

# Exact bosonization in arbitrary dimensions

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## Abstract

We extend the previous results of exact bosonization, mapping from fermionic operators to Pauli matrices, in 2d and 3d to arbitrary dimensions. This bosonization map gives a duality between any fermionic system in arbitrary  $n$  spatial dimensions and a new class of  $(n-1)$ -form  $\mathbb{Z}_2$  gauge theories in  $n$  dimensions with a modified Gauss's law. This map preserves locality and has an explicit dependence on the second Stiefel-Whitney class and a choice of spin structure on the manifold. A new formula for Stiefel-Whitney homology classes on lattices is derived. In the Euclidean path integral, this exact bosonization map is equivalent to introducing a topological “Steenrod square” term to the spacetime action.

## 1 Introduction and Summary

It is well known that every fermionic lattice system in 1d is dual to a lattice system of spins with a  $\mathbb{Z}_2$  global symmetry (and vice versa). The duality is kinematic (independent of a particular Hamiltonian) and arises from the Jordan-Wigner transformation. Recently it has been shown that any fermionic lattice system in 2d is dual to a  $\mathbb{Z}_2$  gauge theory with an unusual Gauss law [1]. The fermion can be identified with the flux excitation of the gauge theory, which is described by the Chern-Simon term  $i\pi \int A \cup \delta A$  in the spacetime action. The 2d duality is also kinematic. This approach has been generalized to 3d [2]. Every fermionic lattice system in 3d is dual to a  $\mathbb{Z}_2$  2-form gauge theory with

an unusual Gauss law. Here “2-form gauge theory” means that the  $\mathbb{Z}_2$  variables live on faces (2-simplices), while the parameters of the gauge symmetry live on edges (1-simplices). 2-form gauge theories in 3+1D have local flux excitations, and the unusual Gauss law ensures that these excitations are fermions. This Gauss’s law can be described by the “Steenrod square” topological action  $i\pi \int B \cup B + B \cup_1 \delta B$ . The form of the modified Gauss law was first observed in [3]: a bosonization of fermionic systems in  $n$  dimensions must have a global  $(n - 1)$ -form  $\mathbb{Z}_2$  symmetry with a particular ’t Hooft anomaly. The standard Gauss law leads to a trivial ’t Hooft anomaly, so bosonization requires us to modify it in a particular way.

In this paper, we extend these results to arbitrary  $n$  dimensions. We show that every fermionic lattice system in  $n$ -dimension is dual to a  $\mathbb{Z}_2$   $(n-1)$ -form gauge theory with a modified Gauss law. Our bosonization map is kinematic and local in the same sense as the Jordan-Wigner map: every local observable on the fermionic side, including the Hamiltonian density, is mapped to a local gauge-invariant observable on the  $\mathbb{Z}_2$  gauge theory side. In the Euclidean picture, we show explicitly that our bosonization map is equivalent to introducing the topological term in the action:

$$S_{\text{top}} = i\pi \int_Y (A_{n-1} \cup_{n-3} A_{n-1} + A_{n-1} \cup_{n-2} \delta A_{n-1}), \quad (1)$$

where  $A_{n-1}$  is  $(n - 1)$ -form gauge fields, a  $(n - 1)$ -cochain  $A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2)$ , and  $Y$  is  $(n + 1)$ -dimensional spacetime manifold. When  $A_{n-1}$  is closed (a cocycle), this term reduces to the Steenrod square operator [4]. This “Steenrod square” term appears in the construction of fermionic symmetry-protected-topological phases [5].

There are already several proposals for an analog of the Jordan-Wigner map in arbitrary dimensions [6, 7, 8, 9]. Our construction is most similar to that of Bravyi and Kitaev [6]. One advantage of our construction is that we can clearly identify the kind of  $n$ -dimensional bosonic systems that are dual to fermionic systems: they possess global  $(n-1)$ -form  $\mathbb{Z}_2$  symmetry with a specific ’t Hooft anomaly, as proposed in [3]. It is also manifest in our approach that the bosonization map depends on a choice of spin structure.

## 2 Notations and Coventions

We will always work with an arbitrary triangulation of a simply-connected  $n$ -dimensional manifold  $M_n$  equipped with a branching structure (orientations on edges without forming a loop in any triangle). The vertices, edges, faces, and tetrahedra are denoted  $v, e, f, t$ , respectively. The general  $d$ -simplex is denoted as  $\Delta_d$ . We can label the vertices of  $\Delta_d$  as  $0, 1, 2, \dots, d$  such that the directions of edges are from the small number to the larger number. We denote this  $d$ -simplex as  $\Delta_d = \langle 01 \dots d \rangle$ . Its boundaries are  $(d-1)$ -simplices  $\langle 0, \dots, \hat{i}, \dots, d \rangle$  for  $i = 0, 1, \dots, d$ , where  $\hat{i}$  means  $i$  is omitted. A formal sum of  $d$ -simplices modulo 2 forms an element of the chain  $C_d(M_n, \mathbb{Z}_2)$ .

For every  $v$ , we define its dual 0-cochain  $\mathbf{v}$ , which takes value 1 on  $v$ , and 0 otherwise, i.e.  $\mathbf{v}(v') = \delta_{v,v'}$ . Similarly,  $\mathbf{e}$  is an 1-cochain  $\mathbf{e}(e') = \delta_{e,e'}$ , and so forth, i.e.  $\Delta_d$  being a  $d$ -cochain  $\Delta_d(\Delta'_d) = \delta_{\Delta_d, \Delta'_d}$ . All dual cochains will be denoted in bold. An evaluation of a cochain  $\mathbf{c}$  on a chain  $c'$  will sometimes be denoted  $\int_{c'} \mathbf{c}$ . When the integration range is not written,  $\mathbf{c}$  is assumed to be the top dimension and  $\int \mathbf{c} \equiv \int_{M_n} \mathbf{c}$ . A  $d$ -cochain  $\mathbf{c}_d \in C^d(M_n, \mathbb{Z}_2)$  can be identified as  $\mathbb{Z}_2$  fields living on each  $d$ -simplex  $\Delta_d$ , with the value  $\mathbf{c}_d(\Delta_d)$ . The cup product  $\cup$  of a  $p$ -cochain  $\alpha_p$  and a  $q$ -cochain  $\beta_q$  is a  $(p+q)$ -cochain defined as:

$$\begin{aligned} [\alpha_p \cup \beta_q](\langle 0, 1, \dots, p+q \rangle) &= \alpha_p(\langle 01 \dots p \rangle) \beta_q(\langle p, p+1, \dots, p+q \rangle) \\ &= \alpha_p(0 \sim p) \beta_q(p \sim p+q). \end{aligned} \tag{2}$$

The definition of the higher cup product [3, 4] is

$$\begin{aligned} [\alpha_p \cup_a \beta_q](0, 1, \dots, p+q-a) &= \\ \sum_{0 \leq i_0 < i_1 < \dots < i_a \leq p+q-a} &\alpha_p(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \dots) \times \beta_q(i_0 \sim i_1, i_2 \sim i_3, \dots) \end{aligned} \tag{3}$$

where  $i \sim j$  represents the integers from  $i$  to  $j$ , i.e.  $i, i+1, \dots, j$ , and  $\{i_0, i_1, \dots, i_a\}$  are chosen such that the arguments of  $\alpha_p$  and  $\beta_q$  contain  $p+1$  and  $q+1$  numbers separately.

