The Absorption of Sound in Suspensions and Emulsions.* I. Water Fog in Air

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The suspended particles are approximated by spheres and the diffraction problem for a fluid sphere in a fluid medium is solved taking into consideration viscosity and thermal conduction. The results are discussed numerically for water droplets in air and a satisfactory agreement with Knudsen's attenuation measurements in water fog is found.

LIST OF SYMBOLS

A viscous wave potential
a, b, c radius of droplets (relative to l, L, λ)
C_v, C_p specific heats
e_{ij} strain tensor
F surface
g, G coefficients, Eq. (9.3)
j_n, h_n Heine functions, App. C.
k, K wave numbers of compressional and thermal waves
l, L wave lengths of compressional and thermal waves
n concentration of droplets
M coefficient, Eq. (9.8)
N coefficient, Eq. (3.5)
p pressure
p_{ij} viscous stress tensor
P_{ij} total stress tensor
q normal velocity of sound
Q coefficient, Eq. (9.5)
r radial coordinate
R radius of droplet
s, S entropy (specific and total)
t time
T temperature
u specific internal energy
v velocity
V volume
w work (specific)
Y function, Eqs. (1.2) and (1.3)
$y = (\omega/2\sigma)^{\frac{1}{2}}R$ parameter
$z = (\omega/2\nu)^{\frac{1}{2}}R$ parameter
α coefficient of volume expansion
β coefficient of attenuation
$\gamma = C_p/C_v$ ratio of specific heats
$\delta = \rho/\rho'$ ratio of densities
δ_{ij} Kronecker symbol
$\epsilon = \eta/\eta'$ ratio of coefficients of viscosity
η coefficient of shear viscosity
ϑ polar angle
κ wave number of viscous wave
λ wavelength of viscous wave
μ coefficient of compressional viscosity
ν kinematic viscosity

ρ density
$\sigma = \tau/\rho C_p$ thermometric conductivity
τ thermal conductivity
φ compressional potential
Φ thermal potential
ψ azimuthal angle
Ψ dissipation function
$\chi = \tau/\tau'$ ratio of conductivities
$\xi = \tau G/\tau' G'$ ratio
ω angular frequency

1. INTRODUCTION

THE earliest theoretical treatment of the problem of sound absorption in fogs was given by Sewell.[1] In his work the fog particles were approximated by small, rigid, and fixed spheres embedded in a viscous fluid without thermal conductivity. Later a model closer to the actual conditions was analyzed by the senior author.[2] To the free small spherical obstacles representing the particles of a suspension were attributed the properties either of viscous fluids or of elastic solids. However, thermal conductivity was left out of account.

The investigation now in progress has the purpose of making the description complete by taking into consideration both the viscosity and the thermal conductivity. The present paper presents that part of the general theory which is concerned with the effect caused by fluid spheres embedded in a fluid medium. Moreover, the reduction of the general results to explicit formulas of attenuation is restricted in this publication to the case of liquid droplets suspended in gases, with particular attention to water particles in air.

In general, the effects of viscosity and of heat conduction are intermingled; but within the approximation in which a certain small dimensionless parameter, $(\gamma-1)\omega\sigma/q$ (see table of notations), may be treated as negligible, the attenuation β can be divided into two parts

$$\beta = \beta_\eta + \beta_\tau. \tag{1.1}$$

In practice this division is always permissible since the parameter in question is, even at frequencies as high as 10^6 cycles, less than 0.001 in air and considerably smaller in water.

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[1] C. J. T. Sewell, Phil. Trans. (A)210, 239 (1910).
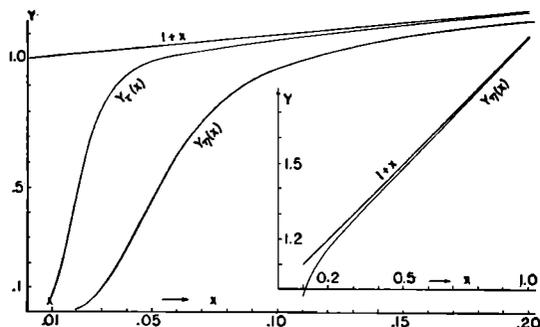[2] Paul S. Epstein in *Applied Mechanics* (Theodore von Karman Anniversary Volume, 1941), p. 162.

FIG. 1. Auxiliary curves.

The first term may be called the coefficient of *viscous attenuation* because it depends on the coefficient of viscosity η. Its form is the same as that obtaining in a viscous and nonconducting medium, so that its mathematical expressions are identical with those found by Epstein in the paper referred to. When the incident acoustical wave is long compared with the radius R of the droplet the explicit formula is with sufficient accuracy (compare Sec. 15)

$$\left.\begin{aligned} \beta_\eta &= (6\pi R/q)n\nu Y_\eta, \\ Y_\eta &= (1+z)\frac{16z^4}{16z^4+72\delta z^3+81\delta^2(1+2z+2z^2)}, \end{aligned}\right\} \quad (1.2)$$

with the abbreviation $z=(\omega/2\nu)^{\frac{1}{2}}R$ (in air).

In a similar way, the second term must be designated as the coefficient of *thermal attenuation*. Under the conditions just mentioned its explicit expression is as follows.

$$\left.\begin{aligned} \beta_\tau &= (4\pi R/q)n(\gamma-1)\sigma Y_\tau, \\ Y_\tau &= 1+y, \end{aligned}\right\} \quad (1.3)$$

with y and y' representing the combination $(\omega/2\sigma)^{\frac{1}{2}}R$ in air and water, respectively. Formula (1.3) is valid only for $y' > 8.0$. A more involved expression for $y' < 1.2$ is given in Sec. 16, formula (16.2). For the intermediate region the values must be inferred from Fig. 1.

The only experimental evidence accurate enough to test our theory is afforded by Knudsen's measurements of the reverberation time in chambers filled with an artificially produced water fog.[3] As discussed in detail in Sec. 17, we find an agreement which we consider entirely sufficient.

It seems appropriate to close the introduction by saying a word about the physical mechanism of the comparatively large attenuation given by Eqs. (1.2) and (1.3). Whenever a compressional acoustic wave falls on a surface of discontinuity, there are produced—in addition to the reflected and transmitted compressional waves—waves of two other kinds spreading from the

[3] Knudsen, Wilson, and Anderson, J. Acoust. Soc. Am. **20**, 849 (1940).

surface to both sides. In either medium there arises a highly damped longitudinal wave and a highly damped transverse wave. The energy loss is due to the practically complete absorption of the four additional waves in very thin layers of the two media.

2. FUNDAMENTALS

The theory of this paper is based on the hydrodynamic equations of conservation subject to certain assumptions and simplifications which it will be well to enumerate.

(1) The equation of conservation of momentum is used in the Stokes-Navier form. The additional terms representing thermal stresses and required by the kinetic theory of gases are of necessity neglected because their exact structure is known only for rarefied gases. It is known, however, that in fluids they are small and would not change the result to a greater extent than 1 to 2 percent.

(2) Phenomena of relaxation have little bearing on our subject and will be considered to lie outside the scope of the investigation.

(3) We shall assume the system to be in a quasi-steady state so that the gradual changes in the mean temperature and pressure (brought about by the absorption) can be neglected over the time of the experiment. The possibility of partial vaporization of the droplets will be set aside also.

(4) In view of the small numerical values of velocity and pressure even in vigorous sound waves, we shall restrict ourselves to first- and second-order terms in these quantities. In keeping with this, it is permissible to neglect the variability of the coefficients of viscosity and heat conduction with temperature and to treat them as constants.

(5) On the other hand, our treatment will be more general than usual in one particular respect. The simplifying assumption will not be made that the compressional (or "second") coefficient of viscosity vanishes. Instead, it will be retained throughout the investigation, and the components of the viscous stress tensor will be written in terms of the strain tensor,

$$e_{ij}=(\partial v_i/\partial x_j)+(\partial v_j/\partial x_i), \quad (2.1)$$

and of the Kronecker symbol δ_{ij} (equal to 1 for $i=j$ and to 0 for $i\neq j$) in the following form

$$p_{ij}^{\bullet}=\eta e_{ij}-\tfrac{2}{3}(\eta-\mu)\nabla\cdot\mathbf{v}\,\delta_{ij}, \quad (2.2)$$

where μ denotes the coefficient of compressional and η of shear viscosity.

