where
\[ p_{\min} = \min_k p_k, \quad p_{\max} = \max_k p_k, \]
october together on every line. Lemma 1 applies and hence the system (7) reduces to one equation,
\[ \delta^2(Kp_{\min} - 1) - \delta(\theta Kp_{\max} - p_{\min}) \theta p_{\max} \geq 0. \] (12)
The coefficient of \( \delta^2 \) and the free term both are negative. Hence, (12) admits positive solutions if and only if the discriminant is positive and the coefficient of \(-\delta\) is negative. We get
\[ \theta^2 K^2 p_{\max}^2 - 2\theta(2p_{\max} - Kp_{\max} p_{\min}) + p_{\min}^2 \geq 0, \] \[ \theta Kp_{\max} - p_{\min} \leq 0, \text{ or } \theta \leq p_{\min}/(Kp_{\max}). \] (13b)
The coefficient of \( \theta^2 \) in (13a) is positive, and the left side of (13a) is negative for \( \theta = p_{\min}/(Kp_{\max}) \). Hence, the solution of (13) is the smaller of the two roots, so
\[ \theta \leq \frac{2 - Kp_{\min} - \sqrt{4(1 - Kp_{\min})}}{K^2 p_{\max}^2}, \]
and \( D^* \) is given by
\[ D^* = \frac{(K - 1) p_{\max}^2 - 2\sqrt{1 - Kp_{\min}}}{K^2 p_{\max}^2}. \] (14)
It is easy to check that \( D^* > 0 \) if \( p_{\min} > 0 \), i.e., if there are no nonzero transitions. For \( K = 2 \), \( D^* \) equals Gray's formula [2] for the binary symmetric source, namely,
\[ D^* = \frac{1}{2} \left( 1 - \frac{\sqrt{2} p_{\max} - 1}{p_{\max}} \right). \]

B. The Asymmetric Binary Source

Write
\[ Q = \begin{pmatrix} 1 & d & c \\ 1 - M & M \end{pmatrix}, \]
where \( M \) is the biggest element. Then note that
\[ \max_{k,j} M_{k,j} = M_{2,2} = M + \delta(M - c), \quad \text{for all } \delta. \]
This obviously is satisfied for \( \delta = 0 \). For other values of \( \delta \), note that the slopes of \( M_{k,j} \) are differences of coefficients in the same column, that the slopes of \( M_{2,2} \) and \( M_{1,1} \) are negative, and that \( M_{2,1} \) has the same slope as \( M_{2,2} \); therefore, none of them can exceed \( M_{2,2} \). It is then obvious that
\[ M_{2,1} = 1 - M + \delta(\text{c} - M) = \min_{k,j} M_{k,j}, \]
for all \( \delta \), since \( M_{2,1} + M_{2,2} = 1 \). Lemma 1 applies again and (11) becomes
\[ \delta^2(1 - M - \theta(M + d)) - \theta M \geq 0, \] (15)
which admits positive solutions if and only if
\[ \theta^2(M + d)^2 - 2\theta[(M + 1)(M + d) - 2M] + (1 - M)^2 \leq 0 \] \[ 1 - M - \theta(M + d) \geq 0. \] (16a)
(16b)
As in (13) we obtain
\[ D^* = \frac{(M + 1)(M + d) - \theta M - \sqrt{4M(\text{c} - M)}}{(M + d)^2}, \]
Again \( D^* \) coincides with Gray’s \( D \) [3, eq. (44)] apart from a typographical error therein.
A simple upper bound to \( D^* \) can be obtained as follows. Let \( R_\theta(D) \) denote the error-frequency rate-distortion function of a source that produces \( n \)-tuples of successive letters independently according to the \( n \)-th order marginal of the Markov source; and let \( D \) denote the value below which the Shannon lower bound to \( R_\theta(D) \) is tight. It is not difficult to show that \((K - 1)\min_{k,j} Q_{k,j}\) is an upper bound to \( D \), and that \( D \) is monotonic nonincreasing with \( n \). It follows that \((K - 1)\min_{k,j} Q_{k,j}\) is an upper bound to \( D_n \) for all \( n \) and hence to \( D \).