The boundary operator is denoted by  $\partial$ . For an  $n$ -simplex  $\Delta_n$ ,  $\partial\Delta_n$  consists of all boundary  $(n-1)$ -simplices of  $\Delta_n$ . The coboundary operator is denoted by  $\delta$  (not to be confused with the Kronecker delta previously). On a 0-cochain  $\mathbf{v}$ ,  $\delta\mathbf{v}$  is an 1-cochain acting on edges, and is 1 if  $\partial e$  contains  $v$  and 0 otherwise:

$$\delta\mathbf{v}(e) = \mathbf{v}(\partial e) = \delta_{v, \partial e}.$$

It is similar for simplices in any dimension.

Finally,  $\Delta_n^1 \supset \Delta_{n'}^2$ , or  $\Delta_{n'}^2 \subset \Delta_n^1$  means that the simplex  $\Delta_n^1$  contains  $\Delta_{n'}^2$  as a subsimplex. A general rule of thumb is that the subset symbol always points to one higher dimension.

### 3 Review of Boson-Fermion Duality in (2+1)D and (3+1)D

We begin by reviewing the duality between fermions and  $\mathbb{Z}_2$  lattice gauge theory in both two spatial dimensions [1] and three spatial dimensions [2]. On each face  $f$  of the 2-manifold  $M_2$ , we place a single pair of fermionic creation-annihilation operators  $c_f, c_f^\dagger$ , or equivalently a pair of Majorana fermions  $\gamma_f, \gamma'_f$ . The algebra of Majorana fermions is

$$\{\gamma_f, \gamma_{f'}\} = \{\gamma'_f, \gamma'_{f'}\} = 2\delta_{f,f'}, \quad \{\gamma_f, \gamma'_{f'}\} = 0 \quad (4)$$

where  $\{A, B\} = AB - BA$  is the anti-commutator. The even fermionic algebra consists of local observables with a trivial fermionic parity (i.e.  $P_F = 1$ ). It is generated by the on-site fermion parity,

$$P_f = -i\gamma_f\gamma'_f,$$

and the fermionic hopping operator on every edge  $e$ ,

$$S_e = i\gamma_{L(e)}\gamma'_{R(e)},$$

where  $L(e)$  and  $R(e)$  are faces to the left and right of  $e$ , with respect to the branching structure of  $e$ . The commutation relation of hopping operators can be expressed as:

$$S_e S_{e'} = (-1)^{\int e \cup e' + e' \cup e} S_e S_{e'} \quad (5)$$

i.e. the sign from the commutation occurs only when the arrows on the two edges follow head to tail and are on the same triangle. In general, for any 1-cochains  $\lambda$  and  $\lambda'$ ,

$$S_{\lambda+\lambda'} \equiv (-1)^{\int \lambda \cup \lambda'} S_{\lambda'} S_\lambda. \quad (6)$$

In other words,  $S_\lambda$  is the product of  $S_e$  over  $\{e | \lambda(e) = 1\}$  and the sign in front is consistent with the commutation relations. If we consider the product of fermionic hopping operators on edges around a vertex

$v$ , the Majorana operators cancel out up to some  $P_f$  terms. The two generators  $P_f$  and  $S_e$  satisfy the following constraint at each vertex  $v$  [1]:

$$(-1)^{\int_{w_2} v} S_{\delta v} \prod_f P_f^{f \cup v} = 1 \quad (7)$$

where  $w_2 \in C_0(M_2, \mathbb{Z}_2)$  is the chain which is Poincaré dual to the second Stiefel-Whitney cohomology class  $w_2(M_2)$ . The explicit expression of  $w_2$  is given in Appendix A. The second Stiefel-Whitney class is the obstruction to a spin structure. The fermion can only be defined on a manifold which admits spin structure  $E \in C_1(M_2, \mathbb{Z}_2)$  such that  $\partial E = w_2$ .

The bosonic dual of this system involves  $\mathbb{Z}_2$ -valued spins on the edges of the triangulation. The bosonic algebra are generated by Pauli matrix on edges:

$$X_e = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y_e = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z_e = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (8)$$

For every face  $f$ , we define the flux operator:

$$W_f = \prod_{e \subset f} Z_e, \quad (9)$$

and for every edge  $e$  we define a unitary operator  $U_e$  which squares to 1:

$$U_e = X_e \left( \prod_{e'} Z_{e'}^{f_{e' \cup e}} \right) \quad (10)$$

where  $X_e, Z_e$  are Pauli matrices acting on a spin at the edge  $e$ . It has been shown in [1] that the sets  $\{U_e, W_f\}$  and  $\{S_e, P_f\}$  satisfy the same commutation relations. The boson-fermion duality map defined on the manifold  $M_2$  is:

$$\begin{aligned} W_f &= \prod_{e \subset f} Z_e \longleftrightarrow P_f = -i \gamma_f \gamma'_f, \\ U_e &= X_e \left( \prod_{e'} Z_{e'}^{f_{e' \cup e}} \right) \longleftrightarrow (-1)^{\int_E e} S_e = (-1)^{\int_E e} i \gamma_{L(e)} \gamma'_{R(e)}, \\ G_v &= \prod_{e \supset v} X_e \left( \prod_{e'} Z_{e'}^{f_{\delta v \cup e'}} \right) \longleftrightarrow (-1)^{\int_{w_2} v} S_{\delta v} \prod_f P_f^{f \cup v} = 1, \\ \prod_f W_f &= 1 \longleftrightarrow \prod_f P_f \end{aligned} \quad (11)$$

where  $w_2 \in C_0(M_2, \mathbb{Z}_2)$  is the chain representation of 2nd Stiefel-Whitney class and  $E \in C_1(M_2, \mathbb{Z}_2)$  denotes a choice of spin structure ( $\partial E = w_2$ ). For the consistency of this duality map, we need to impose the gauge constraints on bosonic side  $\prod_{e \supset v} X_e (\prod_{e'} Z_{e'}^{f \delta v \cup e'}) = 1$ . The gauge invariant subspace in the bosonic Hilbert space is dual to the fermionic system with total fermion parity  $\prod_f P_f = 1$ .

