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TRANSIENTS ON A LINEAR ANTENNA

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1. Transients on a lossless line

We suppose that we have to deal with a transmission line with distributed L and C . The line is finite, of length ℓ ; the end at $x = \ell$ is open; at $x = 0$ a signal $Ae^{j\omega t}$ is applied starting at $t = 0$.

Figure 1.

From the well-known properties of the transmission line the wave front propagates with a velocity $\alpha = 1/\sqrt{LC}$, reflects itself at $x = \ell$ with a change of sign, and reaches 0 . We assume that no reflection occurs for $x = 0$, hence when the wave front reaches 0 for the first time after $t = 0$, the steady state is attained.

For the sake of completeness, let us do the complete computation. The transmission line equations are:

$$\begin{cases} C \dfrac{\partial V}{\partial t} = - \dfrac{\partial I}{\partial x} \\ L \dfrac{\partial I}{\partial t} = - \dfrac{\partial V}{\partial x} \end{cases}$$

with the following boundary conditions:

$$\left.\begin{aligned} V(o,t) &= Ae^{-j\omega t} \\ I(\ell,t) &= 0 \end{aligned}\right. \qquad t > 0 \quad .$$

Both voltage and current intensity satisfy

$$\frac{\partial^2 U}{\partial t^2} = \alpha^2 \frac{\partial^2 U}{\partial x^2} \quad , \qquad \alpha^2 = \frac{1}{LC} \quad .$$

Taking the Laplace transform $\mathcal{L}$ we get:

$$\frac{\partial^2 \mathcal{U}}{\partial x^2} = \left(\frac{p}{\alpha}\right)^2 \mathcal{U}$$

hence

$$\mathcal{U} = A(p)\, e^{\frac{p}{\alpha}x} + B(p)\, e^{-\frac{p}{\alpha}x} \quad .$$

Therefore, if

$$\mathcal{I} = \mathcal{L}(I) = A(p)\, e^{\frac{p}{\alpha}x} + B(p)\, e^{-\frac{p}{\alpha}x}$$

knowing that $C \frac{\partial V}{\partial t} = - \frac{\partial I}{\partial x}$ one gets

$$\mathcal{V} = \mathcal{L}(V) = - \frac{1}{C\,\alpha} \left[A(p)\, e^{\frac{p}{\alpha}x} - B(p)\, e^{-\frac{p}{\alpha}x} \right] .$$

Using the boundary conditions

$$x = 0 \quad , \qquad \mathcal{V} = \frac{V_o p}{p + j\omega}$$

$$x = \ell \quad , \qquad \mathcal{I} = 0$$

one gets:

$$0 = A(p)\, e^{\frac{p}{\alpha}\ell} + B(p)\, e^{-\frac{p}{\alpha}\ell}$$

$$\frac{V_o p}{p + j\omega} = - \frac{1}{C\,\alpha} \left[A(p) - B(p) \right] ;$$

whence, $B(p) = -A(p)\, e^{\frac{2p\ell}{\alpha}}$

$$A(p) = -\frac{C\alpha V_o p}{p + j\omega}\,\frac{1}{1 + e^{2p\ell/\alpha}}$$

$$A(p) = -\frac{V_o \beta p}{p + j\omega}\,\frac{1}{1 + e^{2p\ell/\alpha}} \qquad \text{with} \quad C\alpha = \frac{C}{\sqrt{LC}} = \sqrt{\frac{C}{L}} = \beta .$$

$$B(p) = \frac{V_o \beta p}{p + j\omega}\,\frac{e^{2p\ell/\alpha}}{1 + e^{2p\ell/\alpha}}$$

Therefore,

$$\mathcal{I} = \frac{-V_o \beta p}{(p + j\omega)}\,\frac{1}{1 + e^{2p\ell/\alpha}}\left[e^{\frac{p}{\alpha}x} - e^{-\frac{p}{\alpha}x + \frac{2p\ell}{\alpha}} \right]$$

$$= -\frac{V_o \beta p}{p + j\omega} \cdot \frac{\mathrm{sh}\,\frac{p}{\alpha}(x - \ell)}{\mathrm{ch}\,\frac{p\ell}{\alpha}} \quad .$$