References

Coding Protection for Magnetic Tapes: A Generalization of the Patel–Hong Code

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Abstract—Patel and Hong have constructed a code that can correct any track error or two track erasures in a 6track magnetic tape. Here the construction is extended to a code that can correct a track error and a track erasure or three track erasures. A generalization is given.

I. INTRODUCTION

Patel and Hong [1], [2] devised an error-correcting scheme that was successfully used in the IBM 3420 series tape units with a recording density of 6250 b/in. This error-correcting scheme is capable of correcting any error pattern on a single track or any error patterns on two tracks provided that the erroneous tracks \( i \) and \( j \) are identified by some external pointers (that is, two track erasures). Here we shall present a subcode of the Patel–Hong code capable of correcting a track error together with a track erasure, or three track erasures. An IBM 3420 series tape unit writes characters in parallel across nine tracks on a half-inch tape, as shown in Fig. 1. Each character consists of eight information bits and one overall parity-check bit.

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The rows and columns of this array will be considered as elements of the Galois field of order $2^8$, GF($2^8$). As in the case of the Patel–Hong code the irreducible polynomial used to define GF($2^8$) is $g(x) = 1 + x^3 + x^4 + x^5 + x^8$. Denote the first eight bits of each column by $B_i$, $0 \leq i \leq 7$, and each row by $Z_j$, $0 \leq j \leq 8$. $Z_8$ is also denoted $Q$ and is a parity-check row. In our code $B_0$, $B_1$, and $Z_8$ will contain parity-check bits; hence, the rate of the code is $2/3$. The code is defined as

$$S_0 = \sum_{i=0}^{7} x^i Z_i,$$
$$S_1 = \sum_{i=0}^{7} x^i Z_i^2,$$

where

$$Z_i = \sum_{k=0}^{7} b_{ik} x^k \in \text{GF}(2^8),$$
$$B_j = \sum_{k=0}^{7} b_{jk} x^k \in \text{GF}(2^8).$$

Of course, the "polynomial" operations in (1), (2), and (3) are taken modulo $g(x)$, i.e., they are operations in GF($2^8$). Equations (1) and (2) define the Patel–Hong code, so our code is a subcode of the Patel–Hong code.

II. ENCODING

$Z_8$ is easily obtained using the procedure described in [1]. $B_2$, $B_3$, $B_4$, $B_5$, $B_6$, and $B_7$ are given, since they contain the information symbols. From (2) and (3)

$$B_0 + xB_1 = \sum_{i=2}^{7} x^i B_i,$$
$$B_0 + x^2 B_1 = \sum_{i=2}^{7} x^{2i} B_i.$$  (4)

Solving system (4) we obtain

$$B_0 = \sum_{i=2}^{7} x^{i-1} (x^{i-2} + \ldots + 1) B_i,$$  (5)
$$B_1 = \sum_{i=2}^{7} x^{i-1} (x^{i-1} + \ldots + 1) B_i.$$  (6)

Circuits performing operations (5) and (6) are easily constructed.

III. DECODING

Assume rows $Z_0$, $Z_1$, $\ldots$, $Z_8$ are received (columns $B_0$, $B_1$, $\ldots$, $B_7$, respectively). The decoder's first step is to calculate the three syndromes

$$S_0 = \sum_{i=0}^{8} Z_i,$$  (7)
$$S_1 = \sum_{i=0}^{7} x^i Z_i,$$  (8)
$$S_2 = \sum_{i=0}^{7} x^{2i} Z_i.$$  (9)

If no errors occur, by (1), (2), and (3) we have $S_0 = S_1 = S_2 = 0$. The following lemma.

Lemma 1:

$$S_1 = \sum_{i=0}^{7} x^i Z_i,$$  (10)
$$S_2 = \sum_{i=0}^{7} x^{2i} Z_i.$$  (11)

Proof: Equation (10) was proved by Patel and Hong [2]. Let us prove (11). From (9)

$$S_2 = \sum_{i=0}^{7} x^{2i} Z_i = \sum_{i=0}^{7} x^{2i} \sum_{j=0}^{7} b_{ij} x^j.$$  (12)
$$= \sum_{i=0}^{7} x^i \left( \sum_{j=0}^{7} b_{ij} x^j \right)^2.$$  (since the field has characteristic 2)
$$= \sum_{i=0}^{7} x^i Z_i^2.$$  (13)

As we stated at the beginning the code can correct either a track error and a track erasure, or three track erasures. Hence we need two decoding modes.

