn}|$ for time large (see Eq. 46) and the wave energy per unit area is $E_n = \frac{1}{2}\rho g A_n^2$, the work done by the disturbance in unit time must be

$$W = \sum_{n=1}^{4} |c_{gn}| \; E_n = \frac{\rho g^3 B^2}{4U^3} \left(1 + \sqrt{1-\alpha}\right)\left(1 + \frac{\alpha}{4\sqrt{1-\alpha}}\right) \quad \text{for} \quad 0<\alpha<1 \,. \tag{54}$$

Because of the complicated interference between the translatory and oscillatory motion, it is not clear how to divide W into two parts, each separately to be associated with these two types of motion.

(2) $\underline{\alpha>1 \text{ and } t\rightarrow+\infty}$.

In this case the waves 3 and 4 in (49) and (50) disappear while the other two waves remain. Aside from this point, the rest of the discussion given above still holds true. The work done in unit time to maintain this type of motion is

$$W = \sum_{n=1}^{2} c_{gn} \, E_n = \frac{\rho g^3 B^2}{4U^3} \left(1 + \frac{\alpha}{4}\right) \quad \text{for} \quad \alpha>1 \,. \tag{55}$$

The value $W/(\frac{1}{4}\rho g^3 B^2 U^{-3})$ given by (54) and (55) is plotted in Fig. 3 as a function of α. Its value increases from 2 very slowly for $0<\alpha<0.8$ and then increases rapidly as α approaches unity; for $\alpha>1$, it increases linearly from the value 5/4.

(3) $\alpha = 1$ and $\tau = gt/4U$ large, the "resonance" case.

In this case the first two terms with ω_1 and ω_2 in (12b) (combined to give H_1 in Sec. 3) still, of course, lead to the same asymptotic solution as before, which are the first two waves in (46) except that now

$$\kappa_1 = \kappa_o(\sqrt{2}-1)^2, \quad \kappa_2 = \kappa_o(\sqrt{2}-1)^2, \quad a_1 = U(2+\sqrt{2}), \quad a_2 = U(2-\sqrt{2}).$$

The other two terms in (12b), combined to become H_2, has the asymptotic representation (35). Thus, with the order terms omitted,

$$\begin{aligned}
\zeta(x,t) \cong\ & iB\kappa_o \frac{3\sqrt{2}-4}{2} \exp\left\{-i\kappa_1\left[x-(3+2\sqrt{2})Ut\right]\right\} \quad \text{in } 0<x\ll(2+\sqrt{2})Ut, \\
& - iB\kappa_o \frac{3\sqrt{2}+4}{2} \exp\left\{-i\kappa_2\left[x-(3-2\sqrt{2})Ut\right]\right\} \quad \text{in } 0<x\ll(2-\sqrt{2})Ut, \\
& + B\kappa_o\left(\frac{gt}{\pi U}\right)^{1/2} \exp\left\{i\left[\frac{g}{4U^2}(x+Ut)+\frac{\pi}{4}\right]\right\} \quad \text{in } |x|\ll 2U\left(\frac{Ut}{g}\right)^{1/2}.
\end{aligned} \tag{56}$$

The corresponding solution of φ is

$$\begin{aligned}
\varphi(x,t) \cong\ & -Bg\frac{2-\sqrt{2}}{4U} \exp\left\{\kappa_1 y - i\kappa_1\left[x-(3+2\sqrt{2})Ut\right]\right\} \quad \text{in } 0<x\ll(2+\sqrt{2})Ut, \\
& - Bg\frac{2+\sqrt{2}}{4U} \exp\left\{\kappa_2 y - i\kappa_2\left[x-(3-2\sqrt{2})Ut\right]\right\} \quad \text{in } 0<x\ll(2-\sqrt{2})Ut, \\
& - \frac{Bg}{2U}\left(\frac{gt}{\pi U}\right)^{1/2} \exp\left\{\kappa_o y + i\kappa_o(x+Ut) - i\frac{\pi}{4}\right\} \quad \text{in } |x|\ll 2U\left(\frac{Ut}{g}\right)^{1/2}.
\end{aligned} \tag{57}$$