We expand $(\mathrm{Re}\;\; p > 0)$ the second ratio:

$$\frac{\mathrm{sh}\,\frac{p}{\alpha}(\ell - x)}{\mathrm{ch}\,\frac{p}{\alpha}\,\ell} = e^{-\frac{p}{\alpha}x} \sum_{n=0}^{\infty} (-1)^n\, e^{-2n\frac{p\ell}{\alpha}} - e^{\frac{p}{\alpha}x} \sum_{n=0}^{\infty} (-1)^n\, e^{-2(n+1)\frac{p\ell}{\alpha}} ,$$

and taking now the inverse Laplace transform, one gets

$$i(x,t) = V_o \beta \left[\sum_0^{\infty} (-1)^n\, \underset{\sim}{e}^{-j\omega(t - \frac{x}{\alpha} - \frac{2\ell n}{\alpha})} - \sum_0^{\infty} (-1)^n\, \underset{\sim}{e}^{-j\omega\left[t + \frac{x}{\alpha} - 2(n+1)\frac{\ell}{\alpha}\right]} \right]$$

where $\underset{\sim}{e}^{-j\omega z} = e^{-j\omega z} cu(z)$, $cu(z)$ = complementary unit step function.

Actually, we are only interested in the behavior of the line between $t = 0$ and $t = 2\ell/\alpha$. Therefore for $0 < t \leq \ell/\alpha$ we have the first term in the series:

$$i(x,t) = V_o\beta\, e^{-j\omega(t - \frac{x}{\alpha})} \quad cu(t - \frac{x}{\alpha}) \quad .$$

For $\ell/\alpha \leq t \leq 2\ell/\alpha$ we have

$$i(x,t) = V_o\beta \left[e^{-j\omega(t - \frac{x}{\alpha})} - e^{-j\omega(t + \frac{x}{\alpha} - \frac{2\ell}{\alpha})} \quad cu(t + \frac{x}{\alpha} - \frac{2\ell}{\alpha}) \right]$$

and that is all, for we suppose there is no reflection at the origin. The steady state is attained for $t = 2\ell/\alpha$ and is then

$$i_s(x,t) = V_o\beta\, e^{-j\omega t} \left[e^{j\omega\frac{x}{\alpha}} - e^{-j\omega\frac{x}{\alpha}} \cdot e^{j\omega\frac{2\ell}{\alpha}} \right]$$

$$= 2j\, V_o\beta\, e^{-j\omega(t - \frac{\ell}{\alpha})} \left[\sin\frac{\omega}{\alpha}(\ell - x) \right] \quad .$$

2. Radiation pattern of the transient

We now suppose that we open the transmission line, (Figure 2), and we assume no substantial change in LC . The magnetic field radiated by

