[1] G. Kallianpur and H. Robbins, "Ergodic Property of the Brownian Motion Process," these PROCEEDINGS, 39, 525–533, 1953, and "The Sequence of Sums of Independent Random Variables," *Duke Math. J.* 21, 285–308, 1954.

[2] H. Robbins, "On the Equidistribution of Sums of Independent Random Variables," *Proc. Am. Math. Soc.*, 4, 786–799, 1953.

[3] T. E. Harris and H. Robbins, "Ergodic Theory of Markov Chains Admitting an Infinite Invariant Measure," these PROCEEDINGS, 39, 860–864, 1953.

[4] K. L. Chung, "Contributions to the Theory of Markov Chains. II," *Trans. Am. Math. Soc.*, 76, 397–419, 1954.

[5] W. Doeblin, "Sur deux problèmes de M. Kolmogoroff concernant les chaines dénombrables," *Bull. Soc. Math. France*, 52, 210–220, 1938.

[6] P. Lévy, Processus stochastiques et mouvement Brownien (Paris, 1948), p. 211.

[7] C. Derman, "A Solution to a Set of Fundamental Equations in Markov Chains," *Proc. Am. Math. Soc.*, 5, 332–334, 1954.

ON THE POSSIBILITY OF ELECTROMAGNETIC SURFACE WAVES

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1. *Introduction.*—The question of electromagnetic surface waves propagating along a plane of discontinuity between two different media was the subject of some diversity of opinion. It is related to the equally unsettled question about the merits of the conflicting solutions given by A. Sommerfeld[1] and H. Weyl[2] for the field of an electric dipole oscillating at the surface of a plane earth. A few remarks about this relation will be made in section 5, but its complete discussion will be postponed until a later date, and the present paper will be devoted to the analysis of the *independent surface wave.* It will be well, therefore, to define what is meant by this term. It is a matter of common knowledge that an ordinary plane wave, falling under an angle on a surface of discontinuity and suffering total reflection in the process, produces in the second medium an inhomogeneous wave with propagation parallel to the surface of discontinuity. But this may be called a *dependent* surface wave because it is only an adjunct of the space waves in the first medium and would not exist without them. On the other hand, an *independent* surface wave is one which consists of two inhomogeneous waves—one in each medium—running along the dividing surface. Its intensity is appreciable only in a thin layer along the surface of discontinuity and decreases exponentially to both sides.

On the whole, the results of the analysis presented in the following sections are negative: an independent surface wave is supported only by a medium with peculiar properties that are hardly available in nature (sec. 4); therefore, it is only of theoretical interest. The whole investigation hinges on the consideration of the physical constants of the materials involved. Let the complex dielectric constants of the first and second medium be noted, respectively, by

$$k^2 = \kappa = \epsilon + i\sigma, \qquad k'^2 = \kappa' = \epsilon' + i\sigma'. \tag{1}$$

In every known medium the constant σ (which is proportional to the conductivity) is positive, and this fact has a decisive influence on the results. Moreover, it is arbitrary which medium is chosen as the primed; therefore, the convention

$$\sigma' > \sigma > 0 \tag{2}$$

does not involve any loss of generality.

In most substances the real part, ϵ, is also positive; however there are exceptions like metals with $\epsilon < 0$. In order not to restrict the generality, we shall leave the sign of ϵ, ϵ' open. On the other hand, the consideration of materials with magnetic properties does not add much of interest to the analysis and only renders it cumbersome. Therefore, we shall restrict ourselves to nonmagnetic media.

If a Cartesian system of co-ordinates x, y, z is used, so that the (x, y)-plane is the plane of discontinuity and the y-axis is the direction of propagation of the surface wave, the first or *unprimed* medium, κ, will be chosen so as to fill the upper half-space $(z > 0)$, and the second or *primed* medium, κ', the lower half-space $(z < 0)$. It will be shown in the next section that the space dependence of the field components in the two media is, respectively, expressed by the two factors

$$F = \exp\left[i\alpha y - (\alpha^2 - k^2)^{1/2} z\right], \tag{3}$$

$$F' = \exp\left[i\alpha y + (\alpha^2 - k'^2)^{1/2} z\right], \tag{4}$$

omitting the time factor $\exp(-i\omega t)$, where ω denotes the angular frequency and α is the same in both functions.