This asymptotic solution shows that, in addition to the two waves on the downstream side, there is a wave propagating with phase velocity U in the upstream direction in a relatively shorter region on both sides of the pressure disturbance, the amplitude and the extent of this wave both increase at the rate proportional to $t^{1/2}$. The physical explanation for this "resonance" phenomenon is believed to be that the energy carried away by the waves is less than that of the total energy input to maintain the external disturbance so that the amplitude of the last wave increases with time. Since the extent of the last wave of (56) is $S = 4U(Ut/g)^{1/2}$ and the wave energy per unit area is $E = \rho g B^2 \kappa_o^2 (gt/2\pi U)$, the total work done in unit time by the external disturbance at $\alpha = 1$ is therefore equal to $d(SE)/dt$, or

$$W = \frac{3\rho g^3 B^2}{8\pi U^3} \left[\left(\frac{gt}{4U}\right)^{1/2} + 0(1) \right] \quad \text{for} \quad gt \gg 4U. \quad (58)$$

Besides the above cases, two other special cases are of interest:

(4) $\underline{\Omega \to 0 \text{ and } U > 0}$.

In this limiting case, $\alpha \to 0$ and

$$\kappa_1, \kappa_3 \to 0, \qquad \kappa_2 \to \kappa_4 \to g/U^2.$$

Hence (49) and (50) reduce to

$$\zeta(x) = -2B(g/U^2)\sin(gx/U^2) \qquad \text{for} \quad x > 0, \quad (59)$$

$$\varphi(x,y) = -2B(g/U)\exp(gy/U^2)\cos(gx/U^2) \quad \text{for} \quad x > 0, \quad (60)$$

and they are both of the order x^{-2} for negative large x. This result is indeed well known (see for example, Ref. 2). The wave resistance R of this motion is readily found to be given by

$$W = R\,U = \rho\, g^3 B^2 U^{-3}\,, \tag{61}$$

which is twice the limit of W in (54) as $\alpha \to 0$. This discrepancy is believed due to the fact that the two waves κ_2 and κ_4 coalesce and become one wave with its amplitude doubled, whereas W is quadratic in the amplitude.

(5) <u>$U = 0$ and $\Omega > 0$.</u>

If U is taken to be zero from the beginning, then from (13) and (16) $\Lambda_1 = \Lambda_2 = -g(k - \Omega^2/g)$; consequently $\kappa_1 = \kappa_3 = \Omega^2/g$ and κ_2 and κ_4 drop out from the analysis. The final result, with the local effect again neglected, is

$$\zeta(x, t) = i\,B\,(\Omega^2/g)\,\exp\left\{ i\,(\Omega t - \Omega^2 |x|/g) \right. \quad , \tag{62}$$

$$\varphi(x, y, t) = -B\Omega\,\exp\left\{ \Omega^2 y/g + i\,(\Omega t - \Omega^2 |x|/g) \right. \tag{63}$$

which is the result given by Green (Ref. 6) and Stoker (Ref. 7). Here one obtains only the out-going waves which propagate with phase velocity $c = g/\Omega$ and group velocity $c_g = c/2$. The work done by the external disturbance in unit time is

$$W = 2\,c_g\,E = \frac{1}{2}\,\rho\, B^2 \Omega^3 \tag{64}$$

which is independent of g.

It should be pointed out, however, that the limit of (49) and (50) as $U \to 0$ for $\Omega > 0$ cannot be directly obtained to agree with (62) and (63), nor can (54) be reduced to (64). But this difficulty can be overcome by taking into account also the effect of superposition of p_o (see Ref. 1) or the capillary effect. If p_o is distributed over an area on the surface, or if surface tension is included, then it can be shown that (49) and (50) tend to (62) and (63) as $U \to 0$.