3. THE CONSERVATION EQUATIONS

The equation of continuity (conservation of mass) and the Stokes-Navier equations (conservation of momentum) are so well known that we shall write them down without further comment. Here, as in the following sections, we shall avail ourselves of Einstein's

summation convention.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0, \qquad (3.1)$$

$$\rho \frac{\partial v_j}{\partial t} + \rho v_i \frac{\partial v_j}{\partial x_i} - \frac{\partial P_{ij}}{\partial x_i} = 0. \qquad (3.2)$$

By P_{ij} is denoted the total stress tensor including the hydrostatic pressure p,

$$P_{ij} = -p\delta_{ij} + p_{ij}. \qquad (3.3)$$

Hence, the second equation can be written in the more explicit form

$$\rho \frac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + N\eta\nabla(\nabla \cdot \mathbf{v}) - \eta\nabla \times \nabla \times \mathbf{v}, \quad (3.4)$$

where the number N is an abbreviation for

$$N = (4/3)(1 + \mu/2\eta). \qquad (3.5)$$

The energy equation is less common but can be also found in literature.[4]

$$\rho \frac{\partial u}{\partial t} + \rho v_i \frac{\partial u}{\partial x_i} + p \frac{\partial v_i}{\partial x_i} - \Psi_\eta - \frac{\partial}{\partial x_i}\left(\tau \frac{\partial T}{\partial x_i}\right) = 0. \quad (3.6)$$

The symbol Ψ_η represents here the viscous dissipation function

$$\Psi_\eta = \tfrac{1}{2} e_{ij} p_{ij}. \qquad (3.7)$$

In the interest of the next section we give an alternative expression of the energy equation obtained by using the relation $\partial v_j/\partial x_j = -(\partial/\partial t + v_j\partial/\partial x_j)\log\rho$, resulting from Eq. (3.1).

$$\rho\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}\right)u + p\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}\right)(1/\rho)$$

$$-\Psi_\eta - \frac{\partial}{\partial x_j}\left(\tau\frac{\partial T}{\partial x_j}\right) = 0. \quad (3.8)$$

4. THE DISSIPATED ENERGY

It is our intention to determine what fraction of the energy of the sound waves is absorbed or converted into a form unavailable to the acoustic field. In the language of thermodynamics this energy is *degraded* or *dissipated*, a process which is accompanied by an irreversible entropy increase. The forces of viscosity produce energy degradation to an extent represented—per unit time and volume—by the viscous dissipation function Ψ_η. Further degradation of energy is caused by the currents of thermal conduction circulating in the field. This process was studied by Tolman and Fine[5]

[4] For example, A. Busemann, "Gasdynamik," *Handbuch der Experimentalphysik* (Edwards Brothers, Inc., Ann Arbor, 1940), Vol. IV, p. 1.
[5] R. C. Tolman and P. C. Fine, Revs. Modern Phys. 20, 51 (1948).

who found that the energy made unavailable by it is given by the *thermal dissipation function*

$$\Psi_\tau = (\tau/T)(\nabla T)^2. \qquad (4.1)$$

It will be necessary to give the derivation for the case when the two processes are taking place simultaneously in order to insure that the two effects are additive. We consider a volume V of the field, limited by the stationary surface F and write the equation of the entropy balance, as used by the authors just mentioned[5]

$$(\partial\ \partial t)\int_V \rho s\,dV = -\int_F \rho s v_N\,dF + \int_F (\tau/T)(\partial T/\partial x_N)\,dF$$

$$+ (\partial/\partial t)\int_V S_{irr}\,d\tau.$$

The left side represents the total rate of change of the entropy in the volume V, which is well defined, since the entropy is a function of the state. The right side represents the sources of this entropy, namely, the entropy carried in by a flow of matter, the entropy due to the heat flow through the surface, and the irreversible entropy S_{irr} created per unit volume through dissipative processes. When all terms are transformed into volume integrals and Eq. (3.1) is used, the condition becomes

$$\rho(\partial/\partial t - v_j\partial/\partial x_j)s - (\partial/\partial x_j)[(\tau/T)(\partial T/\partial x_j)]$$
$$= (\partial/\partial t)S_{irr}. \quad (4.2)$$

The change of entropy s depends only on the initial and final state and must be evaluated by considering a reversible (infinitely slow) process connecting these states. In the entropy differential, $\partial s = \partial u + \partial w$, relating to such a process the stress tensor P_{ij} of Eq. (3.3) is reduced to $-p$ because the velocities vanish. Hence, the element of work becomes $\partial w = p\partial(1/\rho)$ and the first term may be transformed by applying Eq. (3.8) into

$$\frac{\rho}{T}\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}\right)u + p\left(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}\right)(1/\rho)$$

$$= \frac{1}{T}\left[\Psi_\eta + \frac{\partial}{\partial x_j}\left(\tau\frac{\partial T}{\partial x_j}\right)\right].$$

Returning to (4.2), we find

$$(\partial/\partial t)S_{irr} = (\Psi_\eta + \Psi_\tau)/T.$$

Because of the relation $T\nabla S_{irr} = \nabla E_{dis}$ the degraded energy E_{dis} is indeed represented by the sum of the two dissipation functions, $\Psi_\eta + \Psi_\tau$.

It is important to notice that the expressions (3.5) and (4.1) are quadratic in the small quantities $\mathbf{v}$ and ∇T. As stated in Sec. 2, we intend to limit the accuracy of computing the dissipation effect to terms of the second order. Therefore, we need to know $\mathbf{v}$ and ∇T themselves only to terms of the first order. In other words, we can obtain all the desired results with *linearized equations*.

5. EQUATIONS OF THE ACOUSTIC FIELD

The acoustic equations are obtained by linearizing the conservation laws of Sec. 3. For this purpose the variables must be expanded to the first order. We characterize by the subscript 0 the normal values of the variables (in the absence of an acoustic field) and by the subscript 1 the small deviations resulting from the field.

$$\rho = \rho_0 + \rho_1, \qquad p = p_0 + p_1, \\ T = T_0 + T_1, \qquad u = u_0 + u_1. \tag{5.1}$$

Further, **v** itself and all the derivatives (for instance, $\partial \rho / \partial t = \partial \rho_1 / \partial t$) are quantities of the first order. Retaining in Eqs. (3.1), (3.4), and (3.6) only first-order terms, we obtain for them the linearized form

$$\partial \rho / \partial t + \rho_0 \nabla \cdot \mathbf{v} = 0, \tag{5.2}$$

$$\rho_0 \partial \mathbf{v} / \partial t + \nabla p - N\eta \nabla (\nabla \cdot \mathbf{v}) + \eta \nabla \times \nabla \times \mathbf{v} = 0, \tag{5.3}$$

$$\rho_0 \partial u / \partial t + p_0 \nabla \cdot \mathbf{v} - \tau \nabla^2 T = 0. \tag{5.4}$$

In the last equation use is made of the simplifying assumption that τ is a constant, as mentioned in Sec. 3.

Our next objective is to eliminate from the system the variables ρ_1, p_1, and u_1. To accomplish this we have at our disposal two additional equations, namely, the thermal and the caloric equations of state, which we write in the general form

$$p = p(\rho, T), \quad u = u(\rho, T). \tag{5.5}$$

It is convenient to replace the symbol of differentiation with respect to time by a dot ($\dot{u} = \partial u / \partial t$). Then, we can write

$$\dot{u} = (\partial u / \partial \rho)_T \dot{\rho} + (\partial u / \partial T)_\rho \dot{T}, \tag{5.6}$$

$$\nabla p = (\partial p / \partial \rho)_T \nabla \rho + (\partial p / \partial T)_\rho \nabla T. \tag{5.7}$$

These expressions must be also linearized, so that the partials of u and p refer to the normal state of the medium and must be regarded as constant over the field. These partials can be reduced to more familiar quantities in the following way. If q is the adiabatic velocity of sound and $\gamma = C_p/C_v$, the ratio of specific heats,

$$(\partial p / \partial \rho)_T = q^2 / \gamma. \tag{5.8}$$

Further, we can make use of the thermodynamic relations

$$(\partial u / \partial T)_\rho = C_v, \tag{5.9}$$

$$(\partial p / \partial T)_\rho = \rho_0 (\gamma - 1) C_v / \alpha T_0, \tag{5.10}$$

where C_v is the specific heat at constant volume and $\alpha = -(1/\rho_0)(\partial \rho / \partial T)_p$, the coefficient of thermal expansion. An alternative expression is

$$(\partial p / \partial T)_\rho = -(\partial p / \partial \rho)_T (\partial \rho / \partial T)_p = q^2 \rho_0 \alpha / \gamma, \tag{5.11}$$

whence comparing with (5.10)

$$\gamma (\gamma - 1) C_v = q^2 \alpha^2 T_0. \tag{5.12}$$

Further,

$$\rho_0^2 (\partial u / \partial \rho)_T = p_0 - T_0 (\partial p / \partial T)_\rho \\ = p_0 - \rho_0 (\gamma - 1) C_v / \alpha. \tag{5.13}$$

To carry out the elimination, it is necessary to differentiate the Eq. (5.7) with respect to time. Besides, $\dot{\rho}$ is expressed from (5.2). Thus, we obtain

$$\dot{u} = -\rho_0 (\partial u / \partial \rho)_T \nabla \cdot \mathbf{v} + C_v \dot{T}, \\ \nabla \dot{p} = -(\rho_0 q^2 / \gamma) \nabla (\nabla \cdot \mathbf{v}) + [\rho_0 (\gamma - 1) C_v / \alpha T_0] \nabla \dot{T}. \tag{5.14}$$

Substituting these expressions into Eq. (5.4) and the equation

$$\rho_0 \frac{\partial^2 \mathbf{v}}{\partial t^2} + \nabla \dot{p} - N\eta \nabla \left(\nabla \cdot \frac{\partial \mathbf{v}}{\partial t} \right) + \eta \nabla \times \nabla \times \frac{\partial \mathbf{v}}{\partial t} = 0$$

(obtained from (5.3) by differentiating with respect to time), we find as our results

$$\gamma \sigma \nabla^2 T - \dot{T} = (\gamma - 1)(1/\alpha) \nabla \cdot \mathbf{v}, \\ \frac{\partial^2 \mathbf{v}}{\partial t^2} - (q^2 / \gamma) \nabla (\nabla \cdot \mathbf{v}) - N\nu \nabla \left(\nabla \cdot \frac{\partial \mathbf{v}}{\partial t} \right) \\ + \nu \nabla \times \nabla \times \frac{\partial \mathbf{v}}{\partial t} = -(\alpha q^2 / \gamma) \nabla \dot{T}. \tag{5.15}$$

The new abbreviations are the kinematic viscosity $\nu = \eta / \rho_0$ and the thermometric conductivity $\sigma = \tau / \rho_0 C_p$.