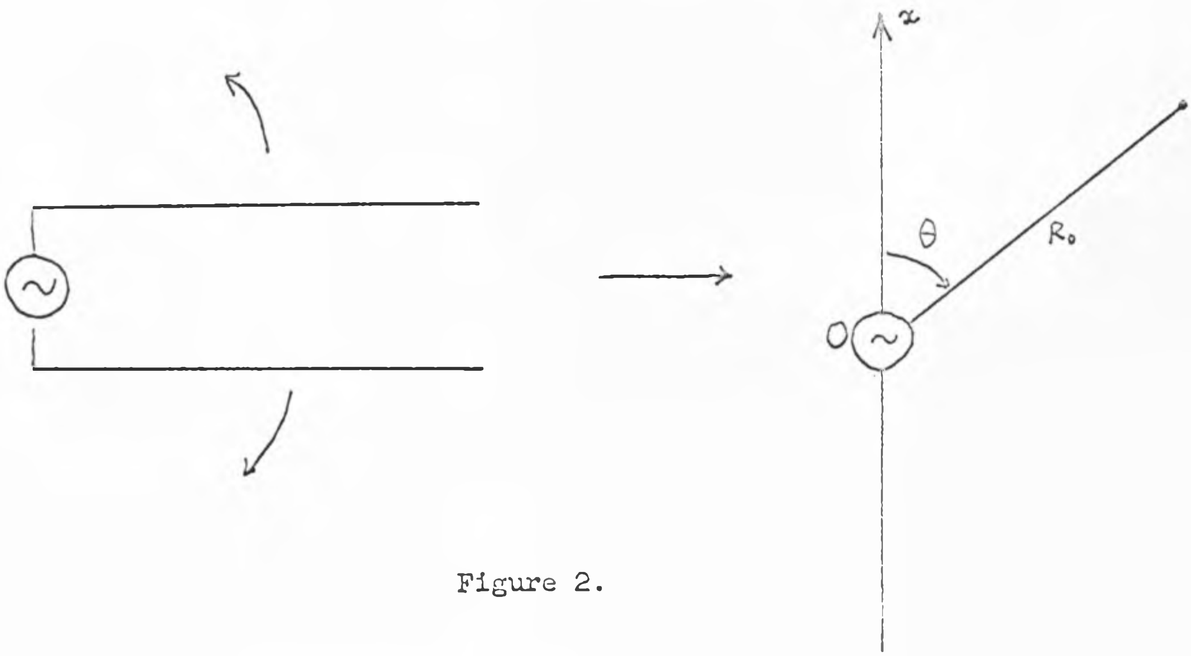


Figure 2.

the line element is:

$$dH_\phi(r,\theta,t) = j\ \frac{[I]\sin\theta\, dx}{2r\lambda}$$

where [I] is the retarded current intensity. The current on the upper wire is then: $i(x,t)$;

on the lower wire: $i(-x,t)$.

Hence for $0 \leq t \leq \ell/\alpha$, we get

$$I_1(x,t) = V_o\beta\, e^{-j\omega(t-\frac{x}{\alpha})} cu(t-\frac{x}{\alpha}) \qquad x > 0$$

$$= V_o\beta\, e^{-j\omega(t+\frac{x}{\alpha})} cu(t+\frac{x}{\alpha}) \qquad x < 0$$

for $\ell/\alpha \leq t \leq 2\ell/\alpha$ we get

$$I_2(x,t) = V_o\beta \left[e^{-j\omega(t-\frac{x}{\alpha})} - e^{-j\omega(t+\frac{x}{\alpha}-\frac{2\ell}{\alpha})} cu(t+\frac{x}{\alpha}-\frac{2\ell}{\alpha}) \right], \quad x > 0$$

$$= V_o\beta \left[e^{-j\omega(t+\frac{x}{\alpha})} - e^{-j\omega(t-\frac{x}{\alpha}-\frac{2\ell}{\alpha})} cu(t-\frac{x}{\alpha}-\frac{2\ell}{\alpha}) \right], \quad x < 0 \ .$$

Thus for $0 \leq t - \frac{r}{c} \leq \frac{\ell}{\alpha}$ we have

$$dH_\phi = j\,\frac{\sin\theta}{2r\lambda}\, V_o\beta\, e^{-j\omega(t-\frac{r}{c})}\, e^{j\frac{\omega x}{\alpha}}\, cu(t-\frac{r}{c}-\frac{x}{\alpha}) \qquad , x > 0$$

$$= j\,\frac{V_o\beta\sin\theta}{2r\lambda}\, e^{-j\omega(t-\frac{r}{c})}\, e^{j\frac{\omega x}{\alpha}} cu(t-\frac{r}{c}-\frac{x}{\alpha}) \quad ;$$

$$dH_\phi = j\,\frac{V_o\beta\sin\theta}{2r\lambda}\, e^{-j\omega(t-\frac{r}{c})}\, e^{-j\frac{\omega x}{\alpha}}\, cu(t-\frac{r}{c}+\frac{x}{\alpha}) \qquad , x < 0 \ .$$

Therefore putting

$$r = R_o - x\cos\theta$$

$$\frac{jV_o\beta e^{-j\omega(t-\frac{R_o}{c})}}{2R_o\lambda} = S$$

and noticing that

$$k = \frac{\omega}{c} = \frac{\alpha}{c}\cdot\frac{\omega}{\alpha} = \frac{1}{\gamma}\cdot\frac{\alpha}{c}$$

one gets

$$H_\phi = S\sin\theta\left[\int_0^{\ell} e^{j(\frac{\omega}{\alpha}-k\cos\theta)x}\, cu(t-\frac{R_o}{c}+\frac{x\cos\theta}{c}-\frac{x}{\alpha})\,dx\right.$$

