Noise Waves and Passive Linear Multiports

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Abstract—A simple derivation is given for the relation between a passive circuit's noise wave correlation matrix and its scattering matrix. It is shown that this relation, referred to as Bosma's theorem, comes readily from the fundamental principle that a passive multiport in thermodynamic equilibrium with reflectionless terminations produces uncorrelated output waves.

I. INTRODUCTION

Noise waves are a powerful means for characterizing noise in microwave circuits. They permit scattering matrix and signal flow graph theory to be used for noise calculations [1], leading to advantages in microwave CAD [2]. In the wave representation of a multiport network, noise waves are outwardly directed correlated sources present at each port, representing the noise originating in the network that is deliverable to its terminations. Denoting incident and output waves by vectors \( a \) and \( b \), respectively, and the scattering matrix by \( S \), the noise waves are given by a source vector \( c \) such that

\[
b = Sa + c.
\]

Characterization of the random noise wave amplitudes is made in terms of a correlation matrix \( C \),

\[
C = cc^\dagger
\]

where the dagger indicates a Hermitian transpose operation, and the overbar the correlation product. For each noise wave \( c \), the correlation matrix yields values for the noise power deliverable to the terminations, given in terms of the statistical expectation of \( c \), and written as \( |c| \)^2, as well as a correlation product with each of the other ports, given by \( c_i^a c^b \). An \( n \)-port linear component's signal and noise properties are completely described by \( n \times n \) scattering and noise wave correlation matrices.

In order to compute the noise properties of a network consisting of passive and active components, the scattering and correlation matrices of each component must be known. For an active device, these are usually found by measurement. The correlation matrix for a passive component is derived by examining how noise power is delivered to reflectionless terminations in thermodynamic equilibrium. The simplest case is that of a passive one-port where (1) simplifies to the scalar equation \( b = sa + c \). The incident wave \( a \) emanates from the termination and is uncorrelated with the noise wave \( c \), and so \( |b|^2 = |s|^2|a|^2 + |c|^2 \). With a reflectionless termination, the incident noise power \( |a|^2 \) in a 1-Hz bandwidth is known to be \( kT \), where \( k \) is Boltzmann's constant. \( T \) is the temperature, and quantum effects have been neglected. Thermodynamic equilibrium requires a balance in power flow such that \( |a|^2 = |b|^2 \) and therefore

\[
|c|^2 = kT(1 - |s|^2).
\]

For the one-port, the correlation matrix (2) is reduced to a scalar equation, the deliverable noise wave power given by (4). The subject of this letter is the derivation of noise wave correlation matrices for passive multiports. Although previously discussed by Bosma [3], the derivation given here is based on simple physical arguments. In all noise calculations, a 1-Hz bandwidth will be assumed.

II. TWO-PORTS IN THERMODYNAMIC EQUILIBRIUM

The properties of the passive multiport are derived by first considering a two-port passive component \( S \) with reflectionless terminations, as shown in Fig. 1. Thermodynamic equilibrium is assumed; the terminations and two-port having reached a common temperature \( T \). Since no external sources are present, the incident waves \( a_1 \) and \( a_2 \) are due solely to the thermal noise waves emanating from the two terminations. These waves will be uncorrelated, and from the derivation of (4) it is known that

\[
|a_1|^2 = |a_2|^2 = kT.
\]

Thermal noise waves generated by network \( S \) and the scattering of noise waves \( a_1 \) and \( a_2 \), contribute to output waves \( b_1 \) and \( b_2 \). Yet, in thermodynamic equilibrium the net power flow must be balanced, so these output waves must satisfy

\[
|b_1|^2 = |b_2|^2 = kT.
\]

As yet unknown is the value of the correlation product \( b_1^* b_2 \), but the properties of noise waves allow a means by which it may be determined. Given in Fig. 2 is the network of Fig. 1 with a directional coupler inserted between the two-port and its terminations. The primed wave variables are those that should be effected by the insertion of the coupler which is assumed to be an ideal (lossless and matched) 3-dB 180° hybrid. The effects on \( b_1 \) and \( b_2 \) are found by applying