6. THE ACOUSTIC POTENTIALS

We shall be interested only in periodic states of our field in which all quantities of the first order have a time dependence expressed by the factor $\exp(-i\omega t)$. Under these circumstances the partial differentiation with respect to time is equivalent to the multiplication by $-i\omega$,

$$\partial / \partial t = -i\omega. \tag{6.1}$$

This reduces our field equations to the form

$$(\gamma - 1)(1/\alpha) \nabla \cdot \mathbf{v} = \gamma \sigma \nabla^2 T + i\omega T, \\ \omega^2 \mathbf{v} + (q^2 / \gamma - i\omega N\nu) \nabla (\nabla \cdot \mathbf{v}) \\ + i\omega \nu \nabla \times \nabla \times \mathbf{v} = -(i\omega q^2 \alpha / \gamma) \nabla T.$$

To simplify writing we shall replace in the following T_1 by T, understanding under T only the deviation of the temperature due to the field. According to Clebsch's theorem, the vector **v** can be always represented in terms of a scalar potential $\bar{\varphi}$ and a vector potential **A** as

$$\mathbf{v} = -\nabla \bar{\varphi} + \nabla \times \mathbf{A}, \tag{6.2}$$

imposing on **A** the condition $\nabla \cdot \mathbf{A} = 0$.

The first equation gives then

$$\gamma \sigma \nabla^2 T + i\omega T + (\gamma - 1)(1/\alpha) \nabla^2 \bar{\varphi} = 0, \tag{6.3}$$

showing that T is a function of $\bar{\varphi}$ only.

The second equation leads to

$$\nabla[\omega^2\bar{\varphi}+(q^2/\gamma-i\omega N\nu)\nabla^2\bar{\varphi}-(i\omega q^2\alpha/\gamma)T]$$
$$= \nabla\times[\omega^2\mathbf{A}+i\omega\nu\nabla\times\nabla\times\mathbf{A}].$$

It is easy to see that the right and the left sides must vanish separately. Without loss of generality we can write (because of $\nabla\cdot\mathbf{A}=0$)

$$\nu\nabla^2\mathbf{A}+i\omega\mathbf{A}=0, \qquad (6.4)$$

$$\alpha T = -(i\omega\gamma/q^2)\bar{\varphi}-[(i/\omega)+N\nu\gamma/q^2]\nabla^2\bar{\varphi}. \quad (6.5)$$

Substituting this into (6.3) and introducing the notations e, f for the two small numbers

$$e=N\nu\omega/q^2, \quad f=\sigma\omega/q^2, \qquad (6.6)$$

we obtain for $\bar{\varphi}$ an equation of the biharmonic type

$$\bar{\varphi}+(q/\omega)^2[1-i(e+\gamma f)]\nabla^2\bar{\varphi}-(q/\omega)^4 f(if+e\gamma)\nabla^4\bar{\varphi}=0.$$

This can be reduced to

$$(\nabla^2+k^2)(\nabla^2+K^2)\bar{\varphi}=0, \qquad (6.7)$$

where

$$\left.\begin{matrix}1/k^2\\1/K^2\end{matrix}\right\} = \frac{q^2}{2\omega^2}\{1-i(e+\gamma f)$$
$$\pm[1-2ie-2if(\gamma-2)-(e-\gamma f)^2]^{\frac{1}{2}}\}. \quad (6.8)$$

7. PROPERTIES OF THE SEVERAL SOUND WAVES

Since k^2 and K^2 are never equal, the general solution of Eq. (6.7) is

$$\bar{\varphi}=\varphi+\Phi, \qquad (7.1)$$

where the two terms satisfy two ordinary wave equations

$$\nabla^2\varphi+k^2\varphi=0, \quad \nabla^2\Phi+K^2\Phi=0. \qquad (7.2)$$

To these two scalar potentials must be joined the vector potential $\mathbf{A}$ whose Eq. (6.4) can be written in the form

$$\nabla^2\mathbf{A}+\kappa^2\mathbf{A}=0. \qquad (7.3)$$

Correspondingly, our medium can support two kinds of longitudinal and one kind of transverse waves. The wave numbers of the longitudinal, k and K, are given by the expression (6.8). To terms of second order in the small numbers e,f, we find

$$1/k^2 = (q/\omega)^2[1-ie-i(\gamma-1)f+(\gamma-1)f(e-f)],$$
$$1/K^2 = -(q/\omega)^2 f[i+(\gamma-1)(e-f)].$$

Even for extremely high frequencies, of the order of 10^6 cycles, e and f are of the order 10^{-3} in air and much smaller in water. Hence, for all practical purposes, the second-order terms may be neglected. Within this approximation we may write

$$k=(\omega/q)\{1+\tfrac{1}{2}i[N\nu+(\gamma-1)\sigma]\omega/q^2\}, \quad (7.4)$$

$$K=(1+i)(\omega/2\sigma)^{\frac{1}{2}}, \qquad (7.5)$$

$$\kappa=(1+i)(\omega/2\nu)^{\frac{1}{2}}. \qquad (7.6)$$

Considering the plane wave solutions of the form $\exp(ikx)$, etc. of Eqs. (7.2) and (7.3), we see that the potential φ represents the ordinary compressional wave with the wavelength $l=2\pi/k=2\pi q/\omega$ and with the slight damping coefficient

$$\beta_D=[N\nu+(\gamma-1)\sigma]\omega^2/q^3, \qquad (7.7)$$

first given by Kirchhoff.

As the dependence on σ implies, the existence of the longitudinal wave given by the potential Φ is the result of thermal conduction and, therefore, we call it the *thermal wave*. The wavelength $L=2\pi(2\sigma/\omega)^{\frac{1}{2}}$ is very much shorter than l of the compressional wave, as is shown in the table of physical constants in Sec. 17. The damping length (or distance in which the intensity declines by $1/e$) is still shorter, being $L/2\pi$.

Similar conditions obtain in the transverse wave of the vector potential $\mathbf{A}$ which, for obvious reasons, we call the *viscous wave*. It has a wavelength, $\lambda=2\pi(2\nu/\omega)^{\frac{1}{2}}$, of the same order as the viscous wave, and its damping length is $\lambda/2\pi$.

8. THE DIFFRACTION PROBLEM

We consider a plane compressional wave falling on a spherical obstacle of radius R. Let the medium outside the sphere be characterized by unprimed variables and parameters and the medium inside by primed. We use for the description of the field polar coordinates r, ϑ, ψ with the origin in the center of the sphere and the polar axis in the direction of propagation of the incident wave, whose potential, $\varphi_0=\exp(ikr\cos\vartheta)$, can be expanded in the standard way (compare Appendix C2 for definitions of spherical and cylindrical harmonics)

$$\varphi_0=\sum_{n=0}^{\infty} i^n(2n+1)j_n(kr)P_n(\cos\vartheta). \qquad (8.1)$$

The presence of the sphere gives rise to six additional waves. Three of them are in the unprimed medium, namely, the reflected compressional wave φ_R, so that

$$\varphi=\varphi_0+\varphi_R,$$

a thermal wave Φ and a viscous wave $\mathbf{A}$. Because of the axial symmetry, of the components A_r, A_ϑ, A_ψ the first two will vanish and only $A_\psi=A$ need be considered. In the usual way we write for them the expansions

$$\varphi_R=\sum_{n=0}^{\infty} i^n(2n+1)A_n h_n(kr)P_n(\cos\vartheta),$$

$$\Phi=\sum_{n=0}^{\infty} i^n(2n+1)B_n h_n(Kr)P_n(\cos\vartheta), \qquad (8.2)$$

$$A=\sum_{n=1}^{\infty} i^n(2n+1)C_n h_n(\kappa r)P_n{}^1(\cos\vartheta).$$

Similarly, for the interior of the sphere

$$\left.\begin{array}{l} \varphi' = \sum_{n=1}^{\infty} i^n (2n+1) A_n' j_n(k'r) P_n(\cos\vartheta), \\[2mm] \Phi' = \sum_{n=0}^{\infty} i^n (2n+1) B_n' j_r(K'r) P_n(\cos\vartheta), \\[2mm] A' = \sum_{n=1}^{\infty} i^n (2n+1) C_n' j_n(\kappa'r) P_n^1(\cos\vartheta). \end{array}\right\} \quad (8.3)$$

The coefficients A_n to C_n' are six series of constants which must be determined by the six boundary conditions which obtain at the surface $r=R$. The velocity components, the stress components acting on the surface elements, the temperature and the heat flow are all continuous for $r=R$. As $v_\psi = P_{r\psi} = 0$ from symmetry, there remain the conditions

$$\left.\begin{array}{lll} v_r = v_r', & v_\vartheta = v_\vartheta', & T = T' \\[1mm] P_{rr} = P_{rr}', & P_{r\vartheta} = P_{r\vartheta}', & \tau \partial T/\partial r = \tau' \partial T'/\partial r. \end{array}\right\} \quad (8.4)$$

9. EXPLICIT BOUNDARY CONDITIONS

We list the expressions in polar coordinates of the quantities entering into the conditions (8.4). From Eq. (6.2)

$$\left.\begin{array}{l} v_r = -\dfrac{\partial\bar\varphi}{\partial r} - \dfrac{1}{r\sin\vartheta}\dfrac{\partial}{\partial\vartheta}(A\sin\vartheta), \\[3mm] v_\vartheta = -\dfrac{1}{r}\dfrac{\partial\bar\varphi}{\partial\vartheta} - \dfrac{1}{r}\dfrac{\partial}{\partial r}(Ar). \end{array}\right\} \quad (9.1)$$