References

1. De Prima, C.R. and Wu, T.Y., "On the Theory of Surface Waves in Water Generated by Moving Disturbances", California Institute of Technology, Engineering Division Report No. 21-23, May, 1957.

2. Lamb, H., Hydrodynamics, 6th Ed., Dover Publication, New York, 1945.

3. Widder, D.V., The Laplace Transform, Princeton University Press, 1946.

4. Jahnke, E. and Emde, F., Tables of Functions, 4th Ed., Dover Publication, 1945.

5. Jeffreys, H. and Jeffreys, B.S., Methods of Mathematical Physics, 2nd Ed., Cambridge University Press, 1950.

6. Green, G., "Waves in Deep Water Due to a Concentrated Surface Pressure", Phil. Mag. 7, Vol. 39, p. 738, 1948.

7. Stoker, J.J., "Some Remarks on Radiation Conditions", Wave Motion and Vibration Theory, Vol. 5, p. 97, Proc. 5th Symposium Appl. Math., Amer. Math. Soc. 1952.

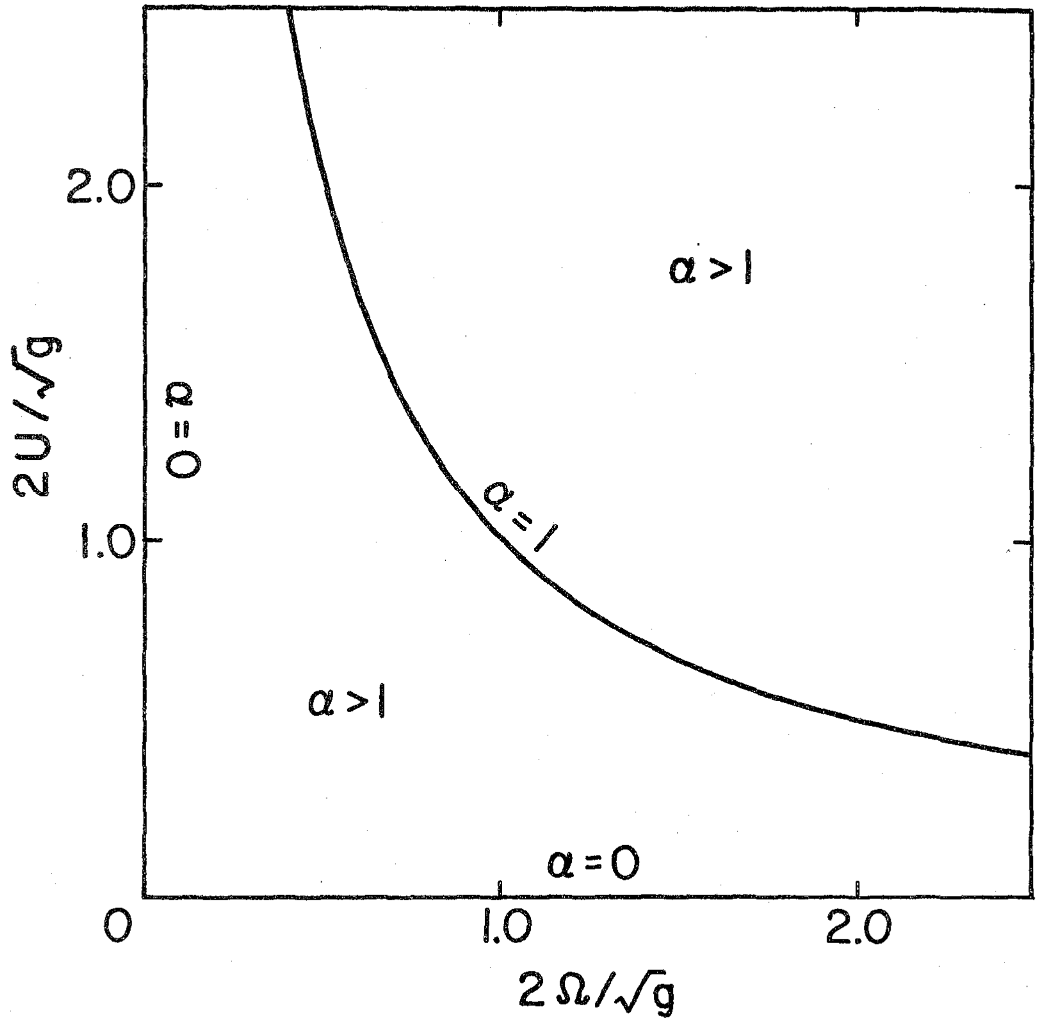
2.0
1.0
0
1.0
2.0
$2U/\sqrt{g}$
$2\Omega/\sqrt{g}$
$\alpha > 1$
$\alpha = 0$
$\alpha = 1$
$\alpha > 1$
$\alpha = 0$