The 3d boson-fermion duality can be done in a similar way [2]. The only difference is that the fermions  $\gamma_t, \gamma'_t$  are at the center of tetrahedra  $t$  and Pauli operators  $X_f, Z_f$  live on faces  $f$ . In three spaitial dimensions, any fermionic system can be mapped to a 2-form  $\mathbb{Z}_2$  gauge theory on the 3-dimensional lattice. The duality dictionary becomes:

$$\begin{aligned}
W_t &= \prod_{f \subset t} Z_f \longleftrightarrow P_t = -i\gamma_t \gamma'_t, \\
U_f &= X_f (\prod_{f'} Z_{f'}^{f' \cup_1 f}) \longleftrightarrow (-1)^{f_E f} S_f = (-1)^{f_E f} i\gamma_{L(f)} \gamma'_{R(f)}, \\
G_e &= \prod_{f \supset e} X_f (\prod_{f'} Z_{f'}^{f \delta e \cup_1 f'}) \longleftrightarrow (-1)^{f_{w_2} e} S_{\delta e} \prod_t P_t^{f e \cup_1 t + t \cup_1 e} = 1, \\
\prod_t W_t &= 1 \longleftrightarrow \prod_t P_t
\end{aligned} \tag{12}$$

where  $w_2 \in C_1(M_3, \mathbb{Z}_2)$  is the chain representative of the second Stiefel-Whitney class, and  $E \in C_2(M_3, \mathbb{Z}_2)$  denotes a choice of spin structure ( $\partial E = w_2$ ).

## 4 Exact bosonization in $n$ dimensions

From the 2d and 3d formulae (11) and (12), it is very natural to conjecture the  $n$ -dimensional boson-fermion duality. The fermions live at the center  $n$ -simplices, i.e.  $\gamma_{\Delta_n}, \gamma'_{\Delta_n}$  for each  $\Delta_n$ . The Pauli matrices live on  $(n-1)$ -simplices, i.e.  $X_{\Delta_{n-1}}$  and  $Z_{\Delta_{n-1}}$  for each  $\Delta_{n-1}$ . The  $n$ -dimensional boson-fermion duality should be:

$$\begin{aligned}
W_{\Delta_n} &\equiv \prod_{\Delta_{n-1} \subset \Delta_n} Z_{\Delta_{n-1}} \longleftrightarrow P_t = -i\gamma_{\Delta_n} \gamma'_{\Delta_n}, \\
U_{\Delta_{n-1}} &\equiv X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \Delta_{n-1}' \cup_{n-2} \Delta_{n-1}} \right) \\
&\longleftrightarrow (-1)^{\int_E \Delta_{n-1}} S_{\Delta_{n-1}} = (-1)^{\int_E \Delta_{n-1}} i \gamma_{L(\Delta_{n-1})} \gamma'_{R(\Delta_{n-1})}, \\
G_{\Delta_{n-2}} &\equiv \prod_{\Delta_{n-1} \supset \Delta_{n-2}} X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta \Delta_{n-2} \cup_{n-2} \Delta_{n-1}'} \right) \\
&\longleftrightarrow (-1)^{\int_{w_2} \Delta_{n-2}} S_{\delta \Delta_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^{\int \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}} = 1, \\
\prod_{\Delta_n} W_{\Delta_n} = 1 &\longleftrightarrow \prod_{\Delta_n} P_{\Delta_n}
\end{aligned} \tag{13}$$

where  $w_2 \in C_{n-2}(M_n, \mathbb{Z}_2)$  is the chain representative of the second Stiefel-Whitney class,  $E \in C_{n-1}(M_n, \mathbb{Z}_2)$  denotes a choice of spin structure ( $\partial E = w_2$ ), and for general  $(n-1)$ -cochain  $\lambda_{n-1}$  and  $\lambda'_{n-1}$ , the product of  $S$  operators is defined as

$$S_{\lambda_{n-1} + \lambda'_{n-1}} \equiv (-1)^{\int \lambda_{n-1} \cup_{n-2} \lambda'_{n-1}} S_{\lambda'_{n-1}} S_{\lambda_{n-1}}. \tag{14}$$

This  $n$ -dimensional boson-fermion duality (13) is the main theorem of this paper, which will be proved by the end of this section.

## 4.1 Commutation relations

Consider an  $n$ -simplex  $\Delta_n = \langle 012 \dots n \rangle$ . Its boundary contains all  $(n-1)$ -simplex  $(\partial \Delta_n)^i = \langle 0 \dots \hat{i} \dots n \rangle$  where  $\hat{i}$  means the vertex  $i$  is omitted. We define the orientation of  $(\partial \Delta_n)^i$  as  $O((\partial \Delta_n)^i) = (-1)^i$ . For “+”-oriented  $\Delta_n$ , if  $O((\partial \Delta_n)^i) = 1$ , the boundary  $(\partial \Delta_n)^i$  is outward, and if  $O((\partial \Delta_n)^i) = -1$ , the boundary  $(\partial \Delta_n)^i$  is inward. For “-”-oriented  $\Delta_n$ , the inward and outward boundaries are opposite.  $S_{\Delta_{n-1}}$  and  $S_{\Delta'_{n-1}}$  anti-commute only when  $\Delta_{n-1}$  and  $\Delta'_{n-1}$  are both inward or both outward boundaries of some  $n$ -simplex, i.e.  $\Delta_{n-1}, \Delta'_{n-1} \in \partial \Delta_n$ .

We are going to prove that this is equivalent to

$$S_{\Delta_{n-1}} S_{\Delta'_{n-1}} = (-1)^{\int \Delta_{n-1} \cup_{n-2} \Delta'_{n-1} + \Delta'_{n-1} \cup_{n-2} \Delta_{n-1}} S_{\Delta'_{n-1}} S_{\Delta_{n-1}}. \quad (15)$$

From the definition of the higher cup product (3), we have

$$\begin{aligned} & [\Delta_{n-1} \cup_{n-2} \Delta'_{n-1}](0, 1, \dots, n) \\ = & \sum_{0 \leq i_0 < i_1 < \dots < i_{n-2} \leq n} \Delta_{n-1}(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \dots) \Delta'_{n-1}(i_0 \sim i_1, i_2 \sim i_3, \dots) \\ = & \sum_{0 \leq j_1 < j_2 \leq n | j_1, j_2 \in \text{even}} \Delta_{n-1}(\langle 0 \dots \hat{j}_2 \dots n \rangle) \Delta'_{n-1}(\langle 0 \dots \hat{j}_1 \dots n \rangle) \\ & + \sum_{0 \leq k_1 < k_2 \leq n | k_1, k_2 \in \text{odd}} \Delta_{n-1}(\langle 0 \dots \hat{k}_1 \dots n \rangle) \Delta'_{n-1}(\langle 0 \dots \hat{k}_2 \dots n \rangle). \end{aligned} \quad (16)$$

The  $\cup_{n-2}$  only contains the product of boundaries  $\Delta_{n-1}^i$  with the same orientation (inward or outward) and each pair of  $\Delta_{n-1}^i, \Delta'_{n-1}^i$  with the same orientation appears exactly once. Therefore, the  $\cup_{n-2}$  expression in (15) captures the commutation relations of fermionic hopping operators  $S_{\Delta_{n-1}}$ . It is easy to check that bosonic operators  $U_{\Delta_{n-1}}$  satisfy the same commutation relations:

$$U_{\Delta_{n-1}} U_{\Delta'_{n-1}} = (-1)^{\int \Delta_{n-1} \cup_{n-2} \Delta'_{n-1} + \Delta'_{n-1} \cup_{n-2} \Delta_{n-1}} U_{\Delta'_{n-1}} U_{\Delta_{n-1}}. \quad (17)$$

Therefore,  $\{S_{\Delta_{n-1}}, P_{\Delta_n}\}$  and  $\{U_{\Delta_{n-1}}, W_{\Delta_n}\}$  in (13) have the same commutation relations.