From (6.5) and (7.2)

$$T = g\varphi + G\Phi. \quad (9.2)$$

The coefficients g, G can be calculated rigorously, but for the purposes of this paper it will be sufficient to apply the same accuracy as used in the expression (7.5) for K, i.e., to neglect terms of the orders e and f compared with 1. This also implies the omission of the second term in the bracket of Eq. (6.5). The result is

$$\left.\begin{array}{l} g = -i\omega(\gamma-1)/\alpha q^2 = -i\omega\rho T_0/\tau G, \\[1mm] G = -(\alpha\sigma)^{-1}, \quad g/G = -(\gamma-1)k^2/K^2. \end{array}\right\} \quad (9.3)$$

The pressure is obtained by substituting (6.2) in (5.3) and canceling the terms depending on the vector potential **A** because of the relation (7.3). We write, moreover, p instead of p_1 and omit the sign ∇ in all terms of the equation

$$p = -i\omega\rho_0\bar\varphi - N\eta\nabla^2\bar\varphi, \quad (9.4)$$

finally omitting terms of order $(k/K)^2$ as in the preced-

ing equations.

$$\left.\begin{array}{l} p = -i\omega\rho_0(\varphi + Q\Phi), \\[1mm] Q = 1 - N\nu/\sigma. \end{array}\right\} \quad (9.5)$$

An alternative way of obtaining the pressure follows from the fact that the operation ∇ may be canceled in Eq. (5.14), and the dots may be replaced by the factor $-i\omega$. This leads to

$$p = (q^2\rho_0/\gamma)[\alpha T - (i/\omega)\nabla\cdot\mathbf{v}]. \quad (9.6)$$

This relation will be useful in Sec. 13. However, to obtain from it the formula (9.5), it is necessary to use more accurate expressions of G and K than those given in Eqs. (9.3) and (7.5).

The stress components expressed in terms of our potentials $\bar\varphi = \varphi + \Phi$ and A are as follows

$$\left.\begin{array}{l} P_{rr} = i\omega\rho_0(\varphi + M\Phi) + 2\eta\left\{ -\dfrac{\partial^2\bar\varphi}{\partial r^2} \right. \\[3mm] \left. + \dfrac{1}{\sin\vartheta}\dfrac{\partial}{\partial\vartheta}\left[\sin\vartheta\left(-\dfrac{A}{r^2} + \dfrac{1}{r}\dfrac{\partial A}{\partial r}\right)\right]\right\}, \\[4mm] P_{r\vartheta} = \eta\left\{ -2\dfrac{\partial}{\partial\vartheta}\left(\dfrac{1}{r}\dfrac{\partial\bar\varphi}{\partial r} - \dfrac{\bar\varphi}{r}\right) - \left(\dfrac{\partial^2 A}{\partial r^2} - \dfrac{2A}{r^2}\right) \right. \\[4mm] \left. + \dfrac{1}{r^2}\dfrac{1}{\partial\vartheta}\left[\dfrac{1}{\sin\vartheta}\dfrac{\partial}{\partial\vartheta}(A\sin\vartheta)\right]\right\}. \end{array}\right\} \quad (9.7)$$

Within our approximation

$$M = 1 - 2\nu/\sigma. \quad (9.8)$$

We introduce the further abbreviations

$$a = kR, \quad b = KR, \quad c = \kappa R, \quad (9.9)$$

moreover, all the quantities relating to the interior of the sphere will be primed.

The further procedure is now obvious. With the help of the series (8.1), (8.2), (8.3) all the quantities of this

TABLE I. ($R = 0.002$ cm)

Medium		Cycles			
		10^3	10^4	10^5	10^6
Air	$\lvert g/G \rvert$	4.42×10^{-7}	4.42×10^{-6}	4.42×10^{-5}	4.42×10^{-4}
	l(cm)	34.4	3.44	0.344	0.0344
	a	3.66×10^{-4}	3.66×10^{-3}	3.66×10^{-2}	0.366
	a^2	1.34×10^{-7}	1.34×10^{-5}	1.34×10^{-3}	0.134
	L(cm)	5.15×10^{-2}	1.63×10^{-2}	5.15×10^{-3}	1.63×10^{-3}
	$\mathrm{Re}b=y$	0.244	0.772	2.44	7.72
	$\lvert b\rvert^2 = 2y^2$	0.119	1.19	11.9	119
	λ(cm)	4.32×10^{-2}	1.40×10^{-2}	4.32×10^{-3}	1.40×10^{-3}
	$\mathrm{Re}c=z$	0.284	0.897	2.84	8.97
	$\lvert c\rvert^2 = 2z^2$	0.161	1.61	16.1	161
Water	$\lvert g'/G'\rvert$	1.44×10^{-12}	1.44×10^{-11}	1.44×10^{-10}	1.44×10^{-9}
	l'(cm)	145	14.5	1.45	0.145
	a'	8.66×10^{-5}	8.66×10^{-4}	8.66×10^{-3}	8.66×10^{-2}
	a'^2	7.52×10^{-9}	7.52×10^{-7}	7.52×10^{-5}	7.52×10^{-3}
	L'(cm)	4.14×10^{-2}	1.35×10^{-2}	4.14×10^{-3}	1.35×10^{-3}
	$\mathrm{Re}b'=y'$	2.93	9.27	29.3	92.7
	$\lvert b'\rvert^2 = 2y'^2$	17.20	172	1720	17 200
	λ'(cm)	1.18×10^{-2}	3.72×10^{-3}	1.18×10^{-3}	3.72×10^{-4}
	$\mathrm{Re}c'=z'$	1.07	3.38	10.7	33.8
	$\lvert c'\rvert^2 = 2z'^2$	2.28	22.8	228	2280

section are converted into expansions with respect to spherical harmonics which are then substituted into the conditions (8.4) setting $r=R$. After the application of Eq. (C1) of Appendix C, only zonal spherical harmonics are left, and their coefficients on both sides are equated. As a result the following six series of equations are obtained, for $n \geqslant 1$.

$$
\begin{aligned}
&a j_n{}'(a)+A_n a h_n{}'(a)+B_n b h_n{}'(b) \\
&\quad -C_n n(n+1) h_n(c)=A_n{}' a' j_n{}'(a') \\
&\quad +B_n{}' b' j_n{}'(b')-C_n{}' n(n+1) j_n(c'); \\[6pt]
&j_n(a)+A_n h_n(a)+B_n h_n(b) \\
&\quad -C_n[h_n(c)+c h_n{}'(c)]=A_n{}' j(a') \\
&\quad +B_n{}' j_n(b')-C_n{}'[j_n(c')+c' h_n{}'(c')]; \\[6pt]
&g[j_n(a)+A_n h_n(a)]+GB_n h_n(b) \\
&\quad =g' A_n{}' j_n(a')+G' B_n{}' j_n(b'); \\[6pt]
&\tau\{g[a j_n{}'(a)+A_n a h_n{}'(a)]+GB_n b h_n{}'(b)\} \\
&\quad =\tau'\{g' A_n{}' a' j_n{}'(a')+G' B_n{}' b' j_n{}'(b')\}; \\[6pt]
&\eta\{[a j_n{}'(a)-j_n(a)]+A_n[a h_n{}'(a)-h_n(a)] \\
&\quad +B_n[b h_n{}'(b)-h_n(b)] \\
&\quad -\tfrac{1}{2}C_n[c^2 h_n{}''(c)+(n^2+n-2) h_n(c)]\} \\
&\quad =\eta'\{A_n{}'[a' j_n{}'(a')-j_n(a')] \\
&\quad +B_n{}'[b' j_n{}'(b')-j_n(b')] \\
&\quad -\tfrac{1}{2}C_n{}'[c'^2 j_n{}''(c')+(n^2+n-2) j_n(c')]\}; \\[6pt]
&\eta\{[c^2 j_n(a)-2a^2 j_n{}''(a)] \\
&\quad +A_n[c^2 h_n(a)-2a^2 h_n{}''(a)] \\
&\quad +B_n[Mc^2 h_n(b)-2b^2 h_n{}''(b)] \\
&\quad +2n(n+1)C_n[c h_n{}'(c)-h_n(c)]\} \\
&\quad =\eta'\{A_n{}'[c'^2 j_n(a')-2a'^2 j_n{}''(a')] \\
&\quad +B_n{}'[M'c'^2 j_n(b')-2b'^2 j_n{}''(b')] \\
&\quad +2n(n+1)C_n{}'[c' j_n{}'(c')-j_n(c')]\}.
\end{aligned}
$$
(9.10)

In the particular case $n=0$ the second and the fifth equations are identically satisfied, while in the remainder the terms with C_n, $C_n{}'$ disappear. Hence, the system is reduced to four equations with four unknowns written out in Appendix A as Eq. (A1.1).

10. EXPLICIT EXPRESSIONS FOR THE COEFFICIENTS

The main purpose of the present paper is to compare the theoretical results with the actual conditions in natural and artificial fogs. The radii of the fog particles for which data are available lie between $R=4\times10^{-4}$ cm and $R=2\times10^{-3}$ cm. The maximum values of a^2 and a'^2 for the different frequencies, are, therefore, apparent from the following table.

We see that in the whole of our range (of up to 10^5 cycles) a^2 and a'^2 are very small numbers which can be safely neglected compared with 1. Other numbers which we shall consider negligible (compared with one) are the ratios k^2/K^2 and k^2/κ^2 (including the case that the numerator or the denominator or both are primed). Indeed these are smaller than 10^{-4} in the entire fre-

quency interval which is of interest to us. This means that we must neglect $a^2/b^2\ll1$, $a^2/c^2\ll1$, $g/G\ll1$ (also g'/G', a'^2/b'^2 etc.). All these ratios are of the order of magnitude k^2/K^2 as follows from the definitions (9.9) and (9.3).