$$\left.+\int_{-\ell}^{0} e^{-j(\frac{\omega}{\alpha}+k\cos\theta)x}\, cu(t-\frac{R_o}{c}+\frac{x\cos\theta}{c}+\frac{x}{\alpha})\,dx\right]$$

$$-S\sin\theta\left[\int_0^{\ell} e^{-j\omega(\frac{x-2\ell}{\alpha})-jkx\cos\theta}\, cu(t-\frac{R_o}{c}+\frac{x\cos\theta}{c}-\frac{x-2\ell}{\alpha})\,dx\right.$$

$$\left.+\int_{-\ell}^{0} e^{j\omega(\frac{x+2\ell}{\alpha})-jkx\cos\theta}\, cu(t-\frac{R_o}{c}+\frac{x\cos\theta}{c}+\frac{x+2\ell}{\alpha})\,dx\right].$$

We have then to distinguish between several intervals of time

$$\left.\begin{array}{lr}
t-\frac{R_o}{c} < 0 & \mathrm{I} \\
0 < t-\frac{R_o}{c} < \frac{\gamma}{\alpha}(\frac{\omega}{\alpha}-k\cos\theta)\ell & \mathrm{II} \\
\frac{\gamma}{\alpha}(\frac{\omega}{\alpha}-k\cos\theta)\ell < t-\frac{R_o}{c} < \frac{\gamma}{\alpha}(\frac{\omega}{\alpha}+k\cos\theta)\ell & \mathrm{III} \\
\frac{\gamma}{\alpha}(\frac{\omega}{\alpha}+k\cos\theta)\ell < t-\frac{R_o}{c} < \frac{2\ell}{\alpha} & \mathrm{IV} \\
t-\frac{R_o}{c} > \frac{2\ell}{\alpha} & \mathrm{V}
\end{array}\right\}$$

for $\cos\theta > 0$. (For $\cos\theta < 0$, complete by symmetry.)

Performing the integration one gets

$$H_\phi = S \sin\theta \left\{ \left[\frac{e^{j\psi_1(t,\theta)} - 1}{j(\frac{\omega}{\alpha} - k\cos\theta)} - \frac{e^{j\psi_2(t,\theta)} - 1}{-j(\frac{\omega}{\alpha} + k\cos\theta)} \right] \right.$$

$$- e^{2j\frac{\omega\ell}{\alpha}} \left[\frac{e^{-j\ell(\frac{\omega}{\alpha} + k\cos\theta)} - e^{-j\psi_3(t,\theta)}}{-j(\frac{\omega}{\alpha} + k\cos\theta)} \right.$$

$$\left.\left. + \frac{e^{-j\psi_4(t,\theta)} - e^{-j\ell(\frac{\omega}{\alpha} - k\cos\theta)}}{j(\frac{\omega}{\alpha} - k\cos\theta)} \right] \right\}$$

with

$$\psi_1(t,\theta) = \begin{cases} 0 & t < \frac{R_o}{c} \\ \frac{\alpha}{\gamma}(t - \frac{R_o}{c}) & \frac{R_o}{c} < t < \frac{R_o}{c} + \frac{\gamma}{\alpha}(\frac{\omega}{\alpha} - k\cos\theta)\ell \\ (\frac{\omega}{\alpha} - k\cos\theta)\ell & t > \frac{R_o}{c} + \frac{\gamma}{\alpha}(\frac{\omega}{\alpha} - k\cos\theta)\ell \end{cases}$$

$$\psi_2(t,\theta) = \begin{cases} 0 & t < \frac{R_o}{c} \\ \frac{\alpha}{\gamma}(t - \frac{R_o}{c}) & \frac{R_o}{c} < t < \frac{R_o}{c} + \frac{\gamma}{\alpha}(\frac{\omega}{\alpha} + k\cos\theta)\ell \\ (\frac{\omega}{\alpha} + k\cos\theta)\ell & t > \frac{R_o}{c} + \frac{\gamma}{\alpha}(\frac{\omega}{\alpha} + k\cos\theta)\ell \end{cases}$$