## 4.2 Gauge constraints

In this section, we will derive the constraints on fermionic operators:

$$(-1)^{\int w_2 \Delta_{n-2}} S_{\delta \Delta_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^{\int \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}} = 1. \quad (18)$$

This follows directly from the following two lemmas.

**Lemma 1.** *The Majorana operators in  $S_{\delta \Delta_{n-2}}$  cancel out with Majorana operators in  $\prod_{\Delta_n} P_{\Delta_n}^{\int \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}}$ .*

**Lemma 2.** *The sign difference of  $S_{\delta \Delta_{n-2}}$  and the product of  $P_{\Delta_n}$  is  $-(-1)^{\sum_{i=1}^d \int \Delta_{n-1}^{i-1} \cup_{n-2} \Delta_{n-1}^i}$  where we order  $(n-1)$ -simplices  $\{\Delta_{n-1} | \Delta_{n-1} \supset$*



$\Delta_{n-2}$  } counterclockwise as  $\Delta_{n-1}^1, \Delta_{n-1}^2, \dots, \Delta_{n-1}^{d-1}, \Delta_{n-1}^d \equiv \Delta_{n-1}^0$ , as shown in Fig. 2. This sign is a chain representative of 2nd Stiefel-Whitney class:

$$-(-1)^{\sum_{i=1}^d} \int \Delta_{n-1}^{i-1} \cup_{n-2} \Delta_{n-1}^i = (-1)^{\int_{w_2} \Delta_{n-2}}. \quad (19)$$

*Proof of Lemma 1.* Let us denote  $\Delta_n = \langle 01 \dots n \rangle$  formed by  $\Delta_{n-2}$  and two  $(n-1)$ -simplex  $\Delta_{n-1}^L$  and  $\Delta_{n-1}^R$ , shown in Fig. 1(a). We know that  $S_{\delta\Delta_{n-2}}$  contains  $\gamma_{\Delta_n} \gamma'_{\Delta_n}$  if and only if  $\Delta_{n-1}^L, \Delta_{n-1}^R$  are one inward boundary and one outward boundary of  $n$ -simplex  $\Delta_n$ , as indicated in Fig. 1(b) and (c).

For the product of  $P_{\Delta_n}$ , we simplify the integral as

$$\int \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2} = \int \delta \Delta_{n-2} \cup_{n-1} \Delta_n. \quad (20)$$

The contribution of  $\Delta_n = \langle 01 \dots n \rangle$  to (20) is

$$\begin{aligned} & [(\Delta_{n-1}^L + \Delta_{n-1}^R) \cup_{n-1} \Delta_n](\langle 01 \dots n \rangle) \\ &= \sum_{0 \leq i_0 < i_1 < \dots < i_{n-1} \leq n} (\Delta_{n-1}^L + \Delta_{n-1}^R)(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \dots) \Delta_n(i_0 \sim i_1, i_2 \sim i_3, \dots) \\ &= \sum_{0 \leq j \leq n | j \in \text{odd}} (\Delta_{n-1}^L + \Delta_{n-1}^R)(\langle 0 \dots \hat{j} \dots n \rangle) \Delta_n(\langle 01 \dots n \rangle) \\ &= \sum_{0 \leq j \leq n | j \in \text{odd}} (\Delta_{n-1}^L + \Delta_{n-1}^R)(\langle 0 \dots \hat{j} \dots n \rangle) \end{aligned} \quad (21)$$

which is 1 if and only  $\Delta_{n-1}^L, \Delta_{n-1}^R$  are one inward boundary and one outward boundary of the  $n$ -simplex  $\Delta_n$ . This shows that product of  $P_{\Delta_n}$  contain  $P_{\Delta_n} \sim \gamma_{\Delta_n} \gamma'_{\Delta_n}$  if and only if  $\Delta_{n-1}^L, \Delta_{n-1}^R$  are one inward boundary and one outward boundary of the  $n$ -simplex  $\Delta_n$ . This cancels out with  $S_{\delta\Delta_{n-2}}$  exactly.  $\square$

*Proof of Lemma 2.* We compare the sign between

$$S_{\delta\Delta_{n-2}} = (-1)^{\sum_{\Delta_{n-1} < \Delta'_{n-1} | \Delta_{n-1}, \Delta'_{n-1} \supset \Delta_{n-2}} \Delta_{n-1} \cup_{n-2} \Delta'_{n-1}} \prod_{\Delta_{n-1} \supset \Delta_{n-2}} S_{\Delta_{n-1}} \quad (22)$$

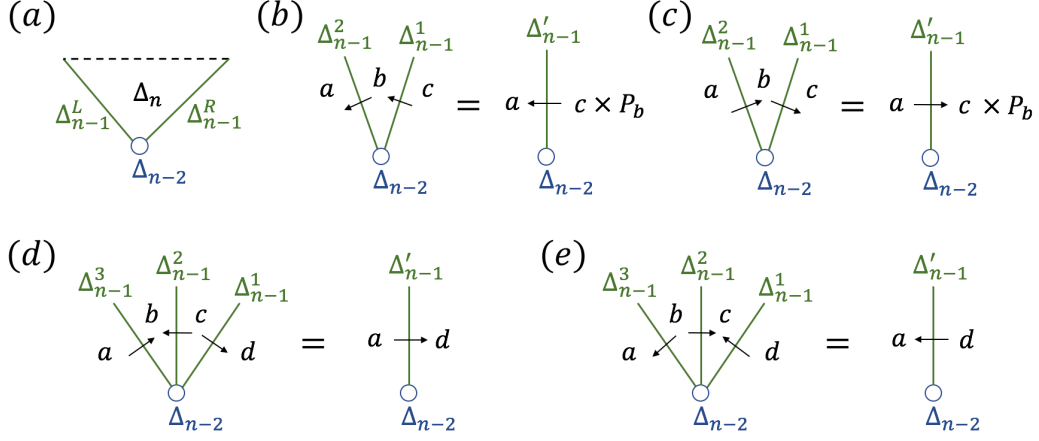


Figure 1: (a) The  $n$ -simplex  $\Delta_n$  is formed by  $\Delta_{n-2}$  and two  $(n-1)$ -simplex  $\Delta_{n-1}^L$  and  $\Delta_{n-1}^R$ . (b) The product of  $S_{\Delta_{n-2}}$  is  $(i\gamma_b\gamma'_a)(i\gamma_c\gamma'_b) = (i\gamma_c\gamma'_a)(-i\gamma_b\gamma'_b) = (i\gamma_c\gamma'_a)P_b$ . (c) The product of  $S_{\Delta_{n-2}}$  is  $(i\gamma_a\gamma'_b)(i\gamma_b\gamma'_c) = (i\gamma_a\gamma'_c)(-i\gamma_b\gamma'_b) = (i\gamma_a\gamma'_c)P_b$ . (d) The product of  $S_{\Delta_{n-2}}$  is  $(i\gamma_a\gamma'_b)(i\gamma_c\gamma'_b)(i\gamma_c\gamma'_d) = i\gamma_a\gamma'_d$ . (e) The product of  $S_{\Delta_{n-2}}$  is  $(i\gamma_b\gamma'_a)(i\gamma_b\gamma'_c)(i\gamma_d\gamma'_c) = i\gamma_d\gamma'_a$ .

