Floquet higher-order topological insulators and superconductors with space-time symmetries

Yang Peng *

Department of Physics and Astronomy, Northridge, California State University, Northridge, California 91330, USA; Institute of Quantum Information and Matter and Department of Physics, California Institute of Technology, Pasadena, California 91125, USA; and Walter Burke Institute for Theoretical Physics, California Institute of Technology, Pasadena, California 91125, USA

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Floquet higher-order topological insulators (HOTIs) and superconductors (SCs) with an order-2 space-time symmetry or antisymmetry are classified. This is achieved by considering unitary loops, whose nontrivial topology leads to the anomalous Floquet topological phases, subject to a space-time symmetry/antisymmetry. By mapping these unitary loops to static Hamiltonians with an order-2 crystalline symmetry/antisymmetry, one is able to obtain the K groups for the unitary loops and thus complete the classification of Floquet HOTIs and SCs. Interestingly, we find that for every order-2 nontrivial space-time symmetry/antisymmetry involving a half-period time translation, there exists a unique order-2 static crystalline symmetry/antisymmetry such that the two symmetries/antisymmetries give rise to the same topological classification. Moreover, by exploiting the frequency-domain formulation of the Floquet problem, a general recipe that constructs model Hamiltonians for Floquet HOTIs and SCs is provided, which can be used to understand the classification of Floquet HOTIs and SCs from an intuitive and complementary perspective.

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I. INTRODUCTION

The interplay between symmetry and topology leads to various of topological phases. For a translationally invariant noninteracting gapped system, the topological phase is characterized by the band structure topology, as well as the symmetries the system respects. Along with these observations, a classification was obtained for topological insulators (TIs) and superconductors (SCs) [1–3] in the ten Altland-Zirnbauer (AZ) symmetry classes [4–8], which is determined by the presence or absence of three types of nonspatial symmetries, i.e., the time-reversal, particle-hole, and chiral symmetries.

One nice feature of these tenfold-way phases is the bulk-boundary correspondence, namely, a topologically nontrivial bulk band structure implies the existence of codimension-1 gapless boundary modes on the surface, irrespective of the surface orientation. (The codimension is defined as the difference between the bulk dimension and the dimension of the boundary where the gapless mode propagates.)

When considering more symmetries beyond the nonspatial ones, the topological classification is enriched. Topological crystalline insulators [9–13] are such systems protected by crystalline symmetries. They are able to host codimension-1 gapless boundary modes only when the boundary is invariant under the crystalline symmetry operation. For example, topological crystalline insulators protected by reflection symmetry [10] can support gapless modes only on the reflection invariant boundary. On the other hand, inversion symmetric topological crystalline insulators do not necessarily give rise to codimension-1 gapless boundary modes [14,15], because no boundary is invariant under inversion.

Remarkably, it was recently demonstrated that a crystal with a crystalline symmetry compatible bulk topology may manifest itself through protected boundary modes of codimension greater than 1 [16–26]. Such insulating and superconducting phases are called higher-order topological insulators (HOTIs) and superconductors. In particular, an nth-order TI and SC can support codimension-n boundary modes. (The strong TIs and SCs in the tenfold-way phases with protected boundary modes at codimension 1 can be called first-order TIs and SCs according to this definition.) A higher-order bulk-boundary correspondence between the bulk topology and gapless boundary modes at different codimensions was derived in Ref. [26] based on K theory.

Beyond equilibrium or static conditions, it is known that topological phases also exist, and one of the famous examples is the Floquet topological insulator, which is proposed to be brought from a static band insulator by applying a periodic drive, such as a circularly polarized radiation or an alternating Zeeman field [27–31]. A complete classification of the Floquet topological insulators (as well as superconductors) in the AZ symmetry classes has been obtained in Refs. [32,33], which can be regarded as a generalization of the classification for static tenfold-way TIs and SCs.

In a periodically driven, or Floquet, system, the nontriviality can arise from the nontrivial topology of the unitary time-evolution operator $U(t)$ (with period T), which can be decomposed into two parts as $U(t) = e^{-iH_F t}P(t)$. Here the

*yang.peng@csun.edu

first part describes the stroboscopic evolution at the time of multiples of T in terms of a static effective Hamiltonian H_F and the second part is known as the micromotion operator $P(t) = P(t + T)$ describing the evolution within a given time period [34]. (We will make this decomposition more explicit later.) Thus, the nontrivial topology can separately arise from H_F as in a static topological phase or from the nontrivial winding of $P(t)$ over one period. Whereas the Floquet topological phase in the former situation is very similar to a static topological phase as it has a static limit, the latter is purely dynamical and cannot exist if the time-periodic term in the Hamiltonian vanishes. Therefore, systems belonging to the latter case are more interesting and are known as the anomalous Floquet topological phases.

In a Floquet system, energy is not conserved because of the explicit time dependence of the Hamiltonian. However, one can define quasienergies as eigenvalues of $H_F = \frac{i}{T} \ln U(T)$, which are only defined modulo the periodic driving frequency $\omega = 2\pi/T$. This can be intuitively understood due to the existence of energy quanta ω that can be absorbed and emitted. Similar to static topological phases, the quasienergy spectrum can be different with different boundary conditions. In particular, inside a bulk quasienergy gap (when a periodic boundary condition is applied), there may exist topologically protected boundary modes.

In Floquet topological phases protected by nonspatial symmetries (tenfold-way phases), the bulk-boundary correspondence is also expected to hold [32], namely, the number of boundary modes inside a particular bulk gap can be fully obtained from the topology of the evolution operator $U(t)$, when a periodic boundary condition is applied. Interestingly, when there exists a symmetry relating states at quasienergies ϵ and $-\epsilon$, then the topological protected boundary modes will appear inside the quasienergy gap at 0 and $\omega/2$, since these are quasienergies that are invariant under the above symmetry operation.

In particular, a bulk micromotion operator with nontrivial topology is able to produce gapless Floquet codimension-1 boundary modes at quasienergy $\omega/2$ (which will be made clear later). The natural question to ask is that how we can create Floquet higher-order topological phases, with protected gapless modes at arbitrary codimensions. In particular, we want to have the topological nontriviality arise from the micromotion operator; otherwise we just need to have H_F as a Hamiltonian for a static higher-order topological phase.

Similar to the static situation, when only nonspatial symmetries are involved, the tenfold-way Floquet topological phases are all first-order phases which can only support codimension-1 boundary modes. Higher-order phases are yet possible when symmetries relating different spatial points of the system are involved. These symmetries can be static crystalline symmetries as well as space-time symmetries which relate systems at different times.

Recently, the authors in Refs. [35–40] constructed Floquet second-order TIs and SCs. In particular, the authors of Ref. [38] were able to construct Floquet corner modes by exploiting the time-glide symmetry [41], which combines a half-period time translation and a spatial reflection, as illustrated in the left part of Fig. 1.

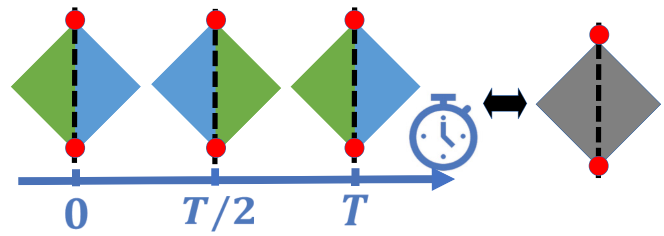

FIG. 1. Floquet second-order TIs and SCs protected by time-glide symmetry/antisymmetry can be mapped to static second-order TIs and SCs protected by reflection symmetry/antisymmetry. The dashed line indicates the reflection (time-glide) plane.

It turns out that the roles played by such space-time symmetries in Floquet systems cannot be trivially replaced by spatial symmetries. As pointed out in Ref. [38], in protecting anomalous Floquet boundary modes, the space-time symmetries generally have different commutation relations with the nonspatial symmetries, compared to what the corresponding spatial symmetries do.