$$\psi_3(t,\theta) = \begin{cases} (\frac{\omega}{\alpha} + k\cos\theta)\ell & t < \frac{\gamma}{\alpha}(\frac{\omega}{\alpha} - k\cos\theta)\ell + \frac{R_o}{c} \\ \dfrac{2\ell - \alpha(t - \frac{R_o}{c})}{\gamma} & \frac{\gamma}{\alpha}(\frac{\omega}{\alpha} - k\cos\theta)\ell + \frac{R_o}{c} < t < \frac{2\ell}{\alpha} + \frac{R_o}{c} \\ 0 & t > \frac{2\ell}{\alpha} + \frac{R_o}{c} \end{cases} .$$

$$
\psi_4(t,\theta) = \begin{cases} \left(\frac{\omega}{\alpha} - k\cos\theta\right)\ell & t < \frac{R_o}{c} + \frac{\gamma}{\alpha}\left(\frac{\omega}{\alpha} + k\cos\theta\right)\ell \\[2ex] = \dfrac{2\ell - \alpha\left(t - \frac{R_o}{c}\right)}{\gamma} & \frac{\gamma}{\alpha}\left(\frac{\omega}{\alpha} - k\cos\theta\right)\ell + \frac{R_o}{c} < t < \frac{2\ell}{\alpha} + \frac{R_o}{c} \\[2ex] = 0 & t > \frac{2\ell}{\alpha} + \frac{R_o}{c} \end{cases} .
$$

If we now perform in detail the integration, we get the following results for the periods I - V .

<u>Period I</u> $\qquad H_\phi = 0 \qquad\qquad$ with $\rho = \frac{\omega}{\alpha k}$

<u>Period II</u>

$$
H_\phi = \frac{S\sin\theta}{jk} \cdot 2\rho\left[\frac{e^{j\omega\left(t - \frac{R_o}{c}\right)} - 1}{\rho^2 - \cos^2\theta}\right]
$$

<u>Period III</u>

$$
H_\phi = \frac{S\sin\theta}{jk}\left\{\left[\frac{e^{j\left(\frac{\omega}{\alpha} - k\cos\theta\right)} - 1}{\rho - \cos\theta} + \frac{e^{j\omega\left(t - \frac{R_o}{c}\right)} - 1}{\rho + \cos\theta}\right] + e^{2j\frac{\omega\ell}{\alpha}}\left[\frac{e^{-j\ell\left(\frac{\omega}{\alpha} + k\cos\theta\right)} - e^{-j\frac{2\ell - \alpha\left(t - \frac{R_o}{c}\right)}{\gamma}}}{\rho + \cos\theta}\right]\right\}
$$

$$
= \frac{S\sin\theta}{jk}\left\{\left[\frac{e^{j\left(\frac{\omega}{\alpha} - k\cos\theta\right)\ell} - 1}{\rho - \cos\theta} + \frac{e^{j\omega\left(t - \frac{R_o}{c}\right)} - 1}{\rho + \cos\theta}\right] + \frac{e^{j\left(\frac{\omega}{\alpha} - k\cos\theta\right)\ell} - e^{j\omega\left(t - \frac{R_o}{c}\right)}}{\rho + \cos\theta}\right\}
$$

$$
\boxed{H_\phi = \frac{S\sin\theta}{jk}\left[e^{j\left(\frac{\omega}{\alpha} - k\cos\theta\right)\ell} - 1\right]\frac{2\rho}{\rho^2 - \cos^2\theta}} \quad .
$$

Period IV

$$H_\phi = \frac{S \sin \Theta}{jk}\left\{\left[\frac{e^{j(\frac{\omega}{\alpha} - k\cos\Theta)\ell} - 1}{\rho - \cos\Theta} + \frac{e^{j(\frac{\omega}{\alpha} + k\cos\Theta)\ell} - 1}{\rho + \cos\Theta}\right]\right.$$

$$+ e^{2j\frac{\omega\ell}{\alpha}}\left[\frac{e^{-j\ell(\frac{\omega}{\alpha} + k\sin\Theta)} - e^{-j\left[\frac{2\ell - \alpha(t - \frac{R_o}{c})}{\gamma}\right]}}{\rho + \cos\Theta}\right.$$

$$\left.\left. - \frac{e^{-j\left[\frac{2\ell - \alpha(t - \frac{R_o}{c})}{\gamma}\right]} - e^{-j\ell(\frac{\omega}{\alpha} - k\cos\Theta)}}{\rho - \cos\Theta}\right]\right\}$$