A. Mode I: Correction of a Track Error and a Track Erasure

Assume that an error pattern $e_i$ occurs in track $i$ and $e_j$ occurs in track $j$, $j$ is known, and all the other tracks are correctly transmitted. If $j \leq 7$, from (7), (10), and (11) we obtain

$$S_0 = e_i + e_j,$$  (14)
$$S_1 = x^i e_i + x^j e_j,$$  (15)
$$S_2 = x^i e_i^2 + x^j e_j^2.$$  (16)

Solving this system we obtain

$$x^i (x^{-j} S_2 + S_j^2) = x^{-j} S_i^2 + S_j.$$  (17)

First we need to construct circuits that will find $x^{-j} S_2 + S_j^2$ and $x^{-j} S_i^2 + S_j$. Then we multiply $x^{-j} S_i^2 + S_j$ by $x$ until we obtain $x^{-j} S_i^2 + S_j$. We now count how many times we had to multiply by $x$; in this way $i$ is obtained. Once we know $i$ we are in the Patel–Hong case of two erasures, i.e., we have to solve the system

$$S_0 = e_i + e_j,$$  (18)
$$S_1 = x^i e_i + x^j e_j.$$  (19)

Circuits to obtain $e_i$ and $e_j$ can be implemented as described in [1].

Assume $j = 8$. Then we have to solve (we are not interested in $e_8$)

$$S_0 = e_i + e_j,$$  (20)
$$S_1 = x^i e_i.$$  (21)

Solving this we obtain

$$e_i = \frac{x^i S_0 + S_i}{x^i + x^j},$$  (22)
$$e_j = \frac{x^j S_0 + S_i}{x^i + x^j}.$$  (23)

Circuits to obtain $e_i$ and $e_j$ can be implemented as described in [1].

Since $x^i S_2 = S_i^2$ we easily obtain $i$, and then $e_i = x^{-i} S_i$ gives us the corresponding track.
B. Mode II: Correction of a Triple Track Erasure

Assume erasure patterns $e_i, e_j, e_k$ occur in tracks $i, j, k$, where $0 < i < j < k < K$. If $k < S$, we have

$$
S_0 = e_i + e_j + e_k
$$

$$
S_1 = x'e_i + x'e_j + x'e_k
$$

$$
S_2 = x'e_i^2 + x'e_j^2 + x'e_k^2.
$$

The solution of this system is given by

$$
e_i^2 = \frac{x'kS_0 + xS_2 + (x' + x^k)S_2}{(x' + x^k)(x' + x^k)}
$$

$$
e_j^2 = \frac{x'kS_0 + xS_2 + (x' + x^j)S_2}{(x' + x^j)(x' + x^j)}
$$

$$
e_k^2 = \frac{x'kS_0 + xS_2 + (x' + x^k)S_2}{(x' + x^k)(x' + x^k)}.
$$

Circuits to solve (19) are more complicated than in the case of two erasures, but still are feasible. To find $e_i, e_j$, and $e_k$, we need to take the square root. But this is easily done, since square root is a linear $1 - 1$ operation.

Finally, if $k = S$, we have to solve the system

$$
S_3 = x'e_i + x'e_j
$$

$$
S_2 + x'e_k = (x' + x^j)(x' + x^j)
$$

and the solution is given by

$$
e_i^2 = \frac{S_3^2 + xS_2^2}{x'(x' + x^j)}
$$

$$
e_j^2 = \frac{S_3^2 + xS_2^2}{x'(x' + x^j)}
$$


IV. Generalization

The construction can be generalized to an $(n + 1) \times n$ array, i.e., an $(n + 1)$-track tape. As before, denote by $Z_i$ the rows $0 \leq i \leq n$, and by $B_j$ the first $n$ bits in each column, $0 \leq j \leq n - 1$. $Z_i$ and $B_j$ are considered elements in GF($2^n$), so we have to choose an irreducible polynomial $g(x)$ of degree $n$ in $\mathbb{Z}_2[x]$. Hence, GF($2^n$) is defined by $g(x)$, $Z_i = x^i \sum_{k=0}^{n-1} b_{jk} x^k$ (notice that our array is now $(b_{jk})$ $0 \leq i \leq n, 0 \leq j \leq n - 1, b_{jk} \in$ GF(2)). Take $0 \leq m \leq n - 1$. Columns $B_0, B_1, \ldots, B_m$ will contain parity-check bits together with row $Z_n$. Hence, the code has rate $n - m - 1/(n + 1)$ and is defined by the $m + 2$ equations in GF($2^n$);

$$
\sum_{i=0}^{n} Z_i = 0
$$

$$
\sum_{i=0}^{n-1} x^i B_i = 0, \quad 0 \leq j \leq m.
$$

(22)