Since the use of space-time symmetries creates possibilities in engineering Floquet topological phases, especially the Floquet HOTIs and SCs, it is important to have a thorough topological classification, as well as a general recipe of model construction for such systems. In this work we completely classify Floquet HOTIs and SCs with an order-2 space-time symmetry/antisymmetry realized by an operator $\hat{\mathcal{O}}$, which can be either unitary or antiunitary. By order-2 we mean that the symmetry/antisymmetry operator trivially acts twice on the time-periodic Hamiltonian $H(t)$, namely,

$$[\hat{\mathcal{O}}^2, H(t)] = 0, \quad \hat{\mathcal{O}} = \hat{\mathcal{U}}, \hat{\mathcal{A}}, \tag{1}$$

where $\hat{\mathcal{O}}$ can be either unitary $\hat{\mathcal{U}}$ or antiunitary $\hat{\mathcal{A}}$.

We further provide a general recipe of constructing tight-binding Hamiltonians for such Floquet HOTIs and SCs in different symmetry classes. Note that the order-2 static crystalline symmetries/antisymmetries considered in Ref. [11] will be a subset of the symmetries/antisymmetries considered in this work.

Our classification and model construction of Floquet HOTIs and SCs involve two complementary approaches. The first approach is based on the classification of gapped unitaries [32,41], namely, the time-evolution operator $U(t)$ at time $t \in [0, T)$, with $U(T)$ gapped in its eigenvalues' phases. It turns out that the gapped unitaries can be (up to homotopy equivalence) decomposed as a unitary loop (which is actually the micromotion operator) and a unitary evolution under the static Floquet Hamiltonian H_F. Thus, a general gapped unitary is classified by separately considering the unitary loop and the static Hamiltonian H_F, where the latter is well known for systems in AZ classes as well as systems with additional crystalline symmetries. The classification of unitary loops, on the other hand, is less trivial since it is responsible for the existence of anomalous Floquet phases [42], especially when we are considering space-time symmetries.

We focus on the classification of Floquet unitary loops in this work. In particular, a Hermitian map between unitary loops and Hermitian matrices is introduced, which is inspired by the dimensional reduction map used in the classification of

TABLE I. Nontrivial space-time symmetry/antisymmetry with subscript $T/2$ vs static spatial symmetry/antisymmetry with subscript 0, sharing the same K groups at the same dimension. Here $\hat{\mathcal{U}}$, $\hat{\mathcal{A}}$, $\bar{\mathcal{U}}$, and $\bar{\mathcal{A}}$ denote unitary symmetry, antiunitary symmetry, unitary antisymmetry, and antiunitary antisymmetry, respectively. The commutation (anticommutation) relations with coexisting nonspatial symmetries are denoted by additional subscripts $+$ $(-)$, while the superscript indicates the square of the operator. In the case of classes BDI, DIII, CII, and CII, the first and second $\pm$ correspond to time-reversal and particle-hole symmetries, respectively.

AZ class	Space-time	Static
A	$\hat{\mathcal{U}}^{+}_{T/2}$	$\hat{\mathcal{U}}^{+}_{0}$
A	$\bar{\mathcal{A}}^{\pm}_{T/2}$	$\bar{\mathcal{A}}^{\mp}_{0}$
AIII	$\hat{\mathcal{U}}^{+}_{T/2,\pm}$	$\hat{\mathcal{U}}^{+}_{0,\mp}$
AIII	$\hat{\mathcal{A}}^{\pm}_{T/2,\pm}$	$\hat{\mathcal{A}}^{\pm}_{0,\mp}$
AI, AII	$\hat{\mathcal{U}}^{+}_{T/2,\pm}$	$\hat{\mathcal{U}}^{+}_{0,\pm}$
AI, AII	$\bar{\mathcal{U}}^{+}_{T/2,\pm}$	$\bar{\mathcal{U}}^{+}_{0,\mp}$
C, D	$\hat{\mathcal{U}}^{+}_{T/2,\pm}$	$\hat{\mathcal{U}}^{+}_{0,\mp}$
BDI, DIII, CII, CI	$\hat{\mathcal{U}}_{T/2,\pm\pm}$	$\hat{\mathcal{U}}_{0,\pm\mp}$

TIs and SCs with scattering matrices [43]. The key observation is that the symmetry constraints on the unitary loops have the same features as the ones on scattering matrices. This Hermitian map has advantages over the one used in earlier works [32,41], because it simply maps a unitary loop with a given order-2 space-time symmetry/antisymmetry to a static Hamiltonian of a topological crystalline insulator with an order-2 crystalline symmetry/antisymmetry. This enable us to exploit the full machinery of K theory, to define K groups, as well as the K subgroup series introduced in Ref. [26], for the unitary loops subject to space-time symmetries/antisymmetries.

Based on this approach, we obtain the first important result of this work, namely, for every order-2 nontrivial space-time (anti)unitary symmetry/antisymmetry, which involves a half-period time translation, there always exists a unique order-2 static spatial (anti)unitary symmetry/antisymmetry such that the two symmetries/antisymmetries correspond to the same K group and thus the same classification. This result is illustrated in Fig. 1 for the case of time-glide vs reflection symmetries. The explicit relations are summarized in Table I. Because of these relations, all results for the classification [17–25] as well as the higher-order bulk-boundary correspondence [26] of static HOTIs and SCs can be applied directly to the anomalous Floquet HOTIs and SCs.

In the second approach, by exploiting the frequency-domain formulation, we obtain the second important result of this work, which is a general recipe of constructing harmonically driven Floquet HOTIs and SCs from static HOTIs and SCs. This recipe realizes the K group isomorphism of systems with a space-time symmetry and systems with a static crystalline symmetry at the microscopic level of Hamiltonians, and therefore provides a very intuitive way of understanding the classification table obtained from the formal K theory.

The rest of the paper is organized as follows. We first introduce the symmetries, both nonspatial symmetries and the order-2 space-time symmetries, for the Floquet system in Sec. III. Then, in Sec. IV, we introduce a Hermitian map which enables us to map the classification of unitary loops to the classification of static Hamiltonians. In Sec. V, by using the Hermitian map, we explicitly map the classification of unitary loops in all possible symmetry classes supporting an order-2 symmetry to the classification of static Hamiltonians with an order-2 crystalline symmetry. In Sec. VI we derive the corresponding K groups for unitary loops in all possible symmetry classes and dimensions. In Sec. VII we introduce the K subgroup series for unitary loops, which enables us to completely classify Floquet HOTIs and SCs. In Sec. VIII the frequency-domain formulation is introduced, which provides a complementary perspective on the topological classification of Floquet HOTIs and SCs. In Sec. IX we introduce a general recipe of constructing harmonically driven Floquet HOTIs and SCs and provide examples in different situations. We summarize our work in Sec. X.

Note that it is possible to skip the K-theory classification parts in Secs. IV–VII and understand the main results in terms of the frequency-domain formulation.

II. FLOQUET BASICS

In a Floquet system, the Hamiltonian

$$H(t+T) = H(t) \tag{2}$$

is periodic in time with period $T = 2\pi/\omega$, where ω is the angular frequency. In a d-dimensional system with translational symmetry and periodic boundary condition, we have a well defined Bloch wave vector $\boldsymbol{k}$ in the d-dimensional Brillouin zone T^d (torus). The system can thus be characterized by a time-periodic Bloch Hamiltonian $H(\boldsymbol{k}, t)$.

In the presence of a d_{def}-dimensional topological defect, the wave vector $\boldsymbol{k}$ is no longer a good quantum number due to the broken translational symmetry. However, the topological properties of the defect can be obtained by considering a large $D = d - d_{\text{def}} - 1$ dimensional surface, on which the translational symmetry is asymptotically restored so that $\boldsymbol{k}$ can be defined, surrounding the defect. We will denote by $\boldsymbol{r}$ the real space coordinate on this surrounding surface, or a D sphere S^D, which will determine the topological classification. Thus, we have a time-periodic ($t \in S^1$) Bloch Hamiltonian $H(\boldsymbol{k}, \boldsymbol{r}, t)$ defined on $T^d \times S^{D+1}$. In the following, we will denote the dimension of such a system with a topological defect by a pair (d, D).