As shown in Appendix A, the coefficients of the order $n=0$ can be expressed in terms of the function

$$
F=\frac{h_0(b)}{bh_1(b)}-\chi\frac{j_0(b')}{b' j_1(b')}.
$$
(10.1)

The only coefficient of interest to us is

$$
A_0=-\tfrac{1}{3}ia(a^2-\delta a'^2)-ia(g/G)(1-\xi)^2 F^{-1}, \quad (10.2)
$$

where

$$
\xi=G\tau/G'\tau'.
$$

Turning to the coefficients of the next order $(n=1)$ the following auxiliary functions must be defined

$$
\left.
\begin{aligned}
&Q(c')=c' j_1(c')-2(1-\epsilon) j_2(c'), \\
&L=h_2(c)Q(c')-\epsilon h_1(c) j_2(c'), \\
&D=[3\delta h_2(c)+2(\delta-1) h_0(c)]Q(c') \\
&\qquad -\epsilon(\delta+2)c h_1(c) j_2(c'),
\end{aligned}
\right\}
$$
(10.3)

with

$$
\epsilon=\eta/\eta'.
$$
(10.4)

The coefficient important to us is

$$
3A_1=-ia^3(\delta-1)L/D.
$$
(10.5)

The orders $n=0$ and $n=1$ are sufficient for our purpose. Every following order contains an additional factor a^2 so that the convergence is rapid when a^2 is small.

11. THE ENERGY DISSIPATION

As shown in Sec. 4, the energy absorption per unit time and volume is represented by the sum of the two dissipation functions $\Psi_\eta+\Psi_\tau$ defined by Eqs. (3.7) and (4.1). Considering first the viscous term, we can write, in view of the definitions (2.1) and (3.3),

$$
\begin{aligned}
\Psi_\eta=(\partial v_j/\partial x_i)p_{ij}&=(\partial v_j/\partial x_i)P_{ij}+p\nabla\cdot\mathbf{v} \\
&=\partial(v_j P_{ij})/\partial x_i-v_j \partial P_{ij}/\partial x_i+p\nabla\cdot\mathbf{v},
\end{aligned}
$$

or with the help of Eq. (3.2), bearing in mind that we need only first-order terms,

$$
\Psi_\eta=\partial(v_j P_{ij})/\partial x_i-\tfrac{1}{2}\partial(\rho v^2)/\partial t+p\nabla\cdot\mathbf{v}. \quad (11.1)
$$

We shall designate by W_η the time average of the total energy loss in a volume V of the acoustic field within the surface F. It is obvious that the second term will give no contribution to this quantity, so that

$$
W_\eta=\int_F \langle v_j P_{Nj}\rangle_{Av} dF+\int \langle p\nabla\cdot\mathbf{v}\rangle_{Av} dV.
$$

The factors entering into the products under the signs of the integral are periodic of the frequency ω. If we

wish to represent them in the complex form, introduced in Sec. 6, omitting the time factor $\exp(-i\omega t)$, we must write [compare Appendix B, Eq. (B1.2)]

$$W_\eta = \tfrac{1}{2}\mathrm{Re}\int_F v_j{}^* P_{nj} dF + \tfrac{1}{2}\mathrm{Re}\int_V p^* \nabla \cdot v dV. \quad (11.2)$$

Here the symbols P_{ij} and p do not contain the hydrostatic pressure. The star indicates the complex conjugate and Re means the real part of the following expression.

Turning to the thermal dissipation, we notice that

$$(\nabla T)^2 = \nabla \cdot (T\nabla T) - T\nabla^2 T. \quad (11.3)$$

Therefore, the total mean thermal dissipation takes the form

$$W_\tau = (\tfrac{1}{2}T_0)\mathrm{Re}\left\{ \int_F T^*\tau(\partial T/\partial N)dF - \tau\int_V T^*\nabla^2 T dV \right\}. \quad (11.4)$$

12. CALCULATION OF THE SURFACE INTEGRALS

We wish to determine the total absorption in the space inside and the space outside the droplet. Hence, the surface integrals must be extended over the surfaces limiting these two spaces. For the inner space, F is the surface of the droplet; the outer space is limited from within again by the surface of the sphere, from without by the infinite surface which we shall choose as a sphere of infinite radius ($r\to\infty$) with its center in the center of the droplet. Examining the structure of the integrands of the two surface integrals in Eqs. (11.2) and (11.4), we notice that the factors of which they consist are all continuous at the surface of the droplet in view of the boundary conditions (8.4). Therefore, the two integrals over this surface are mutually canceled and only the integral over the infinite sphere remains. This fact greatly simplifies the calculation; indeed the strongly damped potentials Φ and A vanish in infinity and give no contribution to the integrals. Hence, only the dependence of the integrands on $\varphi = \varphi_0 + \varphi_R$ needs to be considered.

As shown in Appendix B

$$I_S = \tfrac{1}{2}\int_F v_j{}^* P_{Nj} dF$$
$$= -2\pi\rho(\omega/k)\sum(2n+1)(\mathrm{Re}A_n + A_n A_n{}^*). \quad (12.1)$$

As compared with this expression, the surface integral in Eq. (11.4) is multiplied by a factor of the order $\tau|g|^2/\omega\rho T_0$ which can be written, with help of Eqs. (9.3) and (5.10), as $\sigma\omega(\gamma-1)/a^2 = (\gamma-1)|k/\kappa|^2$. As pointed out in Sec. 10, this is a negligible number, and hence this integral gives no contribution. Therefore, (12.1) represents the total amount contributed by the surface integrals.

13. CALCULATION OF THE VOLUME INTEGRALS AND OF THE FINAL EXPRESSION

The second of the volume integrals, the one contained in Eq. (11.4) can be transformed by using Eq. (5.15) in the modified form

$$\nabla^2 T = (1/\gamma\sigma)\dot{T} + (\rho\alpha q^2 T_0/\gamma\tau)\nabla \cdot v, \quad (13.1)$$

which follows from Eq. (5.12) and the definition of $\tau = \gamma\sigma\rho_0 c_v$. Inasmuch as the time average of the term $T\dot{T}$ vanishes, the sum of the volume integrals in (11.3) and (11.4) becomes

$$I_V = \mathrm{Re}\int_V [p^* - (\rho q^2\alpha/\gamma)T^*]\nabla \cdot v dV,$$

but we see from Eq. (9.6) that this is equal to

$$I_V = \tfrac{1}{2}\mathrm{Re}(iq^2\rho_0/\gamma\omega)\int_V |\nabla \cdot v|^2 dV = O, \quad (13.2)$$

since a purely imaginary number has no real part.

This remarkable result means a very important practical simplification, because the expression (12.1) represents now the total effect. We can write as our final formula

$$W = W_\eta + W_\tau$$
$$= -2\pi\rho(\omega/k)\sum(2n+1)(\mathrm{Re}A_n + |A_n|^2). \quad (13.3)$$

14. EXPLICIT EXPRESSIONS FOR THE ATTENUATION

We know from the calculations of Sec. 10 that of all the coefficients A_n only the first two, A_0 and A_1, are significant. Even these quantities are so small that the products $A_0{}^*A_0$, $A_1{}^*A_1$ are of the order of the neglected higher coefficients and thus must be considered negligible in their turn. Hence, the expression (13.3) reduces to

$$W = -2\pi\rho(\omega/k)\mathrm{Re}(A_0 + 3A_1). \quad (14.1)$$

This represents the energy loss caused by one suspended particle. If our medium contains n particles per cubic centimeter, the total energy loss per unit volume and unit time will be nW. To obtain the relative energy loss or the *attenuation* per unit length of the path, this quantity must be divided by the energy E_0 which the primary wave carries in unit time across a normal unit area. Within the required approximation the primary wave, having the potential φ_0 of Eq. (8.1), may be taken as nonviscous and nonconducting. It is then easily deduced from the field equations (3.1), (3.2), and (3.3) that the mean energy flow is $E_0 = \langle pv \rangle_{Av} q = \tfrac{1}{2}q\mathrm{Re}(p^*v)$. From Eqs. (6.2) and (9.5) this is seen to be $E_0 = \tfrac{1}{2}k\omega\rho$. The coefficient of attenuation takes, therefore, the form

$$\beta = 4\pi nk^{-2}\mathrm{Re}(A_0 + 3A_1). \quad (14.2)$$

This formula implies the assumption that effects of the individual particles are additive. We shall examine

the properties that the medium must possess for the validity of this assumption in Sec. 17.

Let us first consider the term depending on A_1. The method used in Appendix A (Sec. 2) for calculating this term rests on the fact that the ratio g/G (of the same order as k^2/K^2) is in our case a very small number. Under these circumstances the terms arising from the conduction potentials may be neglected in the boundary equations $n=1$. This means that the expression of A_1 is the same as in a nonconducting medium. Indeed the expression that results from the second term of (14.2) by substituting from Eq. (10.5) is

$$\beta_\eta = 4\pi nkR^2(1-\delta)\mathrm{Re}(iL/D). \qquad (14.3)$$

The meaning of the symbols L and D was given in Eqs. (10.3). This expression of the *viscous attenuation* is identical with that derived by Epstein (l.c.) for the total coefficient of attenuation in a nonconducting medium.