Fig 1.


k-plane
k-plane
C_{12}
k = 0
K_1
K_2
K_4
k = 0
K_3
C_{34}

Fig 2.

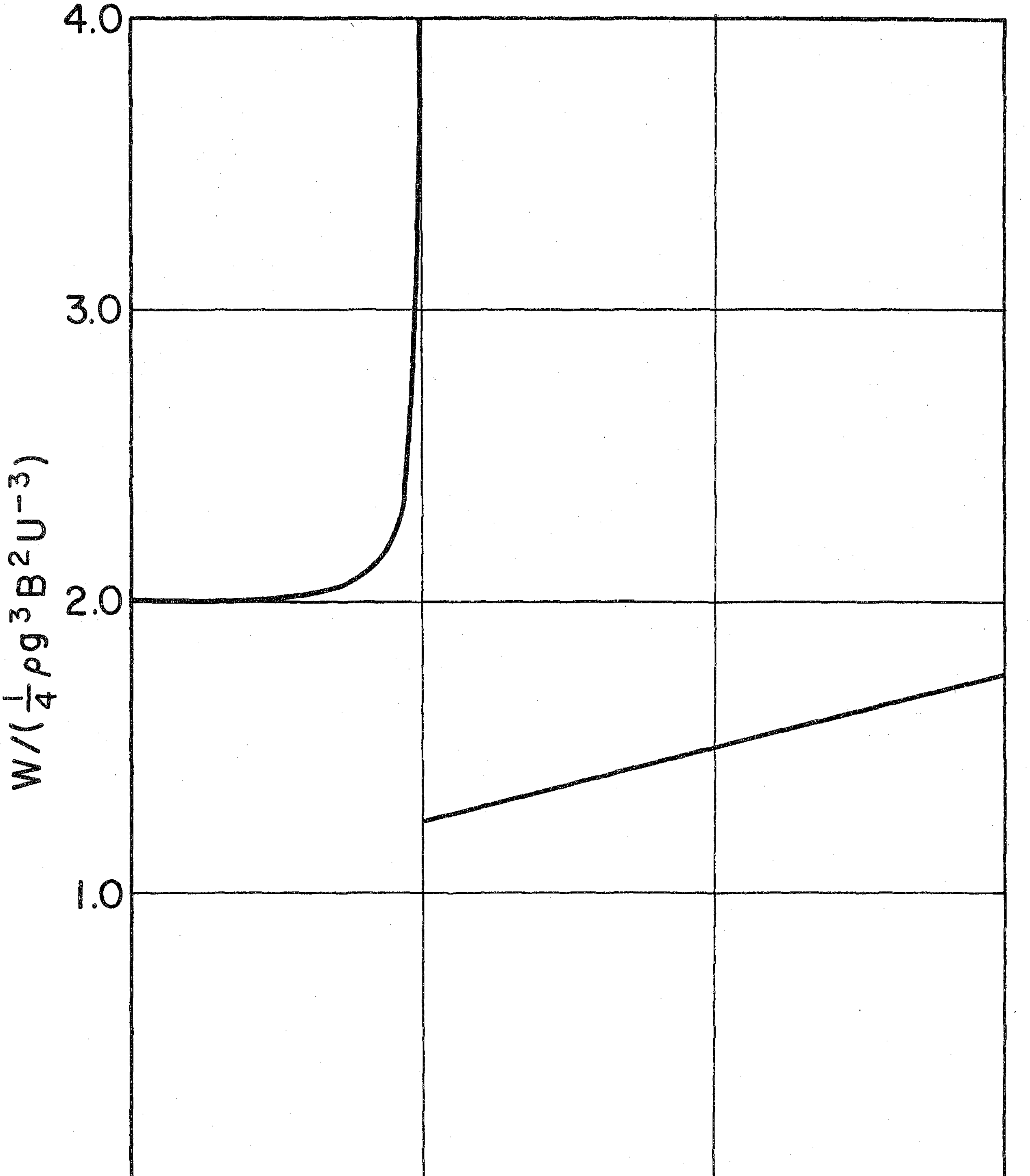

Fig 3.

DISTRIBUTION LIST FOR UNCLASSIFIED
REPORTS ON CAVITATION

Contract Nonr-220(28)

Chief of Naval Research
Navy Department
Washington 25, D.C.
Attn: Code 438 (3)
Code 463 (1)

Commanding Officer
Office of Naval Research
Branch Office
The John Crear Library Bldg.
86 E. Randolph Street
Chicago 1, Illinois (1)

Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, New York (1)

Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, California (1)

Commanding Officer
Office of Naval Research
Navy 100, Fleet Post Office
New York, New York (25)

Director
Naval Research Laboratory
Washington 25, D.C.
Attn: Code 2021 (6)

Chief, Bureau of Aeronautics
Navy Department
Washington 25, D.C.
Attn: Research Division (1)
Aero and Hydro Branch (Code Ad-3) (1)
Appl. Mech. Branch (Code DE-3) (1)

Chief, Bureau of Ordnance
Navy Department
Washington 25, D.C.
Attn: Asst. Chief for Research (Code Re) (1)
Systems Director, Underwater Ord (Code Rexc) (1)
Armor, Bomb, Projectile, Rocket, Guided Missile Warhead and Ballistics Branch (Code Re3) (1)
Torpedo Branch (Code Re6) (1)