The topological properties for a given Hamiltonian $H(\boldsymbol{k}, \boldsymbol{r}, t)$ can be derived from its time-evolution operator

$$U(\boldsymbol{k}, \boldsymbol{r}, t_0 + t, t_0) = \hat{\mathcal{T}} \exp\left[-i \int_{t_0}^{t_0+t} dt' H(\boldsymbol{k}, \boldsymbol{r}, t')\right], \tag{3}$$

where $\hat{\mathcal{T}}$ denotes the time-ordering operator. The Floquet effective Hamiltonian $H_F(\boldsymbol{k}, \boldsymbol{r})$ is defined as

$$U(\boldsymbol{k}, \boldsymbol{r}, T + t_0, t_0) = \exp[-iH_F(\boldsymbol{k}, \boldsymbol{r})T]. \tag{4}$$

Note that different H_F defined at different t_0 are related by unitary transformations, and thus the eigenvalues of the Floquet effective Hamiltonian are uniquely defined independently of t_0, which is independent of t_0. We also introduce $\epsilon_n(\boldsymbol{k}, \boldsymbol{r}) \in [-\pi/T, \pi/T]$ to define the nth eigenvalue of $H_F(\boldsymbol{k}, \boldsymbol{r})$ and call it the nth quasienergy band. Although H_F captures the stroboscopic evolution of the system, it does not produce a complete topological classification of the Floquet phases. It is known that one can have the so-called anomalous Floquet phases even when H_F is a trivial Hamiltonian.

To fully classify the Floquet phases, we need information about the evolution operator at each t within the period. In order to have a well defined phase, we will only consider gapped unitary evolution operators, whose quasienergy bands are gapped at a particular quasienergy ϵ_{gap}. Thus, given a set of symmetries the system respects, one needs to classify these gapped unitaries defined from each gapped quasienergies ϵ_{gap}. The most commonly considered gapped energies in a system with particle-hole or chiral symmetry are 0 and $\omega/2$, since such energies respect the symmetry. Note that the $\epsilon_{\text{gap}} = \omega/2$ case is more interesting since it corresponds to anomalous Floquet phases [42], which has no static analog. When neither of the two above-mentioned symmetries exists, the gapped energy can take any value, but one can always deform the Hamiltonian such that the gapped energy appears at $\omega/2$ without changing the topological classification. Hence, in the following we will set $\epsilon_{\text{gap}} = \omega/2$.

It is evident that the initial time t_0 in the evolution operator does not affect the classification, since it corresponds to different ways of defining the origin of time. Thus, from now on we will set $t_0 = 0$ and define

$$U(\boldsymbol{k}, \boldsymbol{r}, t) = U(\boldsymbol{k}, \boldsymbol{r}, t, 0). \tag{5}$$

A less obvious fact is that one can define the symmetrized time-evolution operator [32] centered around time τ as

$$U_\tau(\boldsymbol{k}, \boldsymbol{r}, t) = \mathscr{T} \exp\left[-i \int_{\tau-t/2}^{\tau+t/2} dt' H(\boldsymbol{k}, \boldsymbol{r}, t')\right], \tag{6}$$

which will also give rise to the same topological classification. This statement is proved in Appendix A. In fact, $U_\tau(\boldsymbol{k}, \boldsymbol{r}, T)$ leads to the same quasienergy band structure independently of the choice of τ. This is because (the explicit $\boldsymbol{k}, \boldsymbol{r}$ dependence is omitted)

$$U_\tau(T) = W U_0(T) W^\dagger, \tag{7}$$

with the unitary matrix $W = U(\tau + T/2)U^\dagger(T/2)$. Thus, the $U_\tau(T)$ at different τ are related by unitary transformation, and we will in the following use $U_\tau(\boldsymbol{k}, \boldsymbol{r}, t)$ to classify Floquet topological phases.

For classification purposes, we need to establish the notion of homotopy equivalence between unitary evolutions. Let us consider evolution operators gapped at a given quasienergy. Following the definition in Ref. [32], we say two evolution operators U_1 and U_2 are homotopic, defined as $U_1 \approx U_2$, if and only if there exists a continuous unitary-matrix-valued function $f(s)$, with $s \in [0, 1]$, such that

$$f(0) = U_1, \quad f(1) = U_2, \tag{8}$$

where $f(s)$ is a gapped evolution operator for all intermediate s. It is worth mentioning that when dealing with symmetrized evolution operators instead of ordinary evolution operators, the definition of homotopy equivalence is similar except one needs to impose that the interpolation function $f(s)$ for all s is also a gapped symmetrized evolution operator. When comparing evolution operators with a different number of bands, the equivalence relation of stable homotopy can be introduced. Such an equivalence relation is denoted by $U_1 \sim U_2$ if there exist two trivial unitaries $U^0_{n_1}$ and $U^0_{n_2}$, with n_1 and n_2 bands, respectively, such that

$$U_1 \oplus U^0_{n_1} \approx U_2 \oplus U^0_{n_2}, \tag{9}$$

where $\oplus$ denotes the direct sum of matrices.

We will now define how to make compositions between two symmetrized evolution operators. Using the notation in Ref. [32], we write the evolution due to $U_{\tau,1}$ followed by $U_{\tau,2}$ as $U_{\tau,1} * U_{\tau,2}$, which is given by the symmetrized evolution under Hamiltonian $H(t)$ given by

$$H(t) = \begin{cases} H_2\left(2t + \frac{T}{2} - \tau\right), & \tau - \frac{T}{2} \leqslant t \leqslant \tau - \frac{T}{4} \\ H_1(2t - \tau), & \tau - \frac{T}{4} \leqslant t \leqslant \tau + \frac{T}{4} \\ H_2\left(2t - \frac{T}{2} - \tau\right), & \tau + \frac{T}{4} \leqslant t \leqslant \tau + \frac{T}{2}, \end{cases} \tag{10}$$

where $H_1(t)$ and $H_2(t)$ are the corresponding Hamiltonians used for the evolution operators $U_{\tau,1}$ and $U_{\tau,2}$, respectively.

As proved in Ref. [32], with such definitions of homotopy and compositions of evolution operators, one can obtain the following two important theorems. First, every gapped symmetrized evolution operator U_τ is homotopic to a composition of a unitary loop L_τ, followed by a constant Hamiltonian evolution C_τ, unique up to homotopy. Here the unitary loop is a special time-evolution operator such that it becomes an identity operator after a full period evolution. Second, $L_{\tau,1} * C_{\tau,1} \approx L_{\tau,2} * C_{\tau,2}$ if and only if $L_{\tau,1} \approx L_{\tau,2}$ and $C_{\tau,1} \approx C_{\tau,2}$, $L_{\tau,1}$ and $L_{\tau,2}$ are unitary loops, and $C_{\tau,1}$ and $C_{\tau,2}$ are constant Hamiltonian evolutions. For completeness, we include the proof of the two theorems in Appendix B.

Because of these two theorems, classifying generic time-evolution operators reduces to classifying separately the unitary loops and the constant Hamiltonian evolutions. Since the latter is exactly the same as classifying static Hamiltonians, we will in this work focus only on the classification of unitary loops. In the following, all the time-evolution operators are unitary loops, which additionally satisfy $U_\tau(\boldsymbol{k}, \boldsymbol{r}, t) = U_\tau(\boldsymbol{k}, \boldsymbol{r}, t + T)$.

III. SYMMETRIES IN FLOQUET SYSTEMS

In this section we will summarize the transformation properties of the time-evolution operator under various symmetry operators.