Call this code $B(n, m)$-code. Using this notation, the Patel–Hong code is a $B(8, 0)$-code, while the code described in the previous section is a $B(8, 1)$-code. Whenever $2s + t \leq m + 2$ the $B(n, m)$-code can correct $s$ track errors and $t$ track erasures.

Assuming $\tilde{Z}_i, 0 \leq i \leq n$, is received, we have the syndromes

$$
S_0 = \sum_{i=0}^{n} \tilde{Z}_i
$$

$$
S_{j+1} = \sum_{i=0}^{n-1} x^{2j} B_i, \quad 0 \leq j \leq m.
$$

(23)

The key property necessary for decoding is

$$
S_{j+1} = \sum_{i=0}^{n-1} x^{2j} \tilde{Z}_i, \quad 0 \leq j \leq m.
$$

(24)

Equation (24) is proved in the same way as Lemma 1. From (23) and (24), if $2s + t \leq m + 2$ and $s$ track errors and $t$ track erasures occur, we have to solve a system of $m + 2$ equations with $m + 2$ unknowns. We will have $(m + 2)/2 + 1$ decoding modes, depending on the number $s = 0, 1, \ldots, (m + 2)/2$ of track errors that $B(n, m)$ can correct.

It remains to be shown that the solution of this system of $m + 2$ equations exists and is unique. To see this observe from (22), (23), and (24) that a $B(n, m)$ code can be defined as the set of vectors $(Z_0, Z_1, \ldots, Z_n)$ satisfying

$$
\sum_{i=0}^{n} Z_i = 0
$$

$$
\sum_{i=0}^{n-1} x^i \tilde{Z}_i = 0, \quad 0 \leq j \leq m.
$$

(25)

This is a linear code over GF($2^n$) of length $n + 1$. The result is proved if we show that this code has minimum distance $m + 3$. From (25) note that the parity-check matrix of the code is $(\lambda_{ij}) - 1 \leq j \leq m, 0 \leq i \leq n$, where

$$
\lambda_{i-1} = 1,
$$

$$
(\lambda_{ij})^{2^t} = x^i, \quad 0 \leq j \leq m, 0 \leq i \leq n - 1,
$$

$$
\lambda_{n+1} = 0, \quad 0 \leq j \leq m.
$$

The code has minimum distance $m + 3$ and dimension $n - m - 1$ if and only if any $m + 2$ columns in the parity-check matrix are linearly independent. Choose any $m + 2$ columns $0 \leq i_0 \leq i_1 < \cdots < i_{m+1} \leq n - 1$. We must show that det$(\lambda_{ij}) \neq 0$.

Taking each $\lambda_{ij}$ to the power $2^m$, this is equivalent to showing that

$$
\det (x^0)^2 (x^1)^2 (x^2)^2 \cdots (x^{m+1})^2 = 0.
$$

(26)

We prove this result by induction. Replace $x^0$ in the first column by the variable $y$. Then

$$
f(y) = \det (y^2 (x^0)^2 (x^1)^2 \cdots (x^{m+1})^2)
$$

is a polynomial of degree $2^m$ in $y$. Since we are in a field of characteristic 2, $f(y)$ is divisible by the following linear factors:

$$
m + 1 \quad \text{factors} \quad (y + x^k), \quad 1 \leq k \leq m + 1
$$

$$
\left( m + 1 \right) \quad \text{factors} \quad (y + a_1 + a_2 + a_3), \quad a_1, a_2, a_3 \in \{ x^0, x^1, \ldots, x^{m+1} \}$$
and so on. This gives a total of 

\[ \left( \frac{m+1}{1} \right) + \left( \frac{m+1}{3} \right) + \left( \frac{m+1}{5} \right) + \ldots \]

