RELATIONS BETWEEN CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Some of the functions mentioned in a recent paper may be expressed in terms of known functions.

The function \( H_{2n}(x) \), which was required to be such that

\[
\int_0^\infty e^{-\lambda x} H_{2n}(x) \, dx = \left( \frac{\lambda + 1}{\lambda - 1} \right)^n \frac{1}{\lambda^2 - 1},
\]

is readily seen to be given by the equation

\[
H_{2n}(x) = xe^{-x} F(n + 1; 2; 2x)
\]  

(1)

where \( F(\alpha; \gamma; z) \) denotes Kummer's function

\[
1 + \frac{\alpha z}{\gamma} + \frac{\alpha(\alpha + 1) z^2}{\gamma(\gamma + 1) 2!} + \ldots
\]

which Kummer himself denotes by the symbol \( \phi(\alpha, \gamma, z) \). Kummer's function \( K(a, b, z) \) may be expressed in terms of Whittaker's function \( W_{k,m}(z) \) by comparing his expression for \( K \) in terms of \( \phi \) with the relation at the end of §16.41 of Whittaker and Watson's "Modern Analysis."

When \( z > 0 \) the relation is

\[
2^{2m} \Gamma(\frac{1}{2} + m + k)K(2m, -2k, z) = \pi z^{m+1/2} W_{k,m}(z)
\]  

(2)

This relation, combined with the relation just mentioned, should take the place of equation (32) of my recent paper, which is marred by some slips. In particular,

\[
k_{2n}(z) \Gamma(1+n) = W_{n,1/2}(2z)
\]  

(3)

and the asymptotic expansion of the function \( k \) is consequently (for \( x > 0 \))

\[
k_{2n}(x) \sim \frac{1}{\Gamma(1+n)} e^{-x} (2x)^n \left\{ 1 + \sum_{m=1}^\infty (-1)^m \frac{n(n-1) \ldots (n-m+1)^2(n-m)}{m! (2x)^m} \right\}
\]  

(4)

When \( n \) is a positive integer the series ends and we have

\[
k_{2n}(x) = (-1)^{n-1} 2x e^{-x} F(1-n; 2; 2x)
\]  

(5)
From the definition of the function \( h \) it is readily seen that when \( a > 0 \)

\[
\int_0^\infty e^{-ax} h_{-2n}(x) \, dx = \int_0^\infty e^{-ax} \, dx \int_0^\infty e^{-x \coth z - 2nx} \, dz
\]

\[
= \int_0^\infty \frac{e^{-2nx} \, dz}{a + \coth z}
\]

\[
= \frac{1}{2n} \frac{1}{a + 1} - \frac{1}{n + 1} \frac{1}{(a + 1)^2} - \frac{1}{n + 2} \frac{a - 1}{(a + 1)^3} - \ldots \quad (6)
\]

Now the last series is known to be equal to the integral

\[
\int_0^\infty e^{-ax} k_{2n}(x) \, dx \cdot \frac{\pi}{2} \cosec (n\pi)
\]

and so it is readily seen that when \( x > 0, n > 0, \)

\[
k_{-2n}(x) = \frac{2}{\pi} \sin (n\pi) h_{2n}(x). \quad (7)
\]

This relation is equivalent to equation (36) of my former paper in which \( k_{-2n}(x) \) should appear instead of \( k_{2n}(x) \). Whether \( n \) is a positive integer or not we may write

\[
h_{2n}(x) = \frac{1}{2} \Gamma(n) W_{-n, \frac{1}{2}}(2x) \quad n > 0 \}
\]

\[
x > 0 \}
\]

and the asymptotic expansion for the function \( h \) is accordingly

\[
h_{2n}(x) \sim \frac{\Gamma(n) e^{-x}}{2^{n+1} x^n} \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m n(n+1) \ldots (n+m-1) (n+m)}{m!} \frac{1}{(2x)^m} \right\}. \quad (9)
\]

This expression fails when \( n = 0 \) for then

\[
h_0(x) = \frac{1}{2} \left[ \Gamma'(1) - \log 2 \right] e^{-x} - e^x \int_x^\infty e^{-t \log \xi} \, d\xi \quad (10)
\]

and the asymptotic expansion is

\[
h_0(x) \sim \frac{1}{2} e^{-x} \left[ \Gamma'(1) - \log (2x) - \frac{1}{(2x)} + \frac{1!}{(2x)^2} - \frac{2!}{(2x)^3} + \ldots \right]. \quad (11)
\]