A. Nonspatial symmetries

Let us first look at the nonspatial symmetries and consider systems belong to one of the ten AZ classes (see Table II), determined by the presence or absence of time-reversal, particle-hole, and chiral symmetries, which are defined by the operators $\hat{\mathcal{T}} = \mathcal{U}_T\hat{\mathcal{K}}$, $\hat{\mathcal{C}} = \mathcal{U}_C\hat{\mathcal{K}}$, and $\hat{\mathcal{S}} = \mathcal{U}_S = \hat{\mathcal{T}}\hat{\mathcal{C}}$, respectively, such that

$$\begin{aligned} \hat{\mathcal{T}} H(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{T}}^{-1} &= H(-\boldsymbol{k}, \boldsymbol{r}, -t), \\ \hat{\mathcal{C}} H(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{C}}^{-1} &= -H(-\boldsymbol{k}, \boldsymbol{r}, t), \\ \hat{\mathcal{S}} H(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{S}}^{-1} &= -H(\boldsymbol{k}, \boldsymbol{r}, -t), \end{aligned} \tag{11}$$

TABLE II. The AZ symmetry classes and their classifying spaces. The top two rows ($s = 0, 1 \bmod 2$) are complex AZ classes, while the remaining eight rows ($s = 0, \ldots, 7 \bmod 8$) are real AZ classes. The third to fifth columns denote the absence (0) or presence ($\epsilon_T, \epsilon_C = \pm 1$ or $\eta_S = 1$) of time-reversal ($\hat{\mathcal{T}}$), particle-hole ($\hat{\mathcal{C}}$), and chiral symmetries ($\hat{\mathcal{S}}$). In addition, $\mathcal{C}_s$ ($\mathcal{R}_s$) denotes the classifying space of the s complex (real) AZ class.

s	AZ class	$\hat{\mathcal{T}}$	$\hat{\mathcal{C}}$	$\hat{\mathcal{S}}$	$\mathcal{C}_s$ or $\mathcal{R}_s$	$\pi_0(\mathcal{C}_s)$ or $\pi_0(\mathcal{R}_s)$
0	A	0	0	0	$\mathcal{C}_0$	$\mathbb{Z}$
1	AIII	0	0	1	$\mathcal{C}_1$	0
0	AI	+1	0	0	$\mathcal{R}_0$	$\mathbb{Z}$
1	BDI	+1	+1	1	$\mathcal{R}_1$	$\mathbb{Z}_2$
2	D	0	+1	0	$\mathcal{R}_2$	$\mathbb{Z}_2$
3	DIII	−1	+1	1	$\mathcal{R}_3$	0
4	AII	−1	0	0	$\mathcal{R}_4$	$2\mathbb{Z}$
5	CII	−1	−1	1	$\mathcal{R}_5$	0
6	C	0	−1	0	$\mathcal{R}_6$	0
7	CI	+1	−1	1	$\mathcal{R}_7$	0

where $\hat{\mathcal{T}} = \mathcal{U}_T\hat{\mathcal{K}}$ and $\hat{\mathcal{C}} = \mathcal{U}_C\hat{\mathcal{K}}$ are antiunitary operators with unitary matrices $\mathcal{U}_T$ and $\mathcal{U}_C$ and complex conjugation operator $\hat{\mathcal{K}}$. In the above equations $\boldsymbol{r}$ is invariant, because of the nonspatial nature of the symmetries. For a Floquet system, the action of symmetry operations $\hat{\mathcal{T}}$, $\hat{\mathcal{C}}$, and $\hat{\mathcal{S}}$ on the symmetrized unitary loops $U_\tau(\boldsymbol{k}, \boldsymbol{r}, t)$ can be summarized as

$$\hat{\mathcal{T}}U_\tau(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{T}}^{-1} = U^\dagger_{-\tau}(-\boldsymbol{k}, \boldsymbol{r}, t), \tag{12}$$

$$\hat{\mathcal{C}}U_\tau(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{C}}^{-1} = U_\tau(-\boldsymbol{k}, \boldsymbol{r}, t), \tag{13}$$

$$\hat{\mathcal{S}}U_\tau(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{S}}^{-1} = U^\dagger_{-\tau}(\boldsymbol{k}, \boldsymbol{r}, t), \tag{14}$$

which follow directly from Eqs. (11). For later convenience, we further introduce notation $\epsilon_T = \mathcal{U}_T\mathcal{U}_T^* = \hat{\mathcal{T}}^2 = \pm 1$, $\epsilon_C = \mathcal{U}_C\mathcal{U}_C^* = \hat{\mathcal{C}}^2 = \pm 1$, and $\epsilon_S = \mathcal{U}_S^2 = \hat{\mathcal{S}}^2 = 1$, respectively.

B. Order-2 space-time symmetry

In addition to the nonspatial symmetries, let us assume that the system supports an order-2 space-time symmetry realized by $\hat{\mathcal{O}}$, as defined in Eq. (1). Moreover, we assume that $\hat{\mathcal{O}}$ commutes or anticommutes with the operators for the nonspatial symmetries of the system. Under the order-2 space-time symmetry operation $\hat{\mathcal{O}}$, the momentum $\boldsymbol{k}$ transforms as [11]

$$\boldsymbol{k} \to \begin{cases} \hat{\mathcal{O}}\boldsymbol{k} = (-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp) & \text{for } \hat{\mathcal{O}} = \hat{\mathcal{U}} \\ -\hat{\mathcal{O}}\boldsymbol{k} = (\boldsymbol{k}_\perp, -\boldsymbol{k}_\parallel) & \text{for } \hat{\mathcal{O}} = \hat{\mathcal{A}}, \end{cases} \tag{15}$$

where the second equality assumes we are in the diagonal basis of $\hat{\mathcal{O}}$, $\boldsymbol{k}_\parallel = (k_1, k_2, \ldots, k_{d_\parallel})$, and $\boldsymbol{k}_\perp = (k_{d_\parallel+1}, k_{d_\parallel+2}, \ldots, k_d)$.

While the nonspatial symmetries leave the spatial coordinate $\boldsymbol{r}$ invariant, the order-2 space-time symmetry transforms $\boldsymbol{r}$ nontrivially. To determine the transformation law, we follow Ref. [11] and consider a D-dimensional sphere S^D surrounding the topological defect, whose coordinates in Euclidean space are determined by

$$\boldsymbol{n}^2 = a^2, \quad \boldsymbol{n} = (n_1, n_2, \ldots, n_{D+1}), \tag{16}$$

with radius $a > 0$. Since $\hat{\mathcal{O}}$ maps S^D to itself, we have

$$\boldsymbol{n} \to (-\boldsymbol{n}_\parallel, \boldsymbol{n}_\perp), \tag{17}$$

with $n_\parallel = (n_1, n_2, \ldots, n_{D_\parallel})$ and $n_\perp = (n_{D_\parallel+1}, n_{D_\parallel+2}, \ldots, n_{D+1})$ in a diagonal basis of $\hat{\mathcal{O}}$. When $D_\parallel \leqslant D$, we can introduce the coordinate $\boldsymbol{r} \in S^D$ by

$$r_i = \frac{n_i}{a - n_{D+1}} \quad (i = 1, \ldots, D), \tag{18}$$

which leads to

$$\boldsymbol{r} \to (-\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp). \tag{19}$$

Here $\boldsymbol{r}_\parallel = (r_1, r_2, \ldots, r_{D_\parallel})$ and $\boldsymbol{r}_\perp = (r_{D_{\parallel+1}}, r_{D_\parallel+2}, \ldots, r_D)$.

Thus, we need to introduce $(d, d_\parallel, D, D_\parallel)$ to characterize the dimension of the system according to the transformation properties of the coordinates, where d and D are defined the same as defined previously, while $d_\parallel$ and $D_\parallel$ denote the dimensions of the flipping momenta and the defect surrounding coordinates, respectively. For example, a unitary symmetry with $(d, d_\parallel, D, D_\parallel) = (2, 1, 1, 1)$ corresponds to the reflection in two dimensions with a point defect on the reflection line, while a unitary symmetry with $(d, d_\parallel, D, D_\parallel) = (3, 2, 2, 2)$ is a twofold rotation in three dimensions with a point defect on the rotation axis.

Next let us consider the action of the order-2 space-time symmetry on the time argument. For unitary symmetries, an action on t can generically have the form $t \to t + s$. Due to the periodicity in t and the order-2 nature of the symmetry, s can be either 0 or $T/2$.

For antiunitary symmetries, we have $t \to -t + s$. When the system does not support time-reversal or chiral symmetry, as in classes A, C, and D, the constraints due to time periodicity and the order-2 nature do not restrict the value s takes. Hence, s is an arbitrary real number in this situation.

However, when the system has at least one of the time-reversal and chiral symmetries, denoted by $\hat{\mathcal{P}}$, s will be restricted to take a few values as shown in the following. The composite operation $\hat{\mathcal{P}}\hat{\mathcal{O}}$ shifts the time as $t \to -s + t$. On the other hand, since $\hat{\mathcal{P}}\hat{\mathcal{O}}$ is another order-2 symmetry, s can be either 0 or $T/2$ (note that s is defined modulo T). To summarize, a Hamiltonian $H(\boldsymbol{k}, \boldsymbol{r}, t)$ existing in dimension $(d, d_\parallel, D, D_\parallel)$, under the action of $\hat{\mathcal{O}}$, transforms as

$$\hat{\mathcal{U}}_s H(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{U}}_s^{-1} = H(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, t + s), \tag{20}$$

$$\hat{\mathcal{A}}_s H(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{A}}_s^{-1} = H(\boldsymbol{k}_\parallel, -\boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, -t + s), \tag{21}$$

in the diagonal basis of $\hat{\mathcal{O}}$, for unitary and antiunitary symmetries.