$$= \frac{S \sin\Theta}{jk}\left[\frac{e^{j(\frac{\omega}{\alpha} - k\cos\Theta)\ell} - 1 - e^{j\omega(t - \frac{R_o}{c})} + e^{j(\frac{\omega}{\alpha} + k\cos\Theta)\ell}}{\rho - \cos\Theta}\right.$$

$$\left. + \frac{e^{j(\frac{\omega}{\alpha} + k\cos\Theta)\ell} - 1 + e^{j(\frac{\omega}{\alpha} - k\cos\Theta)\ell} - e^{j\omega(t - \frac{R_o}{c})}}{\rho + \cos\Theta}\right]$$

$$= \frac{S\sin\Theta}{jk}\left[\frac{2e^{j\frac{\omega}{\alpha}\ell}\cos(k\ell\cos\Theta)\cdot 2\rho}{\rho^2 - \cos^2\Theta} - \frac{\left[1 + e^{j\omega(t - \frac{R_o}{c})}\right]2\rho}{\rho^2 - \cos^2\Theta}\right]$$

$$= \boxed{\frac{S\sin\Theta}{jk}\;\frac{\left[e^{j\frac{\omega\ell}{\alpha}}\cos(k\ell\cos\Theta)\,2 - (1 + e^{j\omega(t - \frac{R_o}{c})})\right]2\rho}{\rho^2 - \cos^2\Theta}}$$

Period V

$$H_\phi = \frac{S \sin\theta}{jk} \frac{\left[e^{j\frac{\omega \ell}{\alpha}} \cos(k\ell \cos\theta)\, 2 - (1 + e^{j\omega\frac{2\ell}{\alpha}})\right] 2\rho}{\rho^2 - \cos^2\theta}$$

$$H_\phi = \frac{S \sin\theta}{jk}\, e^{j\frac{\omega\ell}{\alpha}} \frac{\left[\cos(k\ell\cos\theta) - \cos\frac{\omega\ell}{\alpha}\right] 4\rho}{\rho^2 - \cos^2\theta} \quad .$$

For $\cos\theta < 0$, same results in I', II', IV', and V' . The only change occurs in III' where one has to replace $\cos\theta$ by $-\cos\theta$.

The radiation patterns will then be:

I : $\quad H_\phi = 0$

II : $$\frac{2\rho\sin\theta}{\rho^2 - \cos^2\theta}\left[1 - \cos\omega(t - \frac{R_o}{c})\right]$$