and

$$\prod_{\Delta_n} P_{\Delta_n}^f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2} \quad (23)$$

where we have used the definition of  $S_{\lambda_{n-1}}$  in (14). As shown in Fig. 2,

$$S_{\Delta_{n-1}^d} \cdots S_{\Delta_{n-1}^2} S_{\Delta_{n-1}^1} = S_{\Delta_{n-1}^d} S_{\Delta_{n-1}^1} \prod_{\Delta_n \neq a,b} P_{\Delta_n}^f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}$$

We can check that

$$S_{\Delta_{n-1}^d} S_{\Delta_{n-1}^1} = -(-1)^f \Delta_n^1 \cup_{n-2} \Delta_n^d + \Delta_n^d \cup_{n-2} \Delta_n^1 \prod_{\Delta_n = a,b} P_{\Delta_n}^f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2},$$

and therefore

$$S_{\Delta_{n-1}^d} \cdots S_{\Delta_{n-1}^2} S_{\Delta_{n-1}^1} = -(-1)^f \Delta_n^1 \cup_{n-2} \Delta_n^d + \Delta_n^d \cup_{n-2} \Delta_n^1 \prod_{\Delta_n} P_{\Delta_n}^f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}.$$

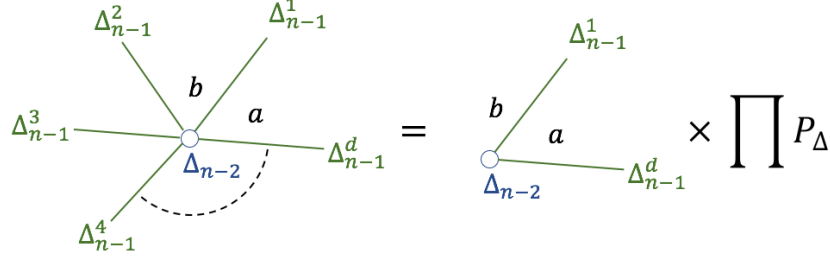


Figure 2: By the operations defined in Fig. 1, we can simplify the product  $S_{\Delta_{n-1}^d} \cdots S_{\Delta_{n-1}^2} S_{\Delta_{n-1}^1} = S_{\Delta_{n-1}^d} S_{\Delta_{n-1}^1} \prod_{\Delta_n \neq a,b} P_{\Delta_n}^f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}$ .

Together with (22), we have

$$\begin{aligned}
& S_{\delta \Delta_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2} \\
&= (-1)^f \int \Delta_{n-1}^1 \cup_{n-2} \Delta_{n-1}^d + \sum_{i=2}^d \int \Delta_{n-1}^{i-1} \cup_{n-2} \Delta_{n-1}^i \left( -(-1)^f \int \Delta_n^1 \cup_{n-2} \Delta_n^d + \Delta_n^d \cup_{n-2} \Delta_n^1 \right) \\
&= -(-1)^{\sum_{i=1}^d} \int \Delta_{n-1}^{i-1} \cup_{n-2} \Delta_{n-1}^i.
\end{aligned} \tag{24}$$

From the definition of  $\cup_{n-2}$  product (16),

$$\begin{aligned}
& \sum_{i=1}^d \int \Delta_{n-1}^{i-1} \cup_{n-2} \Delta_{n-1}^i \\
&= \sum_{i=1}^d \sum_{\Delta_n} \Delta_{n-1}^{i-1} \cup_{n-2} \Delta_{n-1}^i (\Delta_n) \\
&= \sum_{\substack{\text{"-"}\text{-oriented} \\ \Delta_n = \langle 0 \dots n \rangle}} \sum_{j_1 < j_2 | j_1, j_2 \in \text{even}} \Delta_{n-2}(\langle 0 \dots \hat{j}_1 \dots \hat{j}_2 \dots n \rangle) \\
&+ \sum_{\substack{\text{"+"}\text{-oriented} \\ \Delta_n = \langle 0 \dots n \rangle}} \sum_{k_1 < k_2 | k_1, k_2 \in \text{odd}} \Delta_{n-2}(\langle 0 \dots \hat{k}_1 \dots \hat{k}_2 \dots n \rangle).
\end{aligned} \tag{25}$$

The distinct orientations of “-”-oriented  $\Delta_n$  and “+”-oriented  $\Delta_n$  in the summation come from the fact that  $j_1, j_2$  and  $k_1, k_2$  in (16) have opposite orders. Eq. (25) is related to  $w_2$  by the following lemma 3, which is proved in appendix A. Therefore, we derive

$$-(-1)^{\sum_{i=1}^d} \int \Delta_{n-1}^{i-1} \cup_{n-2} \Delta_{n-1}^i = (-1)^{f_{w_2}} \Delta_{n-2}. \tag{26}$$

**Lemma 3.** *In  $n$ -dimension manifold with triangulation and branching structure, the homology class of  $w_2$  can be represented by a  $(n-2)$ -chain  $w_2 \in C_{n-2}(M_n, \mathbb{Z}_2)$ :*

$$w_2 = \sum_{\Delta_{n-2}} c(\Delta_{n-2}) \Delta_{n-2} \quad (27)$$

where

$$\begin{aligned} c(\Delta_{n-2}) = & 1 + \sum_{\substack{\text{"-"}\text{-oriented } \Delta_n = \langle 0 \dots n \rangle \\ j_1 < j_2 | j_1, j_2 \in \text{even}}} \sum_{\Delta_{n-2}} \Delta_{n-2}(\langle 0 \dots \hat{j}_1 \dots \hat{j}_2 \dots n \rangle) \\ & + \sum_{\substack{\text{"+"}\text{-oriented } \Delta_n = \langle 0 \dots n \rangle \\ k_1 < k_2 | k_1, k_2 \in \text{odd}}} \sum_{\Delta_{n-2}} \Delta_{n-2}(\langle 0 \dots \hat{k}_1 \dots \hat{k}_2 \dots n \rangle). \end{aligned} \quad (28)$$

□

We can modify the sign of  $S_{\Delta_{n-1}}$  as

$$S_{\Delta_{n-1}}^E \equiv (-1)^{f_E} \Delta_{n-1} S_{\Delta_{n-1}} \quad (29)$$

where  $E \in C_{n-1}(M_n, \mathbb{Z}_2)$  is a choice of spin structure satisfying  $\partial E = w_2$ . In these modified operators, the constraint on the fermionic operator becomes

$$S_{\delta \Delta_{n-2}}^E \prod_{\Delta_n} P_{\Delta_n}^{f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}} = 1, \quad (30)$$

which is mapped to

$$\begin{aligned} G_{\Delta_{n-2}} &= U_{\delta \Delta_{n-2}} \prod_{\Delta_n} W_{\Delta_n}^{f \Delta_{n-2} \cup_{n-2} \Delta_n + \Delta_n \cup_{n-2} \Delta_{n-2}} \\ &= \prod_{\Delta_{n-1} \supset \Delta_{n-2}} X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{f \delta \Delta_{n-2} \cup_{n-2} \Delta_{n-1}'} \right). \end{aligned} \quad (31)$$

We need to impose this gauge constraint  $G_{\Delta_{n-2}} = 1$  on bosonic operators for every  $(n-2)$ -simplex  $\Delta_{n-2}$ .