Let us suppose that $\hat{\mathcal{O}}^2 = \epsilon_O = \pm 1$ and $\hat{\mathcal{O}}$ commutes or anticommutes with coexisting nonspatial symmetries according to

$$\hat{\mathcal{O}}\hat{\mathcal{T}} = \eta_T\hat{\mathcal{T}}\hat{\mathcal{O}}, \quad \hat{\mathcal{O}}\hat{\mathcal{C}} = \eta_C\hat{\mathcal{C}}\hat{\mathcal{O}}, \quad \hat{\mathcal{O}}\hat{\mathcal{S}} = \eta_S\hat{\mathcal{S}}\hat{\mathcal{O}}, \tag{22}$$

where $\eta_T = \pm 1$, $\eta_C = \pm 1$, and $\eta_S = \pm 1$. Note that when $\hat{\mathcal{O}} = \hat{\mathcal{U}}$, we can always set $\epsilon_O = 1$ with the help of multiplying $\hat{\mathcal{O}}$ by the imaginary unit i, but this changes the (anti)commutation relation with $\hat{\mathcal{T}}$ and/or $\hat{\mathcal{C}}$ at the same time.

One can also consider an order-2 antisymmetry $\bar{\mathcal{O}}$ defined by

$$\bar{\mathcal{U}}_s H(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{U}}_s^{-1} = -H(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,-t+s),$$
$$\bar{\mathcal{A}}_s H(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{A}}_s^{-1} = -H(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t+s), \qquad (23)$$

where $\bar{\mathcal{O}}$ can be either unitary $\bar{\mathcal{U}}$ or antiunitary $\bar{\mathcal{A}}$. Such an antisymmetry can be realized by combining any of order-2 symmetries with chiral or particle-hole symmetry. Similar to $\hat{\mathcal{O}}$, we define $\bar{\mathcal{O}}^2 = \epsilon_{\bar{O}}$, $\bar{\mathcal{O}}\hat{\mathcal{T}} = \bar{\eta}_T\hat{\mathcal{T}}\bar{\mathcal{O}}$, $\bar{\mathcal{O}}\hat{\mathcal{C}} = \bar{\eta}_C\hat{\mathcal{C}}\bar{\mathcal{O}}$, and $\bar{\mathcal{O}}\hat{\mathcal{S}} = \bar{\eta}_S\hat{\mathcal{S}}\bar{\mathcal{O}}$. The values that the time shift s takes are similar to the ones in the case of symmetries. We have $s = 0, T/2$ for $\bar{\mathcal{U}}_s$. For $\bar{\mathcal{A}}_s$, s is arbitrary in classes A, C, and D, whereas $s = 0, T/2$ in the rest of classes.

The actions of symmetry/antisymmetry operators $\hat{\mathcal{O}}$ and $\bar{\mathcal{O}}$, either unitary or antiunitary, on the unitary loops can be summarized as

$$\hat{\mathcal{U}}_s U_\tau(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{U}}_s^{-1} = U_{\tau+s}(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t), \qquad (24)$$

$$\hat{\mathcal{A}}_s U_\tau(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{A}}_s^{-1} = U^\dagger_{s-\tau}(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t), \qquad (25)$$

$$\bar{\mathcal{U}}_s U_\tau(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{U}}_s^{-1} = U^\dagger_{s-\tau}(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t), \qquad (26)$$

$$\bar{\mathcal{A}}_s U_\tau(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{A}}_s^{-1} = U_{s+\tau}(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t). \qquad (27)$$

In the following, we will discuss each symmetry/antisymmetry operator separately and choose a particular value of τ for each case, since we know the classification would not depend on what value τ takes.

For $\hat{\mathcal{U}}_s$ and $\bar{\mathcal{A}}_s$, $s = 0, T/2$ and we set $\tau = T/2$. By using

$$U_{\tau+T/2}(\boldsymbol{k},\boldsymbol{r},t) = U^\dagger_\tau(\boldsymbol{k},\boldsymbol{r},T-t) \qquad (28)$$

and omitting the subscript τ from $U_\tau(\boldsymbol{k},\boldsymbol{r},t)$ from now on for simplicity, we get

$$\begin{aligned}
\hat{\mathcal{U}}_0 U(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{U}}_0^{-1} &= U(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t),\\
\hat{\mathcal{U}}_{T/2} U(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{U}}_{T/2}^{-1} &= U^\dagger(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,T-t),\\
\bar{\mathcal{A}}_0 U(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{A}}_0^{-1} &= U(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t),\\
\bar{\mathcal{A}}_{T/2} U(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{A}}_{T/2}^{-1} &= U^\dagger(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,T-t).
\end{aligned} \qquad (29)$$

When considering $\bar{\mathcal{U}}_s$ and $\hat{\mathcal{A}}_s$ in classes A, C, and D, we can choose $\tau = s/2$, which gives

$$\begin{aligned}
\hat{\mathcal{A}}_s U(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{A}}_s^{-1} &= U^\dagger(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t),\\
\bar{\mathcal{U}}_s U(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{U}}_s^{-1} &= U^\dagger(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t).
\end{aligned} \qquad (30)$$

This implies that the value of s here actually does not play a role in determining topological classification.

In the remaining classes we have $s = 0, T/2$, and we will choose $\tau = T/2$. This leads to

$$\begin{aligned}
\hat{\mathcal{A}}_0 U(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{A}}_0^{-1} &= U^\dagger(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t),\\
\hat{\mathcal{A}}_{T/2} U(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{A}}_{T/2}^{-1} &= U(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,T-t),\\
\bar{\mathcal{U}}_0 U(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{U}}_0^{-1} &= U^\dagger(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t),\\
\bar{\mathcal{U}}_{T/2} U(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{U}}_{T/2}^{-1} &= U(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,T-t).
\end{aligned} \qquad (31)$$

IV. HERMITIAN MAP

One observation that can be made from Eqs. (12)–(14) is that at fixed $\boldsymbol{r}$ and t, the transformation properties for the unitary loops $U(\boldsymbol{k},\boldsymbol{r},t)$ under the actions of $\hat{\mathcal{T}}$, $\hat{\mathcal{C}}$, and $\hat{\mathcal{S}}$ are exactly the same as the ones for unitary boundary reflection matrices introduced in, for example, Refs. [18,43]. In these works, an effective Hermitian matrix can be constructed from a given reflection matrix, which maps the classification of reflection matrices onto the classification of Hermitian matrices.

Here we can borrow the same Hermitian mapping defined as

$$\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t) = \mathcal{U}_S U(\boldsymbol{k},\boldsymbol{r},t) \qquad (32)$$

if $U(\boldsymbol{k},\boldsymbol{r},t)$ has a chiral symmetry and

$$\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t) = \begin{pmatrix} 0 & U(\boldsymbol{k},\boldsymbol{r},t)\\ U^\dagger(\boldsymbol{k},\boldsymbol{r},t) & 0\end{pmatrix} \qquad (33)$$

if $U(\boldsymbol{k},\boldsymbol{r},t)$ does not have a chiral symmetry. In the latter case, $\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)$ acquires a new chiral symmetry

$$\mathcal{U}'_S\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t) = -\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\mathcal{U}'_S, \qquad (34)$$

with $\mathcal{U}_S = \rho_z \otimes \mathbb{I}$, where we have introduced a set of Pauli matrices $\rho_{x,y,z}$ in the enlarged space.

Note that when the unitary loop $\mathcal{U}(\boldsymbol{k},\boldsymbol{r},t)$ does not have a chiral symmetry, our Hermitian map is the same as the one used in Refs. [32,41]. When the unitary loop does have a chiral symmetry, however, we choose a new map which maps the unitary loop into a Hermitian matrix without unitary symmetry.