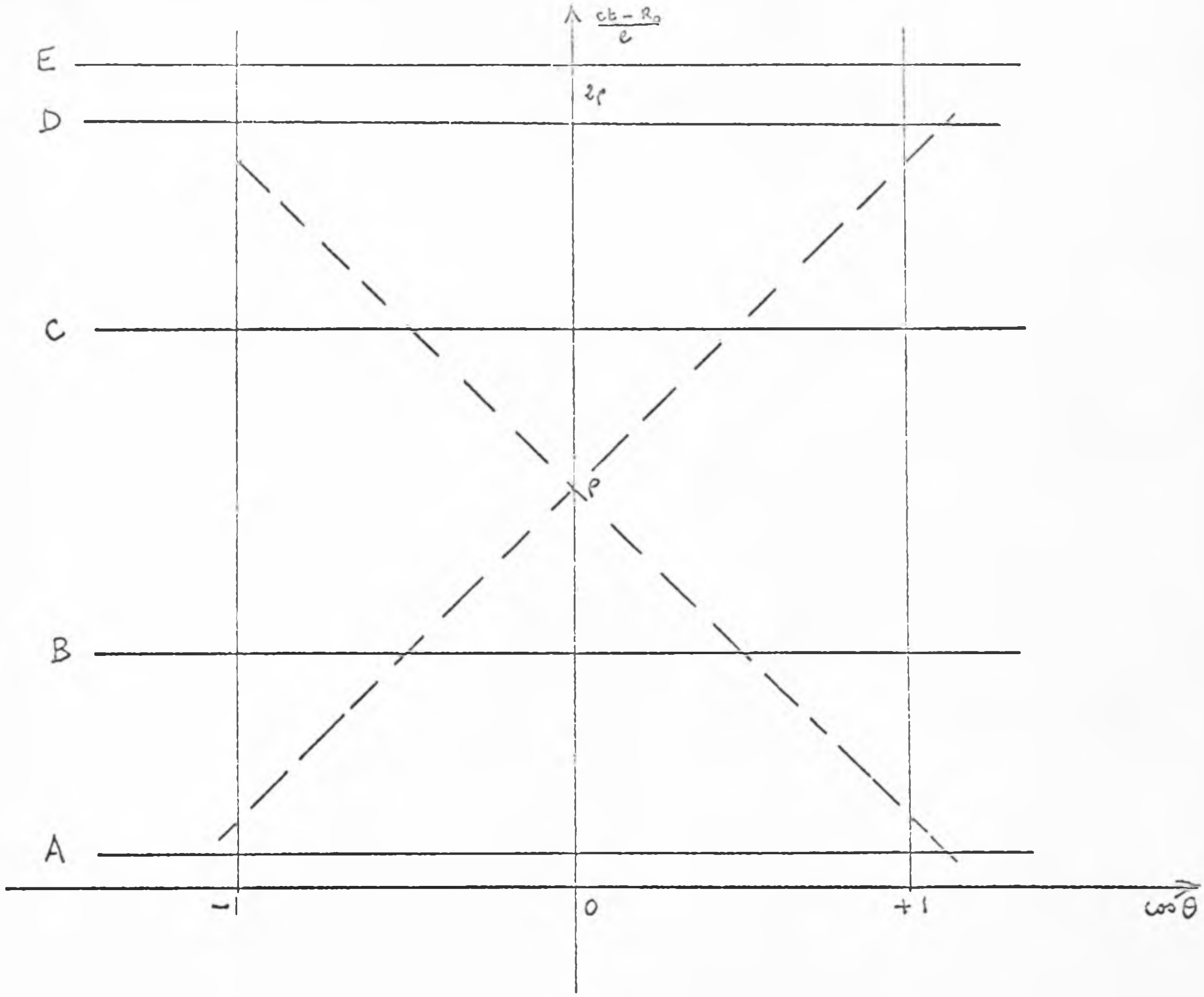
III: $$\frac{2\rho\sin\theta}{\rho^2 - \cos^2\theta}\left\{\cos\left[\omega(t - \frac{R_o}{c}) - (\frac{\omega}{\alpha} - k\cos\theta)\ell\right] - \cos\omega(t - \frac{R_o}{c})\right\} \quad .$$

III': $$\frac{2\rho\sin\theta}{\rho^2 - \cos^2\theta}\left\{\cos\left[\omega(t - \frac{R_o}{c}) - (\frac{\omega}{\alpha} + k\cos\theta)\ell\right] - \cos\omega(t - \frac{R_o}{c})\right\} \quad .$$

IV : $$\frac{2\rho\sin\theta}{\rho^2 - \cos^2\theta}\left\{2\cos(k\ell\cos\theta)\cos\omega(t - \frac{R_o}{c} - \frac{\ell}{\alpha}) - \left[\cos\omega(t - \frac{R_o}{c}) + 1\right]\right\} \quad .$$

V : $$\frac{2\rho\sin\theta}{\rho^2 - \cos^2\theta}\left\{(\cos(k\ell\cos\theta) - \cos\frac{\omega\ell}{\alpha}) \cdot 2\cos\omega(t - \frac{R_o}{c} - \frac{\ell}{\alpha})\right\}$$

In Figure 3 we represent in a (t,θ), or more precisely, in a $(\cos\theta, \frac{ct - R_o}{\ell})$ diagram, the regions corresponding to the various periods.



I

Figure 3.

The five cuts, A,B,C,D,E, give a fair description of the phenomenon.

3. Numerical results

Radiation patterns as shown in Fig. 4 have been drawn for

$$\rho = 1.2$$

$$k\ell = \pi/2$$

Diagram A : $$\frac{ct - R_o}{\ell} = \frac{\rho - 1}{2}$$

Diagram B : $$\frac{ct - R_o}{\ell} = \rho - \frac{1}{2}$$

Diagram C : $$\frac{ct - R_o}{\ell} = \rho + \frac{1}{2}$$

Diagram D : $$\frac{ct - R_o}{\ell} = \frac{3\rho + 1}{2}$$

Diagram E : $$\frac{ct - R_o}{\ell} = 2\rho + \frac{1}{2}$$

Diagrams A and B show that the maximum of the field intensity is not in the direction perpendicular to the antenna.