We also need to impose the even total parity constraint for fermions

$$\prod_{\Delta_n} P_{\Delta_n} = 1 \quad (32)$$

since it is mapped to the bosonic operator  $\prod_{\Delta_n} W_{\Delta_n} = 1$ . After imposing the gauge constraints, the  $n$ -dimensional boson-fermion duality (13) is completed.

## 5 Modified Gauss's law and Euclidean action

### 5.1 Gauss's law as boundary anomaly

First, we consider the standard  $\mathbb{Z}_2$  lattice gauge theory on the  $n$ -dimensional manifold  $M_n$ :

$$H^0 = -A \sum_{\Delta_{n-1}} X_{\Delta_{n-1}} - B \sum_{\Delta_n} W_{\Delta_n} \quad (33)$$

with the gauge constraint (Gauss's law)

$$G_{\Delta_{n-2}}^0 = \prod_{\Delta_{n-1} \supset \Delta_{n-2}} X_{\Delta_{n-1}} = 1. \quad (34)$$

It is well-known that its Euclidean theory is  $(n+1)$ -dimensional Ising model (with some choice of  $A$  and  $B$ ) [10]:

$$S_{\text{Ising}}(A_{n-1}) = -J \sum_{\Delta_n \subset Y} |\delta A_{n-1}(\Delta_n)| \quad (35)$$

where  $A \in C^{n-1}(Y, \mathbb{Z}_2)$  is a  $(n-1)$ -cochain on the spacetime manifold  $Y$ . In this case,  $S_{\text{Ising}}$  is invariant under the gauge transformation  $A_{n-1} \rightarrow A_{n-1} + \delta \Lambda_{n-2}$  for arbitrary  $(n-2)$ -cochain  $\Lambda_{n-2} \in C^{n-2}(Y, \mathbb{Z}_2)$ . Therefore,  $S_{\text{Ising}}$  has no boundary anomaly under the standard Gauss's law.

Now, we propose a new class of  $\mathbb{Z}_2$  lattice gauge theory:

$$H = -A \sum_{\Delta_{n-1}} U_{\Delta_{n-1}} - B \sum_{\Delta_n} W_{\Delta_n} \quad (36)$$

with the modified Gauss's law (gauge constraints) at  $(n-2)$ -simplices

$$G_{\Delta_{n-2}} = \prod_{\Delta_{n-1} \supset \Delta_{n-2}} X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta \Delta_{n-2} \cup_{n-2} \Delta_{n-1}'} \right) = 1. \quad (37)$$

This model describes a free fermion system, since it is dual to

$$\begin{aligned} H_f &= -A \sum_{\Delta_{n-1}} (-1)^{\int_E \Delta_{n-1}} i \gamma_{L(\Delta_{n-1})} \gamma'_{R(\Delta_{n-1})} - B \sum_{\Delta_n} (-i \gamma_{\Delta_n} \gamma'_{\Delta_n}) \\ &= -A \sum_{\Delta_{n-1}} S_{\Delta_{n-1}}^E - B \sum_{\Delta_n} P_{\Delta_n}. \end{aligned} \quad (38)$$

The modified Gauss's law (37) on a  $(n-2)$ -simplex  $\Delta_{n-2}$ , or equivalently on the dual  $(n-2)$ -cochain  $\mathbf{\Delta}_{n-2}$ , can be generalized to an arbitrary  $(n-2)$ -cochain  $\lambda_{n-2} = \sum_i \mathbf{\Delta}_{n-2}^i$ , the Gauss's law is

$$\begin{aligned}
G_{\lambda_{n-2}} &= \prod_i G_{\Delta_{n-2}^i} \\
&= \left( \prod_{\mathbf{\Delta}_{n-1} \in \delta \lambda_{n-2}} X_{\Delta_{n-1}} \right) \left( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta \lambda_{n-2} \cup_{n-2} \mathbf{\Delta}_{n-1}'} \right) (-1)^{\int \lambda_{n-2} \cup_{n-4} \lambda_{n-2} + \lambda_{n-2} \cup_{n-3} \delta \lambda_{n-2}} \\
&= 1
\end{aligned} \tag{39}$$

where the sign comes from anti-commutation of  $X$  and  $Z$  on the same simplex. The derivation uses the following property of higher cup products:

$$A \cup_a B + B \cup_a A = A \cup_{a+1} \delta B + \delta A \cup_{a+1} B + \delta(A \cup_{a+1} B). \tag{40}$$

Consider now the following  $(n-1)$ -form gauge theory defined on a general triangulated  $(n+1)$ -dimensional manifold  $Y$ :

$$S(A_{n-1}) = - \sum_{\Delta_n \subset Y} |\delta A_{n-1}(\Delta_n)| + i\pi \int_Y (A_{n-1} \cup_{n-3} A_{n-1} + A_{n-1} \cup_{n-2} \delta A_{n-1}). \tag{41}$$

where  $A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2)$ , and the gauge symmetry acts by  $A_{n-1} \rightarrow A_{n-1} + \delta \Lambda_{n-2}$  for  $\Lambda_{n-2} \in C^{n-2}(Y, \mathbb{Z}_2)$ . The second term is the generalized Steenrod square term defined in [5]. The action is gauge-invariant up to a boundary term:

$$\begin{aligned}
&S(A_{n-1} + \delta \Lambda_{n-2}) - S(A_{n-1}) \\
&= i\pi \int_{\partial Y} (\Lambda_{n-2} \cup_{n-4} \Lambda_{n-2} + \Lambda_{n-2} \cup_{n-3} \delta \Lambda_{n-2} + \delta \Lambda_{n-2} \cup_{n-2} A_{n-1}) \\
&= i\pi \int_{\partial Y} (\Lambda \cup_{n-4} \Lambda + \Lambda \cup_{n-3} \delta \Lambda + \delta \Lambda \cup_{n-2} A)
\end{aligned} \tag{42}$$

where we have omitted the subscript of  $A_{n-1}$  and  $\Lambda_{n-2}$  for simplicity. This boundary term determines the Gauss law for the wave-function  $\Psi(A)$  on the spatial slice  $M = \partial Y$ :

$$\Psi(A + \delta \Lambda) = (-1)^{\omega(\Lambda, A)} \Psi(A) \tag{43}$$

where  $\omega(\Lambda, A) = \int_M (\Lambda \cup_{n-4} \Lambda + \Lambda \cup_{n-3} \delta \Lambda + \delta \Lambda \cup_{n-2} A)$ . The Gauss law is the same as the gauge constraint (39) if we identify  $Z_{\Delta_{n-1}}$

as  $(-1)^{A_{n-1}(\Delta_{n-1})}$  and  $X_{\Delta_{n-1}}$  acts as the transformation  $A_{n-1} \rightarrow A_{n-1} + \mathbf{\Delta}_{n-1}$ . The modified Gauss's law (37) represents the boundary anomaly of topological action (41) as we claimed.