The advantage of the Hermitian map defined here over the one in the previous works will become clear soon. Note that the Hermitian matrix $\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)$ can be regarded as a static spatially modulated Hamiltonian in $(d, D+1)$ dimension, because the time argument transforms like a spatial coordinate similar to $\boldsymbol{r}$. The classification of unitary loops in (d, D) dimension in a given symmetry class is then the same as the classification of static Hamiltonians in $(d, D+1)$ dimension in the symmetry class shifted upward by one $(s \to s-1)$ (modulo 2 or 8 for complex or real symmetry classes), where s is used to order the symmetry classes according to Table II. Thus, one can directly apply the classification scheme of the static Hamiltonians $H(\boldsymbol{k},\boldsymbol{r})$ using K theory, as was done in Ref. [11]. This is provided by a homotopy classification of maps from the base space $(\boldsymbol{k},\boldsymbol{r}) \in S^{d+D}$ to the classifying space of Hamiltonians $H(\boldsymbol{k},\boldsymbol{r})$ subject to the given symmetries, which we denoted by $\mathcal{C}_s$ or $\mathcal{R}_s$ as shown in the table.

Because of the Bott periodicity in the periodic table of static TIs and SCs [4–8], the classification is unchanged when simultaneously shifting the dimension $D \to D+1$ and the symmetry class upward by one $s \to s-1$ (mod 2 or 8 for complex or real symmetry classes). It turns out that the classification of unitary loops is the same as the classification of the static Hamiltonian in the same symmetry class and with the same dimension (d, D). In the following, we will explicitly derive the action of the Hermitian map on each symmetry classes.

A. Classes A and AIII

We first consider the two complex classes. Under the Hermitian map defined above, classifying unitary loops in (d, D) dimension in class A is the same as classifying Hermitian matrices in $(d, D+1)$ dimension in class AIII. On the other hand, classifying unitary loops in (d, D) dimension in class AIII is the same as classifying Hermitian matrices in $(d, D+1)$ dimension in class A.

B. Classes AI and AII

Now we turn to real symmetry classes. Since classes AI and AII have only time-reversal symmetry, we need to apply the Hermitian map defined in Eq. (33). By using Eq. (12) with $\tau = T/2$, or

$$\mathcal{U}_T U^T(\boldsymbol{k}, \boldsymbol{r}, t) = U(-\boldsymbol{k}, \boldsymbol{r}, t)\mathcal{U}_T, \tag{35}$$

we have effective time-reversal symmetry

$$\mathcal{U}'_T \mathcal{H}^*(\boldsymbol{k}, \boldsymbol{r}, t) = \mathcal{H}(-\boldsymbol{k}, \boldsymbol{r}, t)\mathcal{U}'_T, \tag{36}$$

with $\mathcal{U}'_T = \rho_x \otimes \mathcal{U}_T$, and effective particle-hole symmetry

$$\mathcal{U}'_C \mathcal{H}^*(\boldsymbol{k}, \boldsymbol{r}, t) = -\mathcal{H}^*(-\boldsymbol{k}, \boldsymbol{r}, t)\mathcal{U}'_C, \tag{37}$$

with $\mathcal{U}'_C = i\rho_y \otimes \mathcal{U}_T$.

Note that the effective time-reversal and particle-hole symmetries combine into the chiral symmetry as expected. The types of effective time-reversal and particle-hole symmetries of $\mathcal{H}(\boldsymbol{k}, t)$ are determined from

$$\mathcal{U}'_T \mathcal{U}'^*_T = \rho_0 \otimes (\mathcal{U}_T \mathcal{U}^*_T), \tag{38}$$

$$\mathcal{U}'_C \mathcal{U}'^*_C = -\rho_0 \otimes (\mathcal{U}_T \mathcal{U}^*_T), \tag{39}$$

where ρ_0 is the 2×2 identity matrix in the extended space. Under the Hermitian map, classifying unitary loops in (d, D) dimension in classes AI and AII is the same as classifying Hermitian matrices in $(d, D+1)$ dimension in classes CI and DIII.

C. Classes C and D

Let us consider classes C and D with only particle-hole symmetry. We need to apply the Hermitian map defined in Eq. (33). By using Eq. (13), one can define effective time-reversal symmetry with $\mathcal{U}'_T = \rho_0 \otimes \mathcal{U}_C$ and particle-hole symmetry with $\mathcal{U}'_C = \rho_z \otimes \mathcal{U}_C$ such that

$$\mathcal{U}'_T \mathcal{H}^*(\boldsymbol{k}, \boldsymbol{r}, t) = \mathcal{H}(-\boldsymbol{k}, \boldsymbol{r}, t)\mathcal{U}'_T, \tag{40}$$

$$\mathcal{U}'_C \mathcal{H}^*(\boldsymbol{k}, \boldsymbol{r}, t) = -\mathcal{H}(-\boldsymbol{k}, \boldsymbol{r}, t)\mathcal{U}'_C. \tag{41}$$

Note that $\mathcal{U}'_T$ and $\mathcal{U}'_C$ combine into the chiral symmetry as expected. The types of these effective symmetries are determined by

$$\mathcal{U}'_T \mathcal{U}'^*_T = \rho_0 \otimes (\mathcal{U}_C \mathcal{U}^*_C), \tag{42}$$

$$\mathcal{U}'_C \mathcal{U}'^*_C = \rho_0 \otimes (\mathcal{U}_C \mathcal{U}^*_C). \tag{43}$$

Under the Hermitian map, classifying unitary loops in (d, D) dimension in classes C and D is the same as classifying Hermitian matrices in $(d, D+1)$ dimension in classes CII and BDI.

D. Classes CI, CII, DIII, and BDI

Here we consider symmetry classes where time-reversal, particle-hole, and chiral symmetries are all present. In this case, $\mathcal{U}_S = \mathcal{U}_T \mathcal{U}^*_C$. By $\mathcal{U}^2_S = 1$ we have $\mathcal{U}^*_T \mathcal{U}_C \mathcal{U}^*_S = 1$. This can be used to show that

$$\mathcal{U}_S \mathcal{U}_C = \mathcal{U}_C \mathcal{U}^*_S (\mathcal{U}_C \mathcal{U}^*_C)(\mathcal{U}_T \mathcal{U}^*_T). \tag{44}$$

Notice that $\mathcal{U}_C \mathcal{U}^*_C = \pm1$ and $\mathcal{U}_T \mathcal{U}^*_T = \pm1$ are just numbers.

The effective Hamiltonian $\mathcal{H}(\boldsymbol{k}, t)$ defined in Eq. (32) has the property

$$\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\mathcal{U}_C = (\mathcal{U}_C \mathcal{U}^*_C)(\mathcal{U}_T \mathcal{U}^*_T)\mathcal{U}_C \mathcal{H}(-\boldsymbol{k}, \boldsymbol{r}, t)^*. \tag{45}$$

This gives rise to time-reversal or particle-hole symmetry depending on $(\mathcal{U}_C \mathcal{U}^*_C)(\mathcal{U}_T \mathcal{U}^*_T) = 1$ or -1, respectively. Therefore, under the Hermitian map, the unitary loops in (d, D) dimension in classes CI, CII, DIII, and BDI map to Hermitian matrices in $(d, D+1)$ dimension in classes C, AII, D, and AI, respectively.

V. CLASSIFICATION WITH ADDITIONAL ORDER-2 SPACE-TIME SYMMETRY

After introducing the Hermitian map which reduces the classification of unitary loops to the classification of static Hermitian matrices, or Hamiltonians, in the AZ symmetry classes, let us now assume that the system supports an additional order-2 space-time symmetry/antisymmetry, which is either unitary or antiunitary, as defined in Sec. III B. In the following, we will focus on each class separately.

A. Complex symmetry classes

The complex classes A and AIII are characterized by the absence of time-reversal and particle-hole symmetries.

1. Class A

Let us start with class A, with additional symmetry realized by $\hat{\mathcal{O}}$ or $\bar{\mathcal{O}}$, whose properties are summarized as (A, $\hat{\mathcal{O}}^{\epsilon_O}$) or (A, $\bar{\mathcal{O}}^{\epsilon_{\bar{O}}}$). For unitary symmetry realized by $\hat{\mathcal{U}}$ and $\bar{\mathcal{U}}$, one can set $\epsilon_U = 1$ or $\epsilon_{\bar{U}} = 1$.