The mathematical condition on which the existence of the surface wave depends is then (sec. 2)

$$N(\alpha^2) = k'^2(\alpha^2 - k^2)^{1/2} + k^2(\alpha^2 - k'^2)^{1/2} = 0; \tag{5}$$

this equation is derived with the help of expressions (3), (4), and the square roots in it must have the same properties as the square roots in F and F'.

A surface wave can exist if equation (5) admits of a solution subject to the physical conditions of the problem. These conditions must apply to any value of α having a physical reality and in particular to the root of equation (5) if it exists. We designate the root by α_0 and abbreviate:

$$\beta_0 = \gamma_0 + i\delta_0 = \alpha_0^2. \tag{6}$$

The conditions which α_0 and β_0 must satisfy are then as follows:

1. Since the positive y-direction is taken as the sense of propagation and the time factor is chosen with the negative exponent, $-i\omega t$, both the real and the imaginary part of α_0 must be positive (i.e., α_0 lies in the first quadrant):

$$\operatorname{Re}(\alpha_0) > 0, \qquad \operatorname{Im}(\alpha_0) > 0. \tag{7}$$

2. Condition of finiteness: the factors F, F' should not grow into infinity as z becomes infinite (positively, for $z > 0$; negatively, for $z < 0$). This requires

$$\operatorname{Re}(\beta_0 - \kappa)^{1/2} > 0, \qquad \operatorname{Re}(\beta_0' - \kappa')^{1/2} > 0. \tag{8}$$

3. Radiation condition: the factors F, F' should not contain any part representing waves moving (in the z-direction) toward the plane of discontinuity. Hence

$$\operatorname{Im}(\beta_0 - \kappa)^{1/2} \leq 0, \qquad \operatorname{Im}(\beta_0' - \kappa')^{1/2} \leq 0. \tag{9}$$

Thus the two square roots are quantities of the fourth quadrant.

When ϵ, ϵ' are both positive, the problem becomes trivial because it is easily seen that the real part of $N(\alpha_0^2)$ is then always positive, so that equation (5) cannot be satisfied and the independent surface wave cannot exist. However, we do not wish to restrict ourselves to this special case; besides, related problems may arise in which not all of conditions (7), (8), and (9) must be met. We think it, therefore, worth while to investigate first whether equation (5) admits of a solution if requirement (8) alone is postulated. This investigation is carried out in sections 3 and 4. It is found that the answer depends on the signs of the following combinations of the physical constants

$$\left.\begin{aligned} R &= (\sigma^2 - \epsilon^2)\,(\epsilon + \epsilon') - 2\epsilon\sigma(\sigma + \sigma'), \\ I &= -(\sigma^2 - \epsilon^2)\,(\sigma + \sigma') - 2\epsilon\sigma(\epsilon + \epsilon'), \end{aligned}\right\} \tag{10}$$

and R', I', which are obtained by cyclic substitution (i.e., by interchange of primed and unprimed symbols).

The results are summarized in Table 1. The structure of the functions I, I' is such that, in view of equations (2), in all cases $I' < I$; therefore, the table contains all possible sign combinations. The entries in the last column are the answers to the question: "Does a root of equation (5), subject to conditions (8), exist?" Whenever the result is "Yes," it is contingent on additional conditions that must be satisfied by the constants; they will be enumerated in section 3. In every case when a root $\beta_0 = \alpha_0^2$ exists, α_0 can be chosen so that conditions (7) are also satisfied. It is different with conditions (9), which impose additional restrictions, making it impossible to obtain a solution. Indeed, to fulfil conditions (9), it would be necessary to have $I < 0$, $I' < 0$, but these inequalities are not true in any of the cases of Table 1, where the answer is "Yes." Hence, $N(\alpha^2)$ does not possess a root satisfying the full set of conditions (7), (8), and (9), and this means that an independent surface wave cannot exist.