In the following subsection, we derive the Euclidean action of the modified  $\mathbb{Z}_2$  lattice gauge theory (36) explicitly, which is analogous to (41).

## 5.2 Euclidean path integral of lattice gauge theories

Start with the Hamiltonian of modified  $\mathbb{Z}_2$  lattice gauge theory:

$$\begin{aligned} H &= -A \sum_{\Delta_{n-1}} U_{\Delta_{n-1}} - B \sum_{\Delta_n} W_{\Delta_n} \\ &= -A \sum_{\Delta_{n-1}} X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \mathbf{\Delta}_{n-1}' \cup_{n-2} \mathbf{\Delta}_{n-1}} \right) - B \sum_{\Delta_n} \prod_{\Delta_{n-1} \subset \Delta_n} Z_{\Delta_{n-1}} \end{aligned} \quad (44)$$

with gauge constraints

$$G_{\Delta_{n-2}} = \prod_{\Delta_{n-1} \supset \Delta_{n-2}} X_{\Delta_{n-1}} \left( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta \mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_{n-1}'} \right) = 1. \quad (45)$$

The partition function is:

$$\mathcal{Z} = \text{Tr} e^{-\beta H} = \text{Tr} T^M \quad (46)$$

where we use Trotter-Suzuki decomposition in imaginary time direction and  $T$  is the transfer matrix defined as

$$T = \left( \prod_{\Delta_{n-2}} \delta_{G_{\Delta_{n-2},1}} \right) e^{-\delta\tau H}. \quad (47)$$

The first factor arises from the gauge constraints on the Hilbert space. The spacetime manifold consists of many time slices labelled by layers  $\{i\}$ . In the  $i$ th layer, we insert a complete basis (in Pauli matrix  $Z_{\Delta_{n-1}}$ ):  $b_{n-1}^i \in C^{n-1}(M_n, \mathbb{Z}_2)$  ( $\mathbb{Z}_2$  fields on each  $\Delta_{n-1}$  of the spatial manifold  $M_n$  such that  $Z_{\Delta_{n-1}} = (-1)^{b_{n-1}^i(\Delta_{n-1})}$ ). The transfer matrix  $T$  between the  $i$ th layer and the  $(i+1)$ th layer contains gauge constraints on every spatial  $(n-2)$ -simplex  $\Delta_{n-2}$ :

$$\delta_{G_{\Delta_{n-2},1}} = \frac{1 + G_{\Delta_{n-2}}}{2} = \frac{1}{2} \sum_{a_{n-2}^{i+1/2}=0,1} (G_{\Delta_{n-2}})^{a_{n-2}^{i+1/2}} \quad (48)$$

where we introduce the Lagrangian multiplier  $a_{n-2}^{i+1/2} \in C^{n-2}(M_n, \mathbb{Z}_2)$  ( $\mathbb{Z}_2$  fields on each  $\Delta_{n-2}$  of the spatial manifold  $M_n$ ). Notice that  $a_{n-2}^{i+1/2}$  defined between two time slices lives on the spatial  $(n-2)$ -simplex  $\Delta_{n-2}$ , which can be interpreted as the spacetime  $(n-1)$ -simplex between the two layers. From the same calculation in [2], we have

$$\mathcal{Z} = \sum_{\{\{a_{n-2}^{i+1/2}\}, \{b_{n-1}^i\}\}} \exp([S_{\text{Ising}} + S_{\text{top}}](\{\{a_{n-2}^{i+1/2}\}, \{b_{n-1}^i\}\})) \quad (49)$$

where

$$\begin{aligned} & S_{\text{Ising}}(\{\{a_{n-2}^{i+1/2}\}, \{a_{n-1}^i\}\}) \\ = & \sum_i \left( -J_s \sum_{\Delta_n} |\delta b_{n-1}^i(\Delta_n)| - J_\tau \sum_{\Delta_{n-1}} | [b_{n-1}^i + b_{n-1}^{i+1} + \delta a_{n-2}^{i+1/2}] (\Delta_{n-1}) | \right) \end{aligned} \quad (50)$$

and

$$\begin{aligned} & S_{\text{top}}(\{\{a_{n-2}^{i+1/2}\}, \{b_{n-1}^i\}\}) \\ = & i\pi \sum_i \int_{M_n} a_{n-2}^{i+1/2} \cup_{n-4} a_{n-2}^{i+1/2} + a_{n-2}^{i+1/2} \cup_{n-3} \delta a_{n-2}^{i+1/2} \\ & + \delta a_{n-2}^{i+1/2} \cup_{n-2} b_{n-1}^{i+1} + b_{n-1}^i \cup_{n-2} (b_{n-1}^i + b_{n-1}^{i+1} + \delta a_{n-2}^{i+1/2}). \end{aligned} \quad (51)$$

Here  $J_s, J_\tau$  are constants depending on  $A, B, \delta\tau$  in the original Hamiltonian and we assume  $J_s = J_\tau = J$  for simplicity.  $|\cdots|$  gives the argument's parity 0 or 1. The gauge transformations act as

$$\begin{aligned} a_{n-1}^i & \rightarrow a_{n-1}^i + \delta\lambda^i, \\ a_{n-2}^{i+1/2} & \rightarrow a_{n-2}^{i+1/2} + \delta\mu^i + \lambda^i + \lambda^{i+1}, \end{aligned} \quad (52)$$

where  $\lambda^i$  are arbitrary  $(n-2)$ -cochains and  $\mu^i$  are arbitrary  $(n-3)$ -cochains.

If we interpret  $a_{n-2}^{i+1/2}$  as spacetime  $(n-1)$ -cochains, we can rewrite

$$\{\{a_{n-2}^{i+1/2}\}, \{b_{n-1}^i\}\} \rightarrow A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2), \quad (53)$$

which is  $\mathbb{Z}_2$  fields on  $(n-1)$ -simplices in spacetime manifold  $Y$ . It is natural to write  $S_{\text{Ising}}$  in (50) as

$$S_{\text{Ising}} = - \sum_{\Delta_n \subset Y} |\delta A_{n-1}(\Delta_n)|. \quad (54)$$



The spacetime manifold  $Y = M_n \times [-\infty, 0]$  (spatial and temporal parts) is not a triangulation, since we only triangularize the spatial manifold  $M_n$  under the discretized time. The (higher) cup products are not well-defined in  $Y$ . However, we can still write an expression

$$S_{\text{top}} = i\pi \int_{Y'} (A_{n-1} \cup_{n-3} A_{n-1} + A_{n-1} \cup_{n-2} \delta A_{n-1}). \quad (55)$$

in  $(n+1)$ -dimensional triangulation  $Y'$  such that  $Y'$  is a refinement of  $Y$ . We can check that (51) and (55) produce the same boundary term under gauge transformations.