(a) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_0$. We have

$$\hat{\mathcal{U}}'_0 \mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{U}}'^{-1}_0 = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, t), \tag{46}$$

where $\hat{\mathcal{U}}'_0 = \rho_0 \otimes \hat{\mathcal{U}}_0$ behaves as an order-2 crystalline symmetry if one regards $t \in S^1$ as an additional defect surrounding parameter. Recalling that $\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)$ has chiral symmetry realized by operator the $\hat{\mathcal{S}}' = \mathcal{U}'_S = \rho_z \otimes \mathbb{I}$, we have

$$[\hat{\mathcal{U}}'_0, \hat{\mathcal{S}}'] = 0. \tag{47}$$

This means that under the Hermitian map, unitary loops with symmetry (A, $\hat{\mathcal{U}}^+_0$) in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetry (AIII, $\hat{\mathcal{U}}^+_+$) in dimension $(d, d_\parallel, D+1, D_\parallel)$. Here we use the notation (AIII, $\hat{\mathcal{O}}^{\epsilon_O}_{\eta_\Gamma}$) to denote class AIII with an additional symmetry realized by $\hat{\mathcal{O}}$, which squares to ϵ_O and commutes ($\eta_S = 1$) or anticommutes ($\eta_S = -1$) with the chiral symmetry operator $\hat{\mathcal{S}}'$. One can also replace $\hat{\mathcal{O}}$ by $\bar{\mathcal{O}}$ to define class AIII with an additional antisymmetry in a similar way.

(b) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_{T/2}$. We have

$$\hat{\mathcal{U}}'_{T/2}\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{U}}'^{-1}_{T/2} = \mathcal{H}(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,T-t), \quad (48)$$

where $\hat{\mathcal{U}}'_{T/2} = \rho_x \otimes \hat{\mathcal{U}}_{T/2}$, which satisfies $\{\hat{\mathcal{U}}'_{T/2}, \hat{\mathcal{S}}'\} = 0$ and $\hat{\mathcal{U}}'^2_{T/2} = 1$. Since $t \in S^1$, if we shift the origin by defining $t = \frac{T}{2} + t'$ and use $t' \in S^1$ instead of t, then the map $t \to T - t$ becomes $t' \to -t'$. Now t' can be regarded as an additional defect surrounding the coordinate which flips under the order-2 symmetry. Under the Hermitian map, unitary loops with symmetry $(\text{A}, \hat{\mathcal{U}}^+_{T/2})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetry $(\text{AIII}, \hat{\mathcal{U}}^+_-)$ in dimension $(d, d_\parallel, D+1, D_\parallel+1)$.

(c) $\bar{\mathcal{O}} = \bar{\mathcal{U}}_s$. The unitary antisymmetry $\bar{\mathcal{U}}_s$ leads to an order-2 symmetry on $\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)$ with

$$\bar{\mathcal{U}}'_s\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{U}}'^{-1}_s = \mathcal{H}(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t), \quad (49)$$

where $\bar{\mathcal{U}}'_s = \rho_x \otimes \bar{\mathcal{U}}_s$. Moreover, we have $\bar{\mathcal{U}}'^2_s = 1$ and $\{\bar{\mathcal{U}}'_s, \hat{\mathcal{S}}'\} = 0$. Under the Hermitian map, unitary loops with symmetry $(\text{A}, \bar{\mathcal{U}}^+_s)$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetry $(\text{AIII}, \hat{\mathcal{U}}^+_-)$ in dimension $(d, d_\parallel, D+1, D_\parallel)$.

(d) $\hat{\mathcal{O}} = \hat{\mathcal{A}}_s$. We have

$$\hat{\mathcal{A}}'_s\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{A}}'^{-1}_s = \mathcal{H}(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t), \quad (50)$$

with $\hat{\mathcal{A}}'_s = \rho_x \otimes \hat{\mathcal{A}}_s$. Moreover, we have $\{\hat{\mathcal{A}}'_s, \hat{\mathcal{S}}'\} = 0$ and $\hat{\mathcal{A}}'^2_s = \hat{\mathcal{A}}^2_s$. Thus, under the Hermitian map, unitary loops with symmetry $(\text{A}, \hat{\mathcal{A}}^\pm_s)$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetry $(\text{AIII}, \hat{\mathcal{A}}^\pm_-)$ in dimension $(d, d_\parallel, D+1, D_\parallel)$.

(e) $\bar{\mathcal{O}} = \bar{\mathcal{A}}_0$. We have

$$\bar{\mathcal{A}}'_0\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{A}}'^{-1}_0 = \mathcal{H}(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t), \quad (51)$$

with $\bar{\mathcal{A}}'_0 = \rho_0 \otimes \bar{\mathcal{A}}_0$, which satisfies $\bar{\mathcal{A}}'^2_0 = \bar{\mathcal{A}}^2_0$ and $[\bar{\mathcal{A}}'_0, \hat{\mathcal{S}}'] = 0$. Under the Hermitian map, unitary loops with symmetry $(\text{A}, \bar{\mathcal{A}}^\pm_0)$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetry $(\text{AIII}, \hat{\mathcal{A}}^\pm_+)$ in dimension $(d, d_\parallel, D+1, D_\parallel)$.

(f) $\bar{\mathcal{O}} = \bar{\mathcal{A}}_{T/2}$. We have

$$\bar{\mathcal{A}}'_{T/2}\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\bar{\mathcal{A}}'^{-1}_{T/2} = \mathcal{H}(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,T-t), \quad (52)$$

with $\bar{\mathcal{A}}'_{T/2} = \rho_x \otimes \bar{\mathcal{A}}_{T/2}$, which satisfies $\bar{\mathcal{A}}'^2_{T/2} = \bar{\mathcal{A}}^2_{T/2}$ and $\{\bar{\mathcal{A}}'_{T/2}, \hat{\mathcal{S}}'\} = 0$. Under the Hermitian map, unitary loops with symmetry $(\text{A}, \bar{\mathcal{A}}^\pm_{T/2})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetry $(\text{AIII}, \hat{\mathcal{A}}^\pm_-)$ in dimension $(d, d_\parallel, D+1, D_\parallel+1)$.

2. Class AIII

In class AIII, we have a chiral symmetry realized by $\hat{\mathcal{S}}$. We assume an additional order-2 symmetry $\hat{\mathcal{U}}^{\epsilon_U}_{\eta_S}$ or antisymmetry $\bar{\mathcal{U}}^{\epsilon_{\bar{U}}}_{\eta_S}$. Moreover, we can set $\epsilon_U = 1$ and $\epsilon_{\bar{U}} = 1$ for unitary symmetries and antisymmetries realized by $\hat{\mathcal{U}}$ and $\bar{\mathcal{U}}$, respectively. For unitary (anti)symmetries, note that $\bar{\mathcal{U}}_{\eta_S}$ in class AIII is essentially the same as $\hat{\mathcal{U}}_{\eta_S}$, because they can be converted to each other by $\bar{\mathcal{U}}_{\eta_s} = \hat{\mathcal{S}}\hat{\mathcal{U}}_{\eta_S}$. Similarly, for antiunitary (anti)symmetries, $\hat{\mathcal{A}}^{\epsilon_A}_{\eta_S}$ and $\bar{\mathcal{A}}^{\epsilon_A\eta_S}_{\eta_S}$ are equivalent since $\hat{\mathcal{A}}^{\epsilon_A}_{\eta_S} = \hat{\mathcal{S}}\bar{\mathcal{A}}^{\epsilon_A\eta_S}_{\eta_S}$. Hence, in the following, we discuss only unitary and antiunitary symmetries.

(a) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_0$. We have

$$\hat{\mathcal{U}}_0\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{U}}^{-1}_0 = \eta_S\mathcal{H}(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t). \quad (53)$$

Under the Hermitian map, unitary loops with symmetries $(\text{AIII}, \hat{\mathcal{U}}^+_{0,+})$ and $(\text{AIII}, \hat{\mathcal{U}}^+_{0,-})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{A}, \hat{\mathcal{U}}^+)$ and $(\text{A}, \bar{\mathcal{U}}^+)$ in dimension $(d, d_\parallel, D+1, D_\parallel)$, respectively.