Diagram C shows a pattern which is already more of the broadside type.

Diagrams D and E are completely broadside.

For $|\cos \Theta| = \frac{1}{2}$, diagrams B and C are continuous but all the derivatives are not continuous.

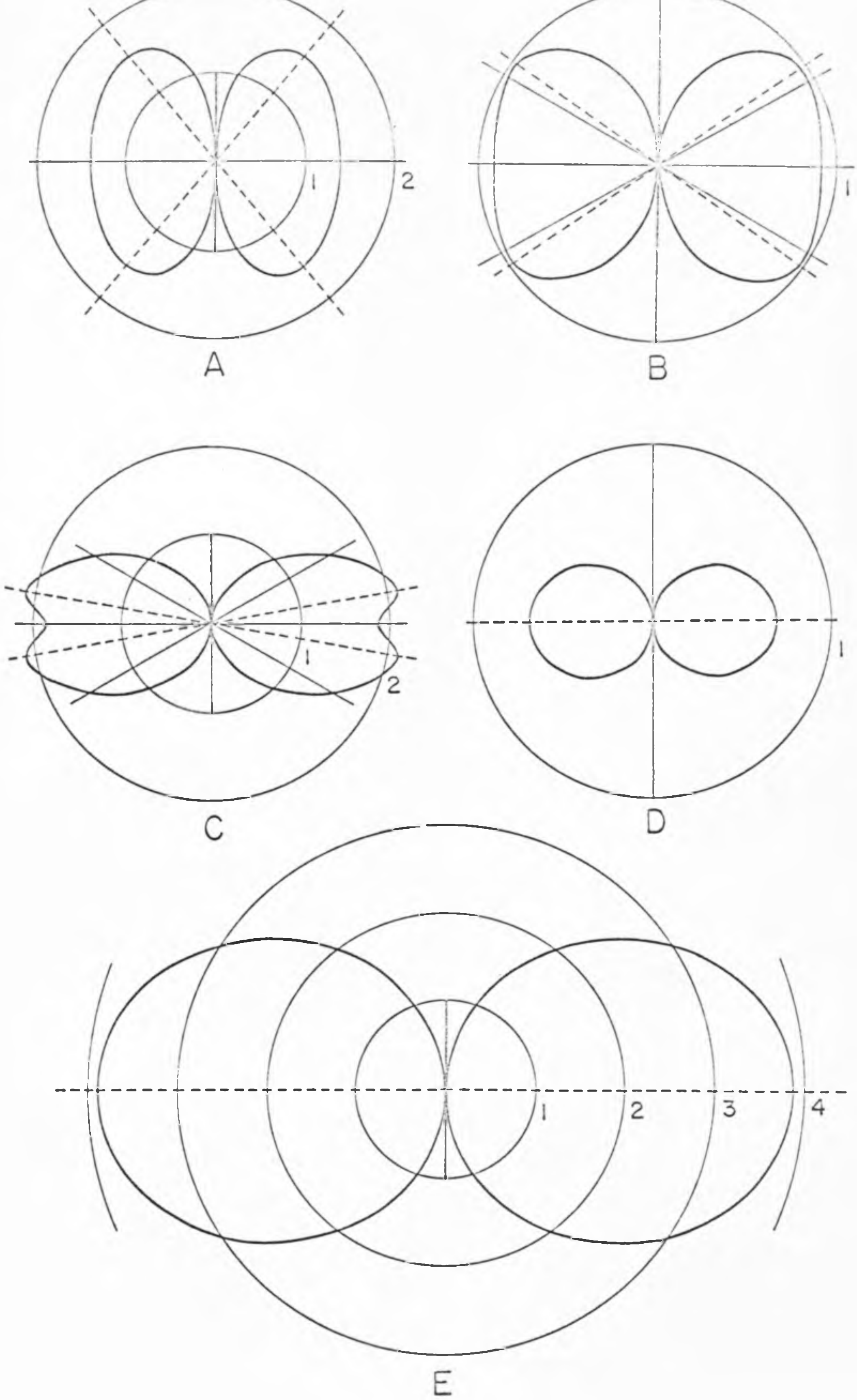


Fig. 4