TABLE 1

No.	R	R'	I	I'	Result
1	Pos.	Pos.	Pos.	Pos.	Yes
2			Pos.	Neg.	Yes
3			Neg.	Neg.	No
4	Pos.	Neg.	Pos.	Pos.	Yes
5			Pos.	Neg.	No
6			Neg.	Neg.	No
7	Neg.	Pos.	Pos.	Pos.	Yes
8			Pos.	Neg.	Yes
9			Neg.	Neg.	No
10	Neg.	Neg.	Pos.	Pos.	Yes
11			Pos.	Neg.	Yes
12			Neg.	Neg.	No

In this form the conclusions apply only to finite I, I'; the limiting case when these functions vanish will be discussed in section 4.

2. *The Electric Field of the Surface Wave.*—The electric field vector $\boldsymbol{E}$ in the first medium ($z > 0$) and the magnetic, $\boldsymbol{H}$, must satisfy the equations

$$\left.\begin{aligned} \nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = 0, \qquad \nabla\cdot\boldsymbol{E} = 0, \\ \boldsymbol{H} = -\left(\frac{c}{\mu\omega}\right)\nabla\times\mathbf{E}, \end{aligned}\right\} \tag{11}$$

where c denotes the velocity of light in vacuum and μ the magnetic permeability. However, we shall deal only with nonmagnetic media, where $\mu = \mu' = 1$.

It is well known and trivial that the state of polarization in which $\boldsymbol{E}$ is parallel to the surface of discontinuity does not permit of a surface wave. Hence the only interesting case is that when E_x vanishes. The components different from zero are then, with the notations of equation (4),

$$\left.\begin{aligned} E_z = AF, \qquad E_y &= -i(\alpha^2 - k^2)\alpha^{-1}AF, \\ H_x &= (ck^2/\omega\alpha)\, AF, \end{aligned}\right\} \tag{12}$$

where A is an arbitrary constant.

In the second (primed) medium ($z < 0$) analogous equations obtain, with the difference that the components and constants are primed:

$$\left.\begin{aligned} E_z' = A'F', \qquad E_y' &= i(\alpha^2 - k'^2)\alpha^{-1}A'F', \\ (H_x' &= ck'^2/\omega\alpha)A'F'. \end{aligned}\right\} \tag{12'}$$

The parameter α is the same in both cases, or else the boundary conditions could not be satisfied, which are (for $z = 0$)

$$E_y = E_y', \qquad H_x = H_x'.$$

The conditions immediately lead to the so-called *determinant equation* (5), which we shall write in the form

$$N(\beta_0) = 0, \tag{13}$$

where

$$N(\beta) = \kappa'(\beta - \kappa)^{1/2} + \kappa(\beta - \kappa')^{1/2}. \tag{14}$$

The equation can be formerly resolved by transferring the second term to the right side and squaring both sides. The result is

$$\beta_0 = \frac{\kappa\kappa'}{\kappa + \kappa'}, \tag{15}$$

but it must be borne in mind that the same expression would follow if $N(\beta)$ were defined as the difference, instead of the sum, of the two terms in formula (14). Hence equation (15) is but a formal expression of the root, provided it exists, but it does not guarantee its existence. This can be put in a different way, if the complex plane of the variable β is considered: the point β_0 in the plane defined by equation (15) may or may not be the root of equation (13). The function $N(\beta)$ defines a two-sheeted Riemann surface for the complex variable β, and conditions (8) impose restrictions upon the region in this surface where the point β_0 can be located. The situation becomes very simple if we choose the branch cuts (starting at the two branch points κ and κ') as indicated in Figure 1, namely, going into infinity in the negative direction at constant imaginary height. Then all points of the first sheet of the Riemann surface are characterized by $-\pi < \varphi < \pi$ $-\pi < \varphi' < \pi$ and satisfy conditions (8), while all points of the second sheet violate them and must be excluded. If the first sheet is chosen as the physically permissible region, equations (8) are taken care of.