On the other hand, the part of the attenuation due to the coefficient A_0 in Eq. (14.2) largely depends on the heat conduction, since it is derived from the boundary conditions of order $n=0$ into which the viscous potentials do not enter. Using Eq. (10.2) for the coefficient A_0 and Eqs. (9.3) defining g and G, we find for the coefficient of *thermal attenuation*

$$\beta_\tau = 4\pi n(\gamma-1)(\sigma/q)R(1-\xi)^2\mathrm{Re}(1/F), \qquad (14.4)$$

the function F being defined by Eq. (10.1). A closer investigation of the expressions (14.3) and (14.4) will be given in the next section.

We should like to point out that the possibility of separating the attenuation into a viscous and a thermal part is entirely due to the smallness of the ratio k^2/K^2. The possibility exists whenever this condition obtains, quite independently of the size of the droplet.

15. ADAPTATION OF THE EXPRESSION FOR VISCOUS ATTENUATION

The coefficient β_η takes a particularly simple form when both δ and ϵ are neglected. Physically, this means that the droplet is assumed to be perfectly rigid and infinitely heavy. This case was treated by Sewell[6] under restriction to a nonconducting medium. Indeed the limiting form of our expression (14.3) with $\delta = \epsilon = 0$,

$$\beta_\eta = 2\pi nkR^2\mathrm{Re}h_2(c)/h_0(c),$$

is identical with Sewell's total results.

In the problem of water droplets in air the actual magnitudes of our parameters are apparent from Table II where those data that are appreciably temperature dependent are given for 22°C.

It is apparent that in our case the parameters δ, ϵ, χ are all small. However, the arguments b, c, of our functions depend themselves on ρ, η, and τ, and, therefore, the question of which terms are negligible cannot be

[6] See reference 1.

TABLE II. Constants in air and in water.

	Air (unprimed)	Water (primed)	Dimension
η	1.82×10^{-4}	1.10×10^{-2}	g cm^{-1} sec^{-1}
ρ	1.17×10^{-3}	1.00	g cm^{-3}
$\nu=\eta/\rho$	0.156	1.10×10^{-2}	cm^2 sec^{-1}
τ	5.80×10^{-5}	1.43×10^{-3}	cal cm^{-1} sec^{-1} deg^{-1}
C_p	0.240	1.00	cal g^{-1} deg^{-1}
$\sigma=\tau/\rho C_p$	0.206	1.43×10^{-3}	cm^2 sec^{-1}
γ	1.40	1.00336	
α	3.66×10^{-3}	1.40×10^{-4}	deg^{-1}
q	3.44×10^{4}	1.45×10^{5}	cm sec^{-1}
G	-1326	-5.05×10^{6}	cm^{-2} sec deg
$\delta=\rho/\rho'$		1.17×10^{-3}	
$\epsilon=\eta/\eta'$		1.66×10^{-2}	
$\chi=\tau/\tau'$		4.07×10^{-2}	
G/G'		2.63×10^{-4}	
$\xi=\tau G/\tau'G'$		1.07×10^{-5}	
ξ/ϵ		6.45×10^{-4}	
g/g'		84.0	

answered offhand but requires a careful examination. Because of the expressions (C 12) for the $h(c)$ functions, the quantities L and D may be replaced without loss of generality by

$$\left.\begin{aligned} L' &= (ic^2-3c-3i)Q(c') + \epsilon(1+ic)c^2j_2(c'), \\ D' &= [(2+\delta)ic^2-9\delta(c+i)]Q(c') \\ &\quad + \epsilon(\delta+2)(1+ic)c^2j_2(c'). \end{aligned}\right\} \qquad (15.1)$$

The important question is as follows: is it permissible to neglect the terms with the factor ϵ whose numerical value is about 0.016? Although we wish to cover a wide range $(0.0004 \leqslant R \leqslant 0.002;\ 1000\pi \leqslant \omega \leqslant 20,000\pi)$ the ratio of $(1+ic)$ to the factors of $Q(c')$ is not greatly different from 1 in the whole range. Everything depends therefore on the ratio of $j_2(c')/Q(c')$. The analysis shows that this ratio is small for large c' and of the order 0.6 for small c'. Hence, the neglect of the terms in question is fully justified in the region of relatively high frequencies and would involve an error of about one percent at low frequencies. Considering that the available measurements are affected with a much larger uncertainty and that the very foundations of our theory are not absolutely rigorous (Sec. 2), it would be pointless to aspire to a greater accuracy. We neglect further δ where it is bracketed with 2 and use the notation

$$c=(1+i)z, \quad c^2=2iz^2, \quad z=(\omega/2\nu)^{\frac{1}{2}}R. \qquad (15.2)$$

The result is the explicit expression for the coefficient β_η in terms of z and R given as the formula (1.2) of the introduction.

16. ADAPTATION OF THE EXPRESSION FOR THERMAL ATTENUATION

To determine β_τ of Eq. (14.4) the function F has to be evaluated. Substituting into its definition (10.1) the expressions of the Heine functions (C 11) and (C 12), we find

$$F(b,b') = \left[\frac{1}{1-ib} - \chi\frac{\sin b'}{\sin b' - b'\cos b'}\right]. \qquad (16.1)$$

TABLE III. Drop size data.

Group j	Mean radius R_j(cm)	Number of drops n_j(cm^{-3})
1	3.75×10^{-4}	55
2	6.25×10^{-4}	89
3	10.0×10^{-4}	121
4	15.0×10^{-4}	38
5	21.5×10^{-4}	21
		$n = 324$

In interpreting the existing observations we have to cover wide ranges of the variables b and b'. If we introduce the notations

$$b = (1+i)y, \quad b^2 = 2iy^2, \quad y = (\omega/2\sigma)^{\frac{1}{2}}R,$$

(with analogous expression for the primed variables) the ranges in which we are interested are

$$0.024 \leqslant y \leqslant 0.782, \qquad 0.296 \leqslant y' \leqslant 9.38.$$

For small values of the parameter y' (up to $y' = 1.20$) the terms of (16.1) may be expanded into power series with the result

$$F = 1 - y - 0.2\chi + i[y - 2y^2 - \chi(1.5/y'^2 + 0.0045y'^2)]. \quad (16.2)$$

The real part of the reciprocal of this expression, which is needed for the attenuation, was determined numerically. The curve of Y_τ given in Fig. 1 was calculated in this way up to $y' = 1.20$ or $y = 0.1$. For larger arguments it was calculated from the unadapted formula (16.1). In this region the results are in practical agreement with the simpler formula (1.3) of the introduction in which the second term of (16.1) is altogether neglected. The difference is nowhere larger than 1 percent of β_τ or than 0.3 percent of the total β.

17. COMPARISON WITH KNUDSEN'S DATA

The absorption of sound in fogs was experimentally studied by many physicists, but most of this work is not suitable for a close comparison with the theory because no analysis of drop size was made. In 1940, however, Knudsen[7] carried out absorption measurements in artificially produced fog using the reverberation-chamber technique, and in this work the distribution in drop size was determined. These authors give a curve which permits of inferring the relative number of drops having radii in any interval of lengths; further they state that the total volume of all drops was 2.00×10^{-6} cm^3 per 1 cm^3 of the fog.

To carry out the comparison, the drops were divided into five groups, the relative numbers of drops in each group were taken from Knudsen's curve and the absolute numbers inferred from them by adjusting them so as to make the sum of their volumes agree with Knudsen's figure. The results are summarized in Table III.

---

[7] See reference 3.

The observed attenuations $\bar{\beta}$ were given in decibels (db) per centimeter—i.e. in tenths of the decimal (instead of natural) logarithms of the transmitted intensities. Hence, the conversion ratio is

$$\bar{\beta} = 4.34\beta = 4.34(\beta_\eta + \beta_\tau).$$

At each frequency the attenuations β_j were calculated for the individual groups, and the several β_j were added. The results and their comparison with the measured attenuations are given in Table IV and, in graphical form, in Fig. 2.

It will be seen that the theoretical and the experimental attenuation curves show a parallel frequency dependence and agree quantitatively in the region of low frequencies. For higher frequencies the calculated points lie below the observed values. We wish to recall here that quite similar conditions obtain in the much simpler phenomenon of the attenuation β_D of compressional sound waves in homogeneous air, mentioned in Sec. 15. Measurements undertaken to test Kirchhoff's formula (7.7) gave values of β_D materially higher than the calculated ones. This lack of agreement is now generally attributed to relaxation and to other secondary effects.[8] It seems probable that the discrepancies in Table IV and in Fig. 2 are due to the same causes and thus rather confirm than weaken our theory.

There remains to say a word in justification of the additive procedure of Sec. 14, where the total absorption was obtained by summing the individual absorptions of the droplets. The absorption is due to the damping of the viscous and thermal waves which takes place in a layer of the approximate thickness of 0.01 cm about the surface of the droplets. On the other hand, the mean distance between neighboring fog droplets was over 0.1 cm so that there is no overlapping of the critical region. The question is only to what extent the primary compressional wave is modified by the presence of the neighboring particles. The influence of the other particles is twofold; in the first place they all cooperate in producing a joint average effect of coherent scattering.[9] It is easy to take this effect into account since it manifests itself merely in dispersion, that is in a change
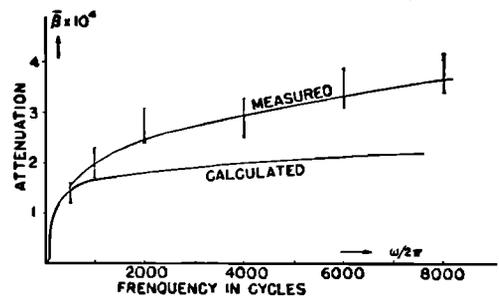

FIG. 2. Comparison of theoretical and experimental attenuations.

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[8] See the summary in Ludwig Bergmann, *Ultrasonics* NAVSHIPS 900, 167, pp. 281–300.