Research and Components Section (Code Re6a) (1)
Mine Branch (Code Re7) (1)

Chief, Bureau of Ships
Navy Department
Washington 25, D.C.
Attn: Research and Development (Code 300) (1)
Ship Design (Code 410) (1)
Preliminary Design and Ship Protection (Code 420) (1)
Scientific, Structural and Hydrodynamics (Code 442) (1)
Submarines (Code 525) (1)
Propellers and Shafting (Code 554) (1)

Chief, Bureau of Yards and Docks, Navy Department
Washington 25, D.C.
Attn: Research Division (1)

Commander
Naval Ordnance Test Station
3202 E. Foothill Blvd.
Pasadena, California
Attn: Head, Underwater Ord. (1)
Head, Research Div. (1)

Commander
Naval Ordnance Test Station
Inyokern, China Lake, Calif.
Attn: Technical Library (1)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D.C.
Attn: Hydromechanics Lab. (1)
Seaworthiness and Fluid Dynamics Div. (1)
Library (1)

Commanding Officer
Naval Ordnance Laboratory
White Oak, Maryland
Attn: Underwater Ord. Dept. (1)

Commanding Officer
Naval Underwater Ordnance Sta.
Newport, Rhode Island (1)

Director
Underwater Sound Laboratory
Fort Trumbull
New London, Connecticut (1)

Librarian
U.S. Naval Postgraduate School
Monterey, California (1)