3. *Roots of the Determinant Equation.*—We shall now evaluate the complex quantities

$$\beta_0 - \kappa = \rho \exp i\varphi, \qquad \beta_0 - \kappa' = \rho' \exp i\varphi', \tag{16}$$

regardless of whether the point β_0 is the root of equation (13) or not. Equations (1) and (2) give, with notations (10),

$$\beta_0 - \kappa = \frac{R + iI}{Q}, \tag{17}$$

where Q is the essentially positive function

$$Q = (\epsilon + \epsilon')^2 + (\sigma + \sigma')^2, \tag{18}$$

whence follows

$$\rho = (\epsilon^2 + \sigma^2)Q^{-1/2}. \tag{19}$$

It is important to notice that the real and imaginary parts of β_0 are expressed by

$$\left.\begin{aligned} \gamma_0 &= \epsilon + \frac{R}{Q} = \frac{\epsilon'(\sigma^2 + \epsilon^2) + \epsilon(\sigma'^2 + \epsilon'^2)}{Q}. \\ \delta_0 &= \sigma + \frac{I}{Q} = \frac{\sigma'(\sigma^2 + \epsilon^2) + \sigma(\sigma'^2 + \epsilon'^2)}{Q}. \end{aligned}\right\} \tag{20}$$

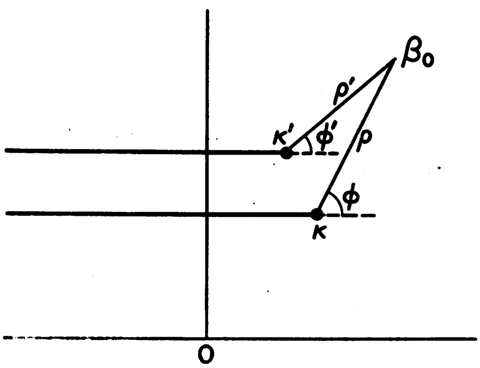

FIG. 1.—Complex plane of the variable β

Because of conditions (2), δ_0 is always positive, and this means that the point $\beta_0 = \alpha_0^2$ always lies in the positive complex half-plane (first or second quadrant). Hence it is always possible to choose the sign of $\alpha_0 = \pm\beta_0^{1/2}$ in such a way as to satisfy conditions (7).

The difference $\beta_0 - \kappa'$ is expressed by equations quite similar to (17) and (19), which are at once obtained by interchanging the primed and unprimed symbols in the above expressions.

We turn now to the quantities directly entering into the function $N(\beta_0)$. Obviously,

$$(\beta_0 - \kappa)^{1/2} = \rho^{1/2} \exp {}^1/_2 i\varphi, \qquad (\beta_0 - \kappa')^{1/2} = \rho'^{1/2} \exp {}^1/_2 i\varphi'. \tag{21}$$

The complex constants κ, κ' can be expressed in terms of these quantities. Indeed, by eliminating β_0 and κ' from equations (15) and (16), we find

$$(\kappa + \rho \exp i\varphi)^2 = \rho\rho' \exp i(\varphi + \varphi'),$$

whence

$$\left.\begin{aligned} \kappa &= \pm(\rho\rho')^{1/2} \exp {}^1/_2 i(\varphi + \varphi') - \rho \exp i\varphi, \\ \kappa' &= \pm(\rho\rho')^{1/2} \exp {}^1/_2 i(\varphi + \varphi') - \rho' \exp i\varphi'. \end{aligned}\right\} \tag{22}$$

Substituting in expression (14) for $N(\beta_0)$,

$$N(\beta_0) = (\pm 1 - 1)\,(\rho\rho')^{1/2} \exp {}^1/_2 i(\varphi + \varphi') \times [\rho^{1/2} \exp {}^1/_2 i\varphi + \rho'^{1/2} \exp {}^1/_2 i\varphi'].$$

It is apparent that only the *upper sign* leads to $N(\beta_0) = 0$ and guarantees that β_0 is a root of the determinant equation. The problem is, therefore, to find out which sign in equations (22) is the correct one under the various circumstances. In view of relations (16), these equations can be thrown into the form

$$\pm(\rho\rho')^{1/2} \exp {}^1/_2(\varphi + \varphi') = \beta_0. \tag{23}$$

Both sides of this equation are reducible to the functions R, R', I, I'; it is, therefore, a routine matter to determine in every particular case which sign is correct. It will be sufficient to illustrate the simple procedure by two examples. (The numbering of them will be that of Table 1.)