[9] First studied by Lord Rayleigh [Phil. Mag. **47**, 377 (1899)] and N. Kasterin [Verslagen Amsterdam, 460 (1898)].

of phase velocity. Under the conditions prevailing in Knudsen's fogs the effect happens to be negligible. On top of this average effect lie the irregularities which are due to the random distribution of the particles. It constitutes the ordinary or noncoherent scattering producing an attenuation which can be calculated from the quadratic terms of our expression (13.3). This contribution is, however, of the order of terms neglected elsewhere in our theory and, therefore, must be also considered as negligible.

APPENDIX A

Calculation of the Expansion of Coefficients

1. Boundary equations of order n=0

In adapting the system (9.9) to the case $n=0$ we notice, in addition to the facts mentioned in text, that in our whole discussion a and a' are assumed to be small numbers so that a^2 and a'^2 may be neglected compared with 1. We can, therefore, use the simplified expression of $j_0(a)$, $h_0(a)$ and of their derivatives listed in Appendix C, as well as the relation $j_0'(x) = -j_1(x)$,

$$\left.\begin{array}{l} A_0 i/a + \tfrac{1}{3}A_0'a'^2 - B_0 bh_1(b) + B_0'b'j_1(b') = \tfrac{1}{3}a^2, \\ A_0(-i/a+1) - A_0'g' + B_0 Gh_0(b) \\ \qquad\qquad - B_0'G'j_0(b') = -g, \\ A_0\tau g i/a + \tfrac{1}{3}A_0'\tau'g'a'^2 - B_0\tau Gbh_1(b) \\ \qquad\qquad + B_0'\tau'G'b'j_1(b') = \tfrac{1}{3}\tau ga^2, \\ A_0\eta[c^2 - (c^2-4)i/a] - A_0'\eta'c'^2 \\ \qquad + B_0\eta[Mc^2h_0(b) - 2b^2h_0''(b)] \\ \qquad - B_0'\eta'[M'c'^2j_0(b') - 2b'^2j_0''(b')] = -\eta c^2. \end{array}\right\} \quad (A1.1)$$

If we divide the third equation by τG, we find that the terms with B_0, B_0' in it are of the same order as in the first, while the remaining terms in the third are of negligible order, being multiplied by the very small factors g/G, and g'/G (compare Table I). Hence, the third equation may be written as $B_0\tau Gbh_1(b) = B_0'\tau'G'b'j_1(b')$, or using the abbreviations

$$\xi = \tau G/\tau'G', \quad \epsilon = \eta/\eta', \quad \delta = \rho/\rho', \quad (A1.2)$$

as

$$B_0' = \xi[bh_1(b)/b'j_1(b')]B_0. \quad (A1.3)$$

It should be pointed out that this relation holds also in the case where the conditions are reversed, so that the primed variables refer to air and the unprimed to water (air bubbles in water). We only would have to divide the third equation by $\tau'G'$ in order to arrive at the same conclusions.

We further introduce the symbols

$$\left.\begin{array}{l} S_1 = (1-\xi)bh_1(b), \\ S_2 = h_0(b) - \chi bh_1(b)j_0(b')/b'j_1(b'), \\ S_3 = (Mc^2 + 2b^2)h_0(b) - (1-\xi/\epsilon)4bh_1(b) \\ \qquad - (\xi/\epsilon)(M'c'^2 + 2b'^2)bh_1(b)j_0(b')/b'j_1(b'), \end{array}\right\} \quad (A1.4)$$

TABLE IV. Comparison of calculated and observed attenuations

$\omega/2\pi$ (cycles)	$\bar\beta\eta$ (db/cm)	$\bar\beta\tau$ (db/cm)	$\bar\beta$(calc.) (db/cm)	$\bar\beta$(obs.) (db/cm)
500	1.00×10^{-4}	0.44×10^{-4}	1.44×10^{-4}	$(1.4\pm0.2)\times10^{-4}$
1000	1.17×10^{-4}	0.47×10^{-4}	1.64×10^{-4}	$(2.0\pm0.3)\times10^{-4}$
2000	1.31×10^{-4}	0.50×10^{-4}	1.81×10^{-4}	$(2.7\pm0.4)\times10^{-4}$
4000	1.44×10^{-4}	0.55×10^{-4}	1.99×10^{-4}	$(2.9\pm0.4)\times10^{-4}$
6000	1.55×10^{-4}	0.59×10^{-4}	2.14×10^{-4}	$(3.5\pm0.4)\times10^{-4}$
8000	1.60×10^{-4}	0.61×10^{-4}	2.21×10^{-4}	$(3.8\pm0.4)\times10^{-4}$

and we find after the elimination of B_0',

$$\left.\begin{array}{l} A_0 i/a + \tfrac{1}{3}A_0'a'^2 - B_0 S_1 = \tfrac{1}{3}a^2, \\ A_0(g/G)(i/a-1) + A_0'(g'/G) - B_0 S_2 = g/G, \\ A_0[(c^2-4)i/a - c^2] + \epsilon^{-1}c'^2 A_0' - B_0 S_3 = c^2. \end{array}\right\} \quad (A1.5)$$

This system is convenient for solution by determinants. Let Δ be the determinant of the system and $\Delta(A_0)$, the determinant which results by replacing the first column by the right sides of the equations. We find then,

$$\left.\begin{array}{l} a\Delta = \{g(i-a)c^2/\delta - g'[c^2(i-a) - 4i]\}S_1/G \\ \qquad + \{ic^2/\delta - \tfrac{1}{3}[c^2(i-a) - 4i]a'^2\}S_2 \\ \qquad + \{\tfrac{1}{3}g(i-a)a'^2 - g'i\}S_3/G; \\ \Delta(A_0) = (g/\delta - g')c^2S_1/G - \tfrac{1}{3}(a^2/\delta - a'^2)c^2S_2 \\ \qquad + \tfrac{1}{3}(ga'^2 - g'a^2)S_3/G \end{array}\right\} \quad (A1.6)$$

In transforming these equations use was made of the relation

$$c'^2/\epsilon = c^2/\delta, \quad (A1.7)$$

which follows from Eqs. (9.9) and (7.6).

In the particular case when we consider water droplets in air, the functions S_1, S_2, S_3 are all of the same order, since the numbers ξ, χ and ξ/ϵ are all small (see Table II in Sec. 15). In view of the extreme smallness of the factors g/G, g'/G and δ, only the terms with S_2 in Δ and with S_1 and S_2 in $\Delta(A_0)$ must be retained, whence

$$\Delta = i(c^2/\delta a)S_2, \quad (A1.8)$$

$$\Delta(A_0) = \tfrac{1}{3}(a^2 - \delta a'^2)c^2S_2/\delta + (g/G\delta)(1-\xi)c^2S_1, \quad (A1.9)$$

since $\delta g'/g = \xi$, according to Eqs. (8.2) and (9.3). The resulting expression for the coefficient A_0 is given in Eq. (10.2) of the text.

2. Boundary Equations of Order n=1

A simplification, similar to that of the case $n=0$, can be carried out for all other orders. Indeed, upon examining the general system of (9.10) boundary equations and, especially, upon comparing the third and fourth equations, respectively, with the first and second, we find that in the third and fourth the coefficients with A_1, A_1' as well as the inhomogeneous terms are of negligible order. This leads to the conclusion that the terms B_1, B_1' are of negligible order in all of the equations. Hence, these terms can be omitted in the first two

and the last two equations, where the remaining terms are of a lower order. As in the preceding section, and for the same reasons as there, the simplification is equally valid in the reversed case when the roles of the primed and the unprimed medium are interchanged.

The problem is thus reduced to a system of four equations instead of six. For $n=1$, the simplified system has the form

$$a j_1'(a) + A_1 a h_1'(a) - 2C_1 h_1(c) = A_1' a' j_1'(a') - 2C_1' j_1(c'),$$
$$j_1(a) + A_1 h_1(a) - C_1[h_1(c) + c h_1'(c)]$$
$$= A_1' j_1(a') - C_1'[j_1(c') + c' j_1'(c')],$$
$$\eta[a j_2(a) + A_1 h_2(a) + \tfrac{1}{2}C_1 c^2 h_1''(c)]$$
$$= \eta'[A_1' j_2(a') + \tfrac{1}{2}C_1' c'^2 j_1''(c')],$$
$$\eta\{[(c^2 + 2a^2) j_1(a) - 4a j_2(a)] - 4C_1 c h_2(c)\}$$
$$= \eta'\{A_1'[(c'^2 + 2a'^2) j_1(a') - 4a' j_2(a')] - 4C_1' c' j_2(c')\}.$$

The equations become simpler if we handle them in the following way. We multiply the second equation, respectively, by -1 and by 2 and add it to the first From the fourth we subtract four times the third taking into account that a^2 is negligible compared with c^2.

$$A_1 a h_2(a) + C_1 c h_2(c) - A_1' a' j_2(a') - C_1' c' j_2(c') = -a j_2(a),$$
$$A_1 a h_0(a) - 2C_1 c h_0(c)$$
$$- A_1' a' j_0(a') + 2C_1' c' j_0(c') = -a j_0(a),$$
$$\eta c^2[A_1 h_1(a) - 2C_1 h_1(c)]$$
$$- \eta' c'^2[A_1' j_1(a') - 2C_1' j_1(c')] = -\eta c^2 j_1(a),$$
$$\eta[A_1 a h_2(a) + \tfrac{1}{2}C_1 c^2 h_1''(c)]$$
$$- \eta'[A_1' a' j_2(a') + \tfrac{1}{2}C_1' c'^2 j_1''(c')] = -\eta a j_2(a).$$