Executive Secretary
Research and Development Board
Department of Defense
The Pentagon
Washington 25, D.C. (1)

Chairman
Underseas Warfare Committee
National Research Council
2101 Constitution Avenue
Washington 25, D.C. (1)

Dr. J. H. McMillen
National Science Foundation
1520 H Street, N.W.
Washington, D.C. (1)

Director
National Bureau of Standards
Washington 25, D.C.
Attn: Fluid Mechanics Section (1)

Dr. G. H. Keulegan
National Hydraulic Laboratory
National Bureau of Standards
Washington 25, D.C. (1)

Director of Research
National Advisory Committee for Aeronautics
1512 H Street, N.W.
Washington 25, D.C. (1)

Director
Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Virginia (1)

Mr. J. B. Parkinson
Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Virginia (1)

Commander
Air Research and Development Command
P.O. Box 1395
Baltimore 18, Maryland
Attn: Fluid Mechanics Div. (1)

Director
Waterways Experiment Station
Box 631
Vicksburg, Mississippi (1)

Beach Erosion Board
U.S. Army Corps of Engineers
Washington 25, D.C. (1)

Office of Ordnance Research
Department of the Army
Washington 25, D.C. (1)

Office of the Chief of Engineers
Department of the Army
Gravelly Point
Washington 25, D.C. (1)

Commissioner
Bureau of Reclamation
Washington 25, D.C. (1)

Director
Oak Ridge National Laboratory
P.O. Box P
Oak Ridge, Tennessee (1)

Director
Applied Physics Division
Sandia Laboratory
Albuquerque, New Mexico (1)

Professor Carl Eckart
Scripps Institute of Oceanography
La Jolla, California (1)

Documents Service Center
Armed Services Technical Information Agency
Knott Building
Dayton 2, Ohio (5)

Office of Technical Services
Department of Commerce
Washington 25, D.C. (1)

Polytechnic Institute of Brooklyn
Department of Aeronautical Engineering and Applied Mech.
99 Livingston Street
Brooklyn 1, New York
Attn: Prof. H. Reissner (1)

Division of Applied Mathematics
Brown University
Providence 12, Rhode Island (1)

California Institute of Technology
Pasadena 4, California
Attn: Hydrodynamics Laboratory
Professor A. Hollander (1)
Professor R. T. Knapp (1)
Professor M. S. Plesset (1)
Professor V. A. Vanoni (1)
GALCIT
Prof. C.B. Millikan, Director (1)
Prof. Harold Wayland (1)

University of California
Berkeley 4, California
Attn: Professor H.A. Einstein
Dept. of Engineering (1)
Professor H. A. Schade,
Dir. of Engr. Research (1)

Case Institute of Technology
Department of Mechanical Engineering
Cleveland, Ohio
Attn: Professor G. Kuerti (1)

Cornell University
Graduate School of Aeronautical Engineering
Ithaca, New York
Attn: Prof. W.R. Sears, Director(1)

Harvard University
Dept. of Mathematics
Cambridge 38, Mass.
Attn: Prof. G. Birkhoff (1)

University of Illinois
Dept. of Theoretical and Applied Mechanics
College of Engineering
Urbana, Illinois
Attn: Dr. J. M. Robertson (1)

Indiana University
Dept. of Mathematics
Bloomington, Indiana
Attn: Professor D. Gilbarg (1)

State University of Iowa
Iowa Institute of Hydraulic Research, Iowa City, Iowa
Attn: Dr. Hunter Rouse, Dir (1)

University of Maryland
Inst. for Fluid Dynamics and Applied Mathematics
College Park, Maryland
Attn: Prof. M. H. Martin,
Director (1)
Prof. J. R. Weske (1)

Massachusetts Institute of Technology
Cambridge 39, Mass.
Attn: Prof. W.M. Rohsenow,
Dept. Mech. Engr. (1)
Prof. A.T. Ippen,
Hydrodynamics Laboratory (1)