1. In this case $0 < \varphi < {}^1/_2\pi$, $0 < \varphi' < {}^1/_2\pi$, whence $0 < {}^1/_2(\varphi + \varphi') < {}^1/_2\pi$, and

$$\cos {}^1/_2(\varphi + \varphi') > 0, \qquad \sin {}^1/_2(\varphi + \varphi') > 0.$$

On the other hand, it is apparent from equations (20) that the point β_0 lies in its complex plane above and to the right of both branch points κ and κ'. Therefore, equation (23) will have the upper sign if β_0 lies in the positive quadrant of the upper half-plane (first quadrant). Since δ_0 is always positive, the only additional requirement is

$$\gamma_0 > 0. \tag{24}$$

2. Here $0 < \varphi < {}^1/_2\pi$, $-{}^1/_2\pi < \varphi' < 0$, whence ${}^1/_4\pi < {}^1/_2(\varphi + \varphi') < {}^1/_4\pi$. The point β_0 lies horizontally to the right of both branch points κ, κ' and vertically between their heights. Two cases must be distinguished:

a) $0 < {}^1/_2(\varphi + \varphi') < \pi/4$ (an additional condition), $\cos {}^1/_2(\varphi + \varphi') > 0$, $\sin {}^1/_2(\varphi + \varphi') > 0$. Again the upper sign is obtained, if requirement (24) is satisfied.

b) $-\pi/4 < {}^1/_2(\varphi + \varphi') < 0$; then $\cos {}^1/_2(\varphi + \varphi') > 0$, $\sin {}^1/_2(\varphi + \varphi') < 0$. Since δ_0 is always positive, the upper sign cannot be obtained.

We list the additional conditions necessary to achieve the upper sign in the cases of Table 1 where the answer is "Yes":

4*a* and 7*a*. When ${}^1/_2\pi < (\varphi + \varphi') < \pi$, $\gamma_0 > 0$, $\epsilon' > \epsilon$, $\epsilon' > 0$.

4*b* and 7*b*. When $\pi < (\varphi + \varphi') < 3\pi/2$, $\gamma_0 < 0$, $\epsilon' > \epsilon$, $\epsilon < 0$.

8. $\gamma_0 > 0$, $\epsilon > \epsilon'$, $\epsilon > 0$.

10. $\gamma_0 < 0$.

11*a*. When $0 < (\varphi + \varphi') < {}^1/_2\pi$, $\gamma_0 > 0$, $\epsilon > 0$, $\epsilon' > 0$.

11*b*. When $-{}^1/_2\pi < (\varphi + \varphi') < 0$, $\gamma_0 < 0$.

The conclusions from these results were drawn in section 1.

4. *Special Case of Nonconducting Media.*—Our general analysis includes also the case that either of the media, or both, are nonconducting, inasmuch as we can consider vanishing conductivity as the limiting case of very small σ, σ'. The case when only one of these parameters vanishes is in no way different from the general and needs no separate discussion, because β_0 remains complex. But it is well to say a word about the case $\sigma = \sigma' = 0$. According to equation (10), there follows,

$$I = I' = 0,$$

$$R = -\epsilon^2(\epsilon + \epsilon'), \qquad R' = -\epsilon'^2(\epsilon + \epsilon'),$$

so that two cases must be distinguished.

a) When $(\epsilon + \epsilon') > 0$, both R and R' are negative, and hence $\gamma_0 < \epsilon$ and $\gamma_0 < \epsilon'$. Therefore, the factors of z in equation (4) reduce to $(\gamma_0 - \epsilon)^{1/2}$, $(\gamma_0 - \epsilon')^{1/2}$ and are purely imaginary. This means that the amplitudes of the waves do not decrease away from $z = 0$. The field does not represent a surface wave but two space waves, each filling the whole of one of the two half-spaces. Consequently, this case is of no interest for our problem.