We use now the approximate expressions for small values of a and neglect terms of order a^2 compared with 1. Moreover, we make use of the abbreviations (A1.2) and (A1.7).

$$\left. \begin{array}{l} -3i A_1/a^2 + C_1 c h_2(c) - C_1' c' j_2(c') = 0, \\ -2C_1 c h_0(c) - A_1' a' + 2C_1' c' j_0(c') = -a', \\ 3\delta i A_1/a^2 + 6\delta C_1 h_1(c) + A_1' a' - 6C_1' j_1(c') = \delta a, \\ -3\epsilon i A_1/a^2 + \tfrac{1}{2}\epsilon C_1 c h_1''(c) \\ \qquad - \tfrac{1}{2}C_1' c'^2 j_1''(c') = 0. \end{array} \right\} \quad \text{(A2.1)}$$

We multiply the first equation by ϵ and subtract from the last. The result is a relation between the coefficients C_1' and C_1 which can be thrown into the form

$$C_1' = [\epsilon c^2 h_1(c)/c' Q(c')] C_1, \quad \text{(A2.2)}$$

where $Q(c')$ is a function defined in Eq. (10.3) of the text. This relation serves for the elimination of C_1'; the term with A_1' is eliminated by adding the second and the third equations. The remaining two equations contain only the variables A_1 and C_1 and can be readily resolved. The result, with respect to A_1 is given in Eqs. (10.3) and (10.5) of the text. Being a rigorous solution of the system (A2.1) these equations can be applied also to the case of bubbles in water.

APPENDIX B

Absorption Calculations

1. Averaging with Respect to Time

As stated in Sec. 6, all the linearized quantities are written in a complex form with omission of the time factor $\exp(-i\omega t)$. The symbol Ref means thus

$$\text{Re}[f \exp(-i\omega t)] = \tfrac{1}{2}[f \exp(-i\omega t) + f^* \exp(i\omega t)]. \quad \text{(B1.1)}$$

If we wish to find the time average of the product fF we have to multiply the expression (B1.1) with the analogous expression for F and then to average the four resulting terms. The two terms with the factors $\exp(\pm 2i\omega t)$ vanish in the averaging and there remains

$$\overline{fF} = \tfrac{1}{4}[fF^* + f^*F] = \tfrac{1}{2}\text{Re}(fF^*) = \tfrac{1}{2}\text{Re}(f^*F). \quad \text{(B1.2)}$$

2. Surface Integrals

The integral in Eq. (11.2) must be extended over the infinite spherical surface. Hence, the only potential that needs to be considered is $\bar{\varphi}$, and it has the form

$$\left. \begin{array}{l} \bar{\varphi} = \sum (2n+1) i^n \varphi_n(kr) P_n(\cos\vartheta), \\ \varphi_n(kr) = j_n(kr) + A_n h_n(kr). \end{array} \right\} \quad \text{(B1.3)}$$

According to Eq. (9.1),

$$\left. \begin{array}{l} v_r = -k \sum (2n+1) i^n \varphi_n'(kr) P_n(\cos\vartheta), \\ v_\vartheta = -(1/r) \sum (2n+1) i^n \varphi_n(kr) dP_n/d\vartheta \end{array} \right\} \quad \text{(B1.4)}$$

and according to Eq. (9.7),

$$\left. \begin{array}{l} P_{rr} = i\omega\rho\bar{\varphi}, \\ P_{r\vartheta} = -(2\eta k/r^2) \\ \qquad \times \sum (2n+1) i^n [kr \varphi_n'(kr) - \varphi_n] dP_n/d\vartheta. \end{array} \right\} \quad \text{(B1.5)}$$

The integrand of the integral in question can be written as $v_r P_{rr} + v_\vartheta P_{r\vartheta}$. We consider the first term, extending the integration over a sphere of radius r. Because of the orthogonality of spherical harmonics,

$$\tfrac{1}{2}\text{Re} \int v_r^* P_{rr} dF$$
$$= 2\pi k\omega\rho r^2 \text{Re}[i \sum (2n+1) \varphi_n'(kr) \varphi_n^*(kr)].$$

But from Eqs. (B2.1)

$$\text{Re}(i\varphi_n'\varphi_n) = \text{Re} i[(j_n'h_n - j_n h_n') A_n + h_n^* h_n' A_n^* A_n].$$

Because of the relations (C8) and the asymptotic expression (C14)

$$\text{Re}(i\varphi_n'\varphi_n^*) = -\text{Re}(A_n + A_n A_n^*)/k^2 r^2.$$

Hence

$$\tfrac{1}{2}\text{Re} \int v_r^* P_{rr} dF$$
$$= -2\pi\omega(\rho/k) \sum (2n+1)(\text{Re} A_n + A_n A_n^*).$$

The term $v_\vartheta P_{r\vartheta}$ gives no contribution because it vanishes in infinity, its integral being of the order $(1/r^2)$.

APPENDIX C

Some Formulas from the Theory of Harmonic Functions

1. Spherical Harmonics

We shall need the following relations

$$P_n{}^1 = -dP_n/d\vartheta, \quad \frac{1}{\sin\vartheta}\frac{d}{d\vartheta}(\sin\vartheta P_n{}^1) = n(n+1)P_n, \quad (C.1)$$

whence $P_n{}^1 = 0$. Further

$$\int_0^\pi (P_n)^2 \sin\vartheta d\vartheta = 2/(2n+1). \quad (C.2)$$

2. Cylindrical Harmonics

The radial factors $R_n(x)$ in the expansions (8.1) and (8.2)—comprising the two special cases $j_n(x)$ and $h_n(x)$—satisfy the equation

$$R_n{}'' + \frac{2}{x}R_n{}' + \left(1 - \frac{n(n+1)}{x^2}\right)R_n = 0. \quad (C.3)$$

They are known as *Heine functions* or as *spherical Bessel functions*. Their connection with the standard cylindrical harmonics $Z_p(x)$ is as follows, $R_n(x) = (\pi/2x)^{\frac{1}{2}}Z_{n+\frac{1}{2}}(x)$. In particular,

$$j_n(x) = (\pi/2x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x),$$
$$h_n(x) = (\pi/2x)^{\frac{1}{2}}H_{n+\frac{1}{2}}{}^{(1)}(x), \quad (C.4)$$

where J_p denotes the Bessel function and $H_p{}^1$ the Hankel function of the first kind.

The following formulas hold for $n \geqslant 0$,

$$h_n(x) = i^{-n-1}\frac{e^{ix}}{x}$$
$$\times\left[1 + \frac{(n+1)!i}{(n-1)!1!2x} + \frac{(n+2)!i^2}{(n-2)!2!(2x)^2} + \cdots\right], \quad (C.5)$$

$$j_n(x) = \frac{x^n}{1.3\cdots(2n+1)}$$
$$\times\left[1 - \frac{x^2}{2(2n+3)} + \frac{x^4}{2.4(2n+3)(2n+5)} - \cdots\right]. \quad (C.6)$$

Of general validity are the formulas

$$\left.\begin{array}{l} xR_n{}' + (n+1)R_n = xR_{n-1}, \\ \quad xR_n{}' - nR_n = -xR_{n+1}, \\ x^2R_n{}'' - 2xR_{n-1} = -x^2R_n + (n+1)(n+2)R_n, \\ x^2R_n{}'' + 2xR_{n+1} = -x^2R_n + n(n-1)R_n. \end{array}\right\} \quad (C.7)$$

Very useful are the following relations

$$\left.\begin{array}{l} j_n(x)h_n{}'(x) - j_n{}'(x)h_n(x) = i/x^2, \\ h_n(x)h_n{}'^*(x) - h_n{}^*(x)h_n{}'(x) = -2i/x^2, \end{array}\right\} \quad (C.8)$$

the second equation applies only to real arguments.

3. Approximate Expressions for Small Arguments

(Neglecting terms of order x^2)

$$\left.\begin{array}{lll} j_0 = 1, & j_0{}' = -\frac{1}{3}x, & j_0{}'' = -\frac{1}{3}, \\ j_1 = \frac{1}{3}x, & j_1{}' = \frac{1}{3}, & j_1{}'' = -\frac{1}{5}x, \\ j_2 = x^2/15, & j_2{}' = 2x/15, & j_2{}'' = 2/15. \end{array}\right\} \quad (C.9)$$

$$\left.\begin{array}{lll} h_0 = -i/x + 1, & h_0{}' = i/x^2, & h_0{}'' = -2i/x^3, \\ h_1 = -i/x^2, & h_1{}' = 2i/x^3, & h_1{}'' = -6i/x^4, \\ h_2 = -3i/x^2, & h_2{}' = 9i/x^4, & h_2{}'' = -36i/x^5. \end{array}\right\} \quad (C.10)$$

4. Functions of Low Order

$$\left.\begin{array}{l} j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{1}{x}\left(\frac{\sin x}{x} - \cos x\right), \\ \\ j_2(x) = \frac{1}{x}\left[\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right], \end{array}\right\}, \quad (C.11)$$

$$\left.\begin{array}{l} h_0(x) = -i\frac{e^{ix}}{x}, \quad h_1(x) = -\frac{e^{ix}}{x}\left(1 + \frac{i}{x}\right), \\ \\ h_2(x) = i\frac{e^{ix}}{x}\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right) \end{array}\right\}. \quad (C.12)$$

5. Asymptotic Expressions for Large Arguments

$$j_n(x) = (-1)^n \sin(x + \frac{1}{2}n\pi)/x, \quad (C.13)$$

$$h_n(x) = i^{-n-1} \exp(ix)/x. \quad (C.14)$$