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the coupler equations

\[ b'_1 = \frac{1}{\sqrt{2}} (b_1 + b_2) \]  

\[ b'_2 = \frac{1}{\sqrt{2}} (b_1 - b_2) \]  

resulting in

\[ |b'_1|^2 = \frac{1}{2} (|b_1|^2 + |b_2|^2) + \text{Re} \left( b_2 b'^*_1 \right) \]  

\[ |b'_2|^2 = \frac{1}{2} (|b_1|^2 + |b_2|^2) - \text{Re} \left( b_2 b'^*_1 \right) \]  

\[ b'_1 b'^*_2 = \frac{1}{2} (|b_1|^2 - |b_2|^2) + j \text{Im} \left( b_2 b'^*_1 \right) \]  

and a measure of \( b_2 b'^*_1 \) is possible by comparing these values. Similar expressions are obtained for \( a'_1 \) and \( a'_2 \), but since (5) remains valid for Fig. 2 as well as for Fig. 1, application of (9)-(11) results in

\[ |a'_1|^2 = |a'_2|^2 = kT \]  

\[ a'_1 a'^*_1 = 0, \]  

\[ a'_2 a'^*_2 = 0, \]  

\[ b'_1 b'^*_2 = 0 \]  

\[ b'_2 b'^*_1 = 0 \]  

Correlation between output waves from the two-port would cause power levels other than \( kT \) to be delivered to the terminations. This cannot be the case. The continued validity of (5) and thermodynamics require that \( |b'_1|^2 = |b'_2|^2 = kT \), and indeed the power in all waves must be \( kT \). The only conclusion is that \( \text{Re} \left( b_2 b'^*_1 \right) = 0 \).

The same reasoning may be repeated with a 3-dB 90° coupler substituted for the 180° hybrid. The result is then that \( \text{Im} \left( b_2 b'^*_1 \right) = 0 \). The statistics of all waves considered are unaffected by the insertion of the directional couplers, leading to the conclusion that

\[ b_2 b'^*_1 = 0. \]  

This result is true for all passive two-ports since no restrictions other than passivity have been placed on the network.

III. Noise from Passive Multiports

The results of Section II can be generalized for the case of a passive multiport with reflectionless terminations as described by matrix equation (1). The terminations will produce uncorrelated waves with thermal noise power \( kT \), written in vector form as

\[ a a^T = kT I. \]  

where \( I \) is the identity matrix. For the multiport, the directional coupler test can be applied to any two ports at a time. The result is the same. For any \( i \) and \( j \), \( |b_i|^2 = |b_j|^2 = kT \) and \( b_i b'^*_j = 0 \). As before, correlation between \( b_i \) and \( b_j \) result in power delivered to the loads differing from \( kT \) and is not possible. This result can be written in vector form as

\[ b b^T = kT I. \]  

This lack of correlation is a rather remarkable outcome. Although input noise waves \( a \) will be scattered, generating correlated waves \( S a \) that contribute to the output wave \( b \), the net correlation in the output waves must vanish. This puzzle is solved upon considering the noise wave correlation matrix for the multiport. Equation (17) allows a simple derivation. From (1)

\[ b b^T = (S a + c)(a S^* + c^*) \]

\[ = S a a^* S^* + c c^* \]  

\[ = S a a^* S^* + c c^* \]  

\[ = S a a^* S^* + c c^* \]  

where \( a c^* = c a^* = 0 \) since the noise waves produced by a multiport and its terminations are uncorrelated. Substitutions from (2), (16), and (17) yield the expression

\[ C_s = kT (I - S S^*) \]  

known as Bosma’s theorem [3]. The noise waves contributed by the network are simply those necessary to cancel the effects of correlation present in the scattered waves \( S a \) in order to maintain the lack of correlation in output waves, and to achieve balance in power flow, both required for thermodynamic equilibrium.

IV. Conclusion

The noise wave correlation matrix for a passive multiport is found by assuming thermodynamic equilibrium with reflectionless terminations. Under these conditions, it has been shown that a passive multiport produces uncorrelated output.
waves. This result allows a simple derivation of Bosma's theorem: the relation between a passive multiport's scattering and noise wave correlation matrices.

REFERENCES