b) When $(\epsilon + \epsilon') < 0$, there follows $R > 0$, $R' > 0$ and hence $\gamma_0 > \epsilon$, $\gamma_0 > \epsilon'$. The factors $(\gamma_0 - \epsilon)^{1/2}$, $(\gamma_0 - \epsilon')^{1/2}$ are both real, so that conditions (9) are satisfied. This case falls under Nos. 1, 2, and 3 of Table 1, with the difference that I, I' go to the limit $I = I' = 0$ as follows:

$$I = \lim\,[\epsilon^2(\sigma + \sigma') + 2\epsilon\sigma|\epsilon + \epsilon|],$$

$$I' = \lim\,[\epsilon'^2(\sigma + \sigma') + 2\epsilon'\sigma'|\epsilon + \epsilon'|].$$

Nos. 1 and 2 require the validity of the additional condition (24), which reduces to

$$\epsilon\epsilon'(\epsilon + \epsilon') > 0.$$

It is clear that the requirements of No. 2 can be satisfied by letting $\epsilon > 0$, $\epsilon' < 0$, and by appropriately adjusting the ratios of all the parameters.

This is the only case in which an independent surface wave can exist, but even this case is purely theoretical and without any practical significance. Apart from the fact that a nonconducting medium with a negative dielectric constant is not readily available, every known material of nature possesses a vestigial conductivity, and the slightest positive values of σ, σ' would invalidate the argument by bringing into play conditions (9).

5. *Bearing on Other Problems.*—The best-known example of a related problem is the propagation of waves emitted by a radio antenna along the surface of the earth, especially in the form given to it by Sommerfeld, as mentioned in the Introduction. The perturbation wave due to the presence of the earth's surface is set up in the form of an integral whose integrand is analogous in type to our expressions (4). However, the integral is extended over *real* values of α, and, in so far as subsidiary conditions of the character of (7), (8), and (9) are necessary, they are required prima facie for real values of α. For the purposes of evaluation the path of integration is then deformed into a contour integral in the complex plane, and the equation $N(\alpha^2) = 0$, identical with our equation (5), becomes important because its root determines the pole of the integrand. The question whether this equation has a root is analogous with the one treated above in section 3, but the writer is not prepared to say that it is identical with it. Indeed, the complex root point has no immediate physical meaning, and it is not obvious that conditions (7), (8), and (9) must be satisfied for it. The problem needs additional analysis and is now being studied.

As the end result of his evaluation, Sommerfeld found two perturbation waves: (1) a space wave with an intensity decreasing with the inverse square of the distance from the antenna and (2) a surface wave propagating along the earth and decreasing with the inverse first power. Such a situation seems questionable[3] in the light of our results stated above; it may be argued that the surface wave outlasts the space wave and becomes independent at large distances, in contradiction to the result that an independent surface wave cannot exist. However, even this conclusion needs further investigation; it is not entirely cogent, because the amplitudes of both waves also contain exponentially decreasing factors, a fact which limits the independence of the surface wave.

[1] A. Sommerfeld, *Ann. Physik*, **28**, 665, 1909.

[2] H. Weyl, *Ann. Physik*, **60**, 481, 1919.

[3] Questions were raised by the writer in these PROCEEDINGS, **33**, 195, 1947; the point of view expressed there is now only partially maintained by him.

ERRATA: OVARIAN HORMONES AND THE IONIC BALANCE OF UTERINE MUSCLE

In the article of the foregoing title appearing in these PROCEEDINGS, **40**, 515–521, 1954, the following corrections should be made:

P. 518, Table 2: Under the headings "Estrogen*" and "Progesterone" read "CV_x" instead of "CV_{sx}"; the note to Table 2 reads correctly: "* x = arithmetic mean Na, K, and ECW, respectively, of samples from one and the same animal; n = number of animals; $\bar{x}$ = arithmetic mean, $\Sigma x/n$; s_x = standard deviation: $\sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$; CV_x = coefficient of variation: $\frac{100\, s_x}{\bar{x}}$."

P. 519, line 12: Read "the difference" instead of "and difference."

P. 519, line 20: Read "is not significantly different" instead of "is significantly different."

P. 520, line 29: Read "...Estrogen: Na $\geq$ 45.6; K $\leq$ 128; progesterone: Na $\leq$ 24.1; K $\geq$ 150."

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