\[ = 2^m \]

linear terms in y. Since the degrees match, \( f(y) \) can be factored as a constant \( c \) times the linear factors described above. The constant \( c \) is given by

\[
c = \det \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\chi_1 & \chi_2 & \cdots & \chi_{m+1} \\
\chi_1^2 & \chi_2^2 & \cdots & \chi_{m+1}^2 \\
\chi_1^{2^{m-1}} & \chi_2^{2^{m-1}} & \cdots & \chi_{m+1}^{2^{m-1}}
\end{pmatrix}
\]

which is nonzero by induction. Replacing \( y \) by \( x^6 \), we have an explicit factorization of (26). Since the remaining factors are polynomials in \( x \) of degree smaller than \( n \), they are all nonzero. This proves our claim.

Example: Consider \( B(8, 2) \). Our field is \( GF(2^8) \) defined by its lowercase letter. For example, \( y_i \) is a realization of \( Y \).

\[
Bify(\ell_1-\varepsilon, \ell_2-\varepsilon, \ldots, \ell_m-\varepsilon) < A \]

\[
\frac{f_y(\mathbf{x})}{f_y(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)} < A, \quad (2)
\]

where \( A \) and \( B \) are the test thresholds and where \( f_y(\mathbf{t}_1, \ldots, \mathbf{t}_n) \) is the \( n \)-variate joint probability density function (pdf) of \( Y_1, \ldots, Y_n \). If the upper (lower) threshold is violated, the test terminates with an acceptance of \( H_0 \) (\( H_1 \)). The sample size, the number of samples at which the test terminates, is a random variable. Let it be denoted by \( N \). Suppose that the actual signal strength is not necessarily the design value \( \theta \) but rather \( \delta \theta \), where \( \delta > 0 \). Note the difference between the actual signal strength (which is \( \delta \theta \)) and the hypothesized signal strength (which is \( \theta \)). The introduction of \( \delta \) serves two purposes. One is to conveniently express the detector performances under both \( H_0 \) and \( H_1 \) by the same expression. The other is to see the effect of signal mismatch between the actual signal strength and the hypothesized signal strength. If \( \delta = 0 \), the hypothesis \( H_0 \) is true, and if \( \delta = 1 \) hypothesis \( H_1 \) is true, i.e., the actual signal coincides with the hypothesized signal strength; \( \delta \neq 0 \) and \( \delta \neq 1 \) means a signal mismatch. The expected value of \( N \) as a function of \( \delta \) is called the average sample number (ASN) function, and is denoted by \( E(N|\delta) \). The probability of accepting \( H_0 \) as a function of \( \delta \) is called the operating characteristic (OC) function, and it is denoted by \( L(\delta) \). We shall use the symbols \( A \) and \( B \) for the error probabilities under \( H_0 \) and \( H_1 \), respectively. Therefore, \( A = 1 - L(0) \) and \( B = L(1) \).

If \( Y_1, Y_2, \ldots \) are independent and identically distributed (iid), then (2) can be written as

\[
\sum_{i=1}^{n} \ln \left[ \frac{f_y(x_i - \theta)}{f_y(x_i, x_2, \ldots, x_n)} \right] \sim \ln(A), \quad (3)
\]

where \( f_y(\cdot) \) is the pdf of \( Y_i \). Wald [1] derived approximations for the ASN and OC functions of (2), and it has been proved [2] that (3) is optimum when \( Y_1, Y_2, \ldots \) are iid. The test (3) is optimum in the sense that it minimizes the ASN under \( H_0 \) and \( H_1 \) among all tests that have error probabilities no larger than \( \alpha \) and \( 1-\beta \). For dependent noise an optimum detector is in the form of a generalized sequential probability ratio test (GSPRT) [3], [4], namely,

\[
B_i < \frac{f_y(x_i - \theta, x_2 - \theta, \ldots, x_n - \theta)}{f_y(x_i, x_2, \ldots, x_n)} < A_i, \quad (4)
\]

where \( B_i \) and \( A_i \) are thresholds that are functions of \( n \). However, the determination of \( A_n \) and \( B_n \) is still an open problem. Equation (4) does not rule out the case that \( A_i \) and \( B_i \) are constants. Since there is no procedure for evaluating \( A_n \) and \( B_n \),