Michigan State College
Hydraulics Laboratory
East Lansing, Michigan
Attn: Prof. H.R. Henry (1)

University of Michigan
Ann Arbor, Michigan
Attn: Director, Engineering
Research Institute (1)
Prof. V.L. Streeter,
Civil Engineering Dept. (1)

University of Minnesota
St. Anthony Falls Hydraulic Lab.
Minneapolis 14, Minnesota
Attn: Dr. L.G. Straub, Dir. (1)

New York University
Institute of Mathematical Sciences
25 Waverly Place
New York 3, New York
Attn: Prof. R. Courant, Dir. (1)

University of Notre Dame
College of Engineering
Notre Dame, Indiana
Attn: Dean K.E. Schoenherr (1)

Pennsylvania State University
Ordnance Research Laboratory
University Park, Pennsylvania
Attn: Prof. G.F. Wislicenus (1)

Rensselaer Polytechnic Inst.
Dept. of Mathematics
Troy, New York
Attn: Dr. Hirsh Cohen (1)

Stanford University
Stanford, California
Attn: Applied Math. and
Statistics Laboratory (1)
Prof. P.R. Garabedian (1)
Prof. L.I. Schiff, Dept. of
Physics (1)
Prof. J.K. Vennard, Dept.
of Civil Engineering (1)

Stevens Institute of Technology
Experimental Towing Tank
711 Hudson Street
Hoboken, New Jersey (1)

Worcester Polytechnic Institute
Alden Hydraulic Laboratory
Worcester, Mass.
Attn: Prof. J.L. Hooper,
Director (1)

Dr. Th. von Karman
1051 S. Marengo Street
Pasadena, California (1)

Aerojet General Corporation
6352 N. Irwindale Avenue
Azusa, California
Attn: Mr. C. A. Gongwer (1)

Dr. J.J. Stoker
New York University
Institute of Mathematical Sciences
25 Waverly Place
New York 3, New York (1)

Prof. C.C. Lin
Dept. of Mathematics
Massachusetts Institute of
Technology
Cambridge 39, Mass. (1)

Dr. Columbus Iselin
Woods Hole Oceanographic Inst.
Woods Hole, Mass. (1)

Dr. A.B. Kinzel, Pres.
Union Carbide and Carbon Research Laboratories, Inc.
30 E. 42nd St.
New York, N.Y. (1)

Dr. F.E. Fox
Catholic University
Washington 17, D.C. (1)

Dr. Immanuel Estermann
Office of Naval Research
Code 419
Navy Department
Washington 25, D.C. (1)

Goodyear Aircraft Corp.
Akron 15, Ohio
Attn: Security Officer (1)

Dr. F.V. Hunt
Director Acoustics Research
Laboratory
Harvard University
Cambridge, Mass. (1)

Prof. Robert Leonard
Dept. of Physics
University of California at
Los Angeles
West Los Angeles, Calif. (1)

Prof. R.E.H. Rasmussen
Buddenvej 47, Lyngby
Copenhagen, Denmark
via: ONR, Pasadena, Calif. (1)

Technical Librarian
AVCO Manufacturing Corp.
2385 Revere Beach Parkway
Everett 49, Mass. (1)

Dr. L. Landweber
Iowa Inst. of Hydraulic Research
State University of Iowa
Iowa City, Iowa (1)

Dr. M. L. Ghai, Supervisor
Heat Transfer/Fluid Mechanics
Rocket Engine-Applied Research
Building 600
Aircraft Gas Turbine Division
General Electric Company
Cincinnati 15, Ohio (1)

Dr. W. W. Clauson
Rose Polytechnic Institute
R.R. No. 5
Terre Haute, Indiana (1)

Mr. Kurt Berman
Rocket Engine Section
Aircraft Gas Turbine
Development Dept.
Malta Test Station
Ballston Spa, New York (1)