fore, $\sqrt{f/A} = 4.8 \cdot 10^9$ cm. Since the radius of the earth is $R = 6.35 \cdot 10^8$, we can say $\sqrt{f/A} = 7.6\ R$.

If we replace in expressions (16), (17), (18) the sign of inequality by that of equality, they will give us directly the maximum angular distance θ_m from the magnetic pole, in which an electron can strike the surface of the earth. We obtain for the three special cases,

(I)
$$7.6 \sin^{1/2}\theta_m = 2.41, \qquad (19)$$
$$\theta_m^{(1)} = 6°.$$

(II*a*)
$$7.6 \sin^{1/2}\theta_m = 1, \qquad (20)$$
$$\theta_m^{(2)} = 1°.$$

(II*b*)
$$7.6 \sin^2 \theta_m = 1 + \sqrt{1 + \sin^2 \theta_m}, \qquad (21)$$
$$\theta_m^{(3)} = 32°.$$

The net result of our considerations is, therefore, that electrons of 10^9 volt energy cannot hit the earth outside of two limited zones around the magnetic poles.[5] Our analysis does not permit us to say whether the maximum distance from the pole of 32° is actually reached.

[1] Bothe and Kolhoerster, *Zs. Physik*, **54**, 686 (1929).

[2] B. Rossi, *Rend. Acc. dei Lincei*, **2**, 478 (1930).

[3] M. A. Tuve, *Phys. Rev.*, **35**, 651 (1930).

[4] L. M. Mott-Smith, *Ibid.*, **35**, 1125 (1930).

[5] For electrons of energy 10^8 volt the maximum distance would be 17°, for 2.10^9 volt it would be 40°.

PERIODICITY IN SEQUENCES DEFINED BY LINEAR RECURRENCE RELATIONS

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A sequence of rational integers

$$u_0,\ u_1,\ u_2,\ \ldots \qquad (1)$$

is defined in terms of an initial set $u_0, u_1, \ldots, u_{k-1}$ by the recurrence relation

$$u_{n+k} + a_1 u_{n+k-1} + \ldots + a_k u_n = a,\ n \geqq 0, \qquad (2)$$

where $a_1, a_2, \ldots, a_k$ are given rational integers. The author examines (1) for periodicity with respect to a rational integral modulus m. Carmichael[1] has shown that (1) is periodic for $(a_k, p) = 1$ and has given periods (mod

m) for the case where the prime divisors of m are greater than k. The present note gives a period for (1) (mod m) without restriction on m. The results include those of Carmichael. The author also shows that if p divides a_k (1) is periodic after a determined number of initial terms and obtains a period.

The algebraic equation

$$F(x) = x^k + a_1 x^{k-1} + \ldots + a_k = 0 \tag{3}$$

is said to be associated with (2). In considering (1) for the prime modulus p we may replace $F(x)$ by any polynomial $f(x)$ which is congruent to $F(x)$ (mod p) and of degree k with leading coefficient unity. The following lemma gives a convenient choice for $f(x)$.

LEMMA 1. We may choose $f(x) \equiv F(x)$ (mod p) with the following properties:

(1) $f(x)$ is irreducible and of degree k with leading coefficient unity.

(2) p does not divide the index of $f(x)$.

(3) If θ is a root of $f(x)$ and p contains precisely the αth power of a prime ideal $\mathfrak{p}$ in $K(\theta)$ then $f'(\theta)$ contains precisely $\mathfrak{p}^{\alpha-1+\rho}$, where $\rho = 1$ or 0 according as α is or is not divisible by p.

(4) $1 - \theta \not\equiv 0$ (mod $\mathfrak{p}^2$) for any prime ideal divisor $\mathfrak{p}$ of p in $K(\theta)$.

If $\theta_1, \theta_2, \ldots, \theta_k$ are the roots of $f(x) = 0$, the general term of any sequence associated with $f(x)$ is given by

$$u_n = \frac{a}{f(1)} + \beta_1\theta_1^n + \beta_2\,\theta_2^n + \ldots + \beta_k\theta_k{}^n, \tag{4}$$

where

$$\beta_j = \frac{\gamma_j}{f^j(\theta^j)} + \frac{a\delta_j}{(1-\theta_j)f'(\theta_j)} \tag{5}$$

and γ_j, δ_j are integers in $K(\theta_j)$.

Let

$$F(x) \equiv \phi_1(x)^{e_1}\phi_2(x)^{e_2} \ldots \phi_r(x)^{e_r} \pmod{p} \tag{6}$$

where the $\phi_i(x)$ are prime functions (mod p) whose degrees we denote by k_i. By the theorem of Dedekind (6) implies the prime ideal decomposition

$$p \equiv \mathfrak{p}_{1j}{}^{e_1}\mathfrak{q}_{2j}{}^{e_2} \ldots \mathfrak{p}^{e_r},\ N\mathfrak{p}_{ij} = p^{k_i}, \tag{7}$$

in $K(\theta_j)$. Lemma 1 gives the power of these ideals dividing the denominators in (5). By use of the theorem of Fermat in algebraic fields we obtain the periodicity of (1) directly from (4).

We say that π is a general period (mod m) of the recurrence (2) if every sequence (1) satisfying (2) has the period π (mod m). The minimum period of any particular sequence (1) will be a divisor of π. We write

$e = \max e_i$ in (6) and l equal to the least common multiple of $p^{k_i} - 1$, $i = 1, 2, \ldots, r$. The following theorems give a general period of (2) for the case $(a_k, p) = 1$.

THEOREM 1. *If* $(a_k, p) \neq 1$, *and either* $a \equiv 0$ *(mod* p*) or* $F(1) \not\equiv 0$ *(mod* p*), then* (2) *has the general period* l *(mod* p*).*

THEOREM 2. *If* $(a_k, p) = 1$ *and* $p^\epsilon \leqq e < p^{\epsilon+1}$, $\epsilon \geqq 0$, *then* (2) *has the general period* $p^{\epsilon+1}\, l$ *(mod* p*).*

If p divides a_k we obtain the following theorem from (4).

THEOREM 3. *If the last s coefficients in* (2) *are divisible by* p, $a^{k-s} \not\equiv 0$ *(mod* p*), then* (1) *is periodic (mod* p*) after s terms and a period is given by Theorems* 1 *and* 2.

A general period π of (2) for the prime power modulus p^α is obtained from a period (mod p) by the following theorem which may be proved quite directly by noting that $(u_{n+\pi} - u_n)/p$ is again a sequence satisfying the recurrence (2) with $a = 0$.

THEOREM 4. *If* $(a_k, p) = 1$, *and the recurrence* (2) *has the general period* π *(mod* p*) then it has the general period* $p^{\alpha-1}\pi$*(mod* p^α*).*

If a_k is divisible by p we obtain the theorem:

THEOREM 5. *If the last s coefficients of* (2) *are divisible by* p, $a_{k-s} \not\equiv 0$ *(mod* p*) then* (1) *is periodic (mod* p^α*) after* αs *terms and a period is given by Theorem* 4.

The following theorem suffices for obtaining a period of (1) for a general rational integral modulus m from the previous results.

THEOREM 6. *If* m *has the prime decomposition* $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t}$, *the least common multiple of a set of general periods* π_i *of* (2) *(mod* $p_i^{\alpha_i}$*),* $i = 1, 2, \ldots t$ *is a general period of* (2) *(mod* m*).*

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[1] R. D. Carmichael, "On Sequences of Integers Defined by Recurrence Relations," *Quart. J. Math.*, **48**, 343–372 (1920).