(b) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_{T/2}$. We have

$$\hat{\mathcal{S}}\hat{\mathcal{U}}_{T/2}\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)(\hat{\mathcal{S}}\hat{\mathcal{U}}_{T/2})^{-1} = \eta_S\mathcal{H}(-\boldsymbol{k}_\parallel,\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,T-t). \quad (54)$$

Under the Hermitian map, unitary loops with symmetries $(\text{AIII}, \hat{\mathcal{U}}^+_{T/2,+})$ and $(\text{AIII}, \hat{\mathcal{U}}^+_{T/2,-})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{A}, \hat{\mathcal{U}}^+)$ and $(\text{A}, \bar{\mathcal{U}}^+)$ in dimension $(d, d_\parallel, D+1, D_\parallel+1)$, respectively.

(c) $\hat{\mathcal{O}} = \hat{\mathcal{A}}_0$. We have

$$\hat{\mathcal{S}}\hat{\mathcal{A}}_0\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)(\hat{\mathcal{S}}\hat{\mathcal{A}}_0)^{-1} = \eta_S\mathcal{H}(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t). \quad (55)$$

Under the Hermitian map, unitary loops with symmetries $(\text{AIII}, \hat{\mathcal{A}}^\pm_{0,+})$ and $(\text{AIII}, \hat{\mathcal{A}}^\pm_{0,-})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{A}, \hat{\mathcal{A}}^\pm)$ and $(\text{A}, \bar{\mathcal{A}}^\mp)$ in dimension $(d, d_\parallel, D+1, D_\parallel)$, respectively.

(d) $\hat{\mathcal{O}} = \hat{\mathcal{A}}_{T/2}$. We have

$$\hat{\mathcal{A}}_{T/2}\mathcal{H}(\boldsymbol{k},\boldsymbol{r},t)\hat{\mathcal{A}}^{-1}_{T/2} = \eta_S\mathcal{H}(\boldsymbol{k}_\parallel,-\boldsymbol{k}_\perp,-\boldsymbol{r}_\parallel,\boldsymbol{r}_\perp,t). \quad (56)$$

Under the Hermitian map, unitary loops with symmetries $(\text{AIII}, \hat{\mathcal{A}}^\pm_{T/2,+})$ and $(\text{AIII}, \hat{\mathcal{A}}^\pm_{T/2,-})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{A}, \hat{\mathcal{A}}^\pm)$ and $(\text{A}, \bar{\mathcal{A}}^\pm)$ in dimension $(d, d_\parallel, D+1, D_\parallel+1)$, respectively.

B. Real symmetry classes

Now let us consider real symmetry classes, where at least one antiunitary symmetry is present.

In classes AI and AII, only time-reversal symmetry is present. We have the equivalence relations between the additional order-2 symmetries/antisymmetries

$$\hat{\mathcal{U}}^{\epsilon_U}_{\eta_T} = i\hat{\mathcal{U}}^{-\epsilon_U}_{-\eta_T} = \hat{\mathcal{T}}\hat{\mathcal{A}}^{\eta_T\epsilon_T\epsilon_U}_{\eta_T} = i\hat{\mathcal{T}}\hat{\mathcal{A}}^{\eta_T\epsilon_T\epsilon_U}_{\eta_T}, \quad (57)$$

$$\bar{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_T} = i\bar{\mathcal{U}}^{-\bar{\epsilon}_U}_{-\bar{\eta}_T} = \hat{\mathcal{T}}\bar{\mathcal{A}}^{\bar{\eta}_T\epsilon_T\bar{\epsilon}_U}_{\bar{\eta}_T} = i\hat{\mathcal{T}}\bar{\mathcal{A}}^{-\bar{\eta}_T\epsilon_T\bar{\epsilon}_U}_{-\bar{\eta}_T}, \quad (58)$$

where $\epsilon_U = \hat{\mathcal{U}}^2$, $\epsilon_{\bar{U}} = \bar{\mathcal{U}}^2$, and $\epsilon_T = \hat{\mathcal{T}}^2$. We only need to consider four cases $\hat{\mathcal{U}}^+_+, \hat{\mathcal{U}}^+_-, \bar{\mathcal{U}}^+_+$, and $\bar{\mathcal{U}}^+_-$.

In classes C and D, the particle-hole symmetry leads to the equivalence relations between the additional order-2 symmetries/antisymmetries

$$\hat{\mathcal{U}}^{\epsilon_U}_{\eta_C} = i\hat{\mathcal{U}}^{-\epsilon_U}_{-\eta_C} = \hat{\mathcal{C}}\bar{\mathcal{A}}^{\eta_C\epsilon_C\epsilon_U}_{\eta_C} = i\hat{\mathcal{C}}\bar{\mathcal{A}}^{\eta_C\epsilon_C\epsilon_U}_{-\eta_C}, \quad (59)$$

$$\bar{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_C} = i\bar{\mathcal{U}}^{-\bar{\epsilon}_U}_{-\bar{\eta}_T} = \hat{\mathcal{C}}\hat{\mathcal{A}}^{\bar{\eta}_C\epsilon_C\bar{\epsilon}_U}_{\bar{\eta}_C} = i\hat{\mathcal{C}}\hat{\mathcal{A}}^{\bar{\eta}_C\epsilon_C\bar{\epsilon}_U}_{-\bar{\eta}_C}, \quad (60)$$

where $\epsilon_C = \hat{\mathcal{C}}^2$. We just need to consider four cases $\hat{\mathcal{U}}^+_+, \hat{\mathcal{U}}^+_-$, $\bar{\mathcal{U}}^+_+$, and $\bar{\mathcal{U}}^+_-$.

Finally, in classes BDI, DIII, CII, and CI, with time-reversal, particle-hole, and chiral symmetries all together, we have

$$\hat{\mathcal{U}}^{\epsilon_U}_{\eta_T,\eta_C} = i\hat{\mathcal{U}}^{-\epsilon_U}_{-\eta_T,-\eta_C} = \hat{\mathcal{T}}\hat{\mathcal{A}}^{\eta_T\epsilon_T\epsilon_U}_{\eta_T,\eta_C} = i\hat{\mathcal{T}}\hat{\mathcal{A}}^{\eta_T\epsilon_T\epsilon_U}_{-\eta_T,\eta_C} = \hat{\mathcal{C}}\bar{\mathcal{A}}^{\eta_C\epsilon_C\epsilon_U}_{\eta_T,\eta_C} = i\hat{\mathcal{C}}\bar{\mathcal{A}}^{\eta_C\epsilon_C\epsilon_U}_{-\eta_T,-\eta_C}, \tag{61}$$

$$\bar{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_T,\bar{\eta}_C} = i\bar{\mathcal{U}}^{-\bar{\epsilon}_U}_{-\bar{\eta}_T,-\bar{\eta}_C} = \hat{\mathcal{T}}\bar{\mathcal{A}}^{\bar{\eta}_T\epsilon_T\bar{\epsilon}_U}_{\bar{\eta}_T,\bar{\eta}_C} = i\hat{\mathcal{T}}\bar{\mathcal{A}}^{\bar{\eta}_T\epsilon_T\bar{\epsilon}_U}_{-\bar{\eta}_T,-\bar{\eta}_C} = \hat{\mathcal{C}}\hat{\mathcal{A}}^{\bar{\eta}_C\epsilon_C\bar{\epsilon}_U}_{\bar{\eta}_T,\bar{\eta}_C} = i\hat{\mathcal{C}}\hat{\mathcal{A}}^{\bar{\eta}_C\epsilon_C\bar{\epsilon}_U}_{-\bar{\eta}_T,-\bar{\eta}_C}. \tag{62}$$

Hence, only four cases $\hat{\mathcal{U}}^{+}_{+,+}$, $\hat{\mathcal{U}}^{+}_{+,-}$, $\hat{\mathcal{U}}^{+}_{-,-}$, and $\hat{\mathcal{U}}^{+}_{-,+}$ need to be considered.

1. Classes AI and AII

(a) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_0$. The new Hermitian matrix $\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)$ under the Hermitian map defined by Eq. (33) acquires new time-reversal and particle-hole symmetries, realized by $\hat{\mathcal{T}}' = \rho_x \otimes \hat{\mathcal{T}}$ and $\hat{\mathcal{C}}' = i\rho_y \otimes \hat{\mathcal{T}}$