## 6 Conclusions

We have extended the the exact bosonization (11) in 2d and (12) in 3d to arbitrary dimensions. The dictionary for  $n$ -dimensional boson-fermion duality is given in (13). This bosonization is a duality between any fermionic system in arbitrary  $n$  spatial dimensions and  $(n-1)$ -form  $\mathbb{Z}_2$  gauge theories in  $n$  dimensions with gauge constraints (modified Gauss's law). This map preserves locality: every local even fermionic observable is mapped to a local gauge-invariant bosonic operator. The formula has an explicit dependence on the second Stiefel-Whitney class of the manifold, and a choice of spin structure is needed. As a side product, we discover a new formula (19) for Stiefel-Whitney homology classes on lattices. In the Euclidean picture, we have shown that the Euclidean path integral of the  $n$ -dimensional  $\mathbb{Z}_2$  gauge theory with modified Gauss's law is the  $(n+1)$ -dimensional Ising model with an additional topological Steenrod square (41) term.

We thank Anton Kapustin and Po-Shen Hsin for many very helpful discussions.

## A A formula for Stiefel-Whitney homology classes

In this section, we prove Lemma 3, (28). First, let us recall the theorem proved in [11]. Let  $s$  be a  $p$ -simplex, say  $s = \langle v_0, v_1, \dots, v_p \rangle$ . Let  $k$  be another simplex which has  $s$  as a face; i.e.,  $s \subset k$  ( $s$  may be equal to

$k$ ). Let

$$\begin{aligned}
B_{-1} &= \text{set of vertices of } k \text{ less than } v_0, \\
B_0 &= \text{set of vertices of } k \text{ between } v_0 \text{ and } v_1, \\
B_m &= \text{set of vertices of } k \text{ between } v_m \text{ and } v_{m+1}, \\
B_p &= \text{set of vertices of } k \text{ greater than } v_p.
\end{aligned} \tag{56}$$

We say that  $s$  is regular in  $k$ , if  $\#(B_m) = 0$  for every odd  $m$ . Let  $\partial_p(k)$  denote the mod 2 chain which consists of all  $p$ -dimensional simplices  $s$  in  $k$  so that  $s$  is regular in  $k$ . For example,  $\langle 012 \rangle$  and  $\langle 023 \rangle$  are regular in  $\langle 0123 \rangle$  and therefore  $\partial_2(\langle 0123 \rangle) = \langle 012 \rangle + \langle 023 \rangle$ . The theorem is [11]:

**Theorem 1.**  $\sum_{k|\dim k \geq (n-2)} \partial_{n-2}(k)$  is a  $(n-2)$ -chain which represents  $w_2$ .

In particular, for any  $n'$ -simplex  $\Delta_{n'} = \langle 0 \dots n' \rangle$ , all  $(n' - 1)$ -simplices regular in  $\Delta_{n'}$  are

$$\langle 0 \dots \hat{i} \dots n \rangle \quad \forall i \in \text{odd} \tag{57}$$

and all  $(n' - 2)$ -simplices regular in  $\Delta_{n'}$  are

$$\langle 0 \dots \hat{i} \dots \hat{j} \dots n \rangle \quad \forall i \in \text{odd}, j \in \text{even}, i < j. \tag{58}$$

We now use this theorem to prove lemma 3.

*Proof of Lemma 3.* For every  $(n-2)$ -simplex  $\Delta_{n-2}$ , it is regular in itself. This contributes the 1 in the coefficient of  $c(\Delta_{n-2})$  in (28).

For every  $(n-1)$ -simplex  $\Delta_{n-1}$ , it is a boundary of two  $n$ -simplices  $\Delta_n^L$  and  $\Delta_n^R$ , with  $\Delta_{n-1}$  being an outward boundary of  $\Delta_n^L$  and an inward boundary of  $\Delta_n^R$ . We define that  $\Delta_{n-1}$  belongs to  $\Delta_n^R$  and the summation of  $\dim k = n-1$ ,  $n$  in theorem 1 can be written as:

$$\begin{aligned}
& \sum_{\Delta_{n-1}} \partial_{n-2}(\Delta_{n-1}) + \sum_{\Delta_n} \partial_{n-2}(\Delta_n) \\
&= \sum_{\Delta_n} \left[ \partial_{n-2}(\Delta_n) + \sum_{\Delta_{n-1} \in \Delta_n | \Delta_{n-1} \text{ is inward}} \partial_{n-2}(\Delta_{n-1}) \right].
\end{aligned} \tag{59}$$

If  $\Delta_n = \langle 0 \dots n \rangle$  is “+”-oriented, the terms in the summation is

$$\begin{aligned}
& \partial_{n-2}(\langle 0 \dots n \rangle) + \sum_{0 \leq i \leq n | i \in \text{odd}} \partial_{n-2}(\langle 0 \dots \hat{i} \dots n \rangle) \\
= & \sum_{i,j | i < j, i \in \text{odd}, j \in \text{even}} \langle 0 \dots \hat{i} \dots \hat{j} \dots n \rangle \\
& + \sum_{0 \leq i \leq n | i \in \text{odd}} \left( \sum_{j < i | j \in \text{odd}} \langle 0 \dots \hat{j} \dots \hat{i} \dots n \rangle + \sum_{j > i | j \in \text{even}} \langle 0 \dots \hat{i} \dots \hat{j} \dots n \rangle \right) \\
= & \sum_{i,j | i < j, i \in \text{odd}, j \in \text{odd}} \langle 0 \dots \hat{i} \dots \hat{j} \dots n \rangle
\end{aligned} \tag{60}$$

where we have used the definition of regular simplex defined above. Similarly, we can derive that if  $\Delta_n = \langle 0 \dots n \rangle$  is “-”-oriented, the term is

$$\sum_{i,j | i < j, i \in \text{even}, j \in \text{even}} \langle 0 \dots \hat{i} \dots \hat{j} \dots n \rangle. \tag{61}$$

Combining (60) and (61) with the 1 from  $\dim k = n - 2$  in theorem 1, we have

$$w_2 = \sum_{\Delta_{n-2}} c(\Delta_{n-2}) \Delta_{n-2} \tag{62}$$

where

$$\begin{aligned}
c(\Delta_{n-2}) = & 1 + \sum_{\text{“-”-oriented } \Delta_n = \langle 0 \dots n \rangle} \sum_{j_1 < j_2 | j_1, j_2 \in \text{even}} \Delta_{n-2}(\langle 0 \dots \hat{j}_1 \dots \hat{j}_2 \dots n \rangle) \\
& + \sum_{\text{“+”-oriented } \Delta_n = \langle 0 \dots n \rangle} \sum_{k_1 < k_2 | k_1, k_2 \in \text{odd}} \Delta_{n-2}(\langle 0 \dots \hat{k}_1 \dots \hat{k}_2 \dots n \rangle).
\end{aligned} \tag{63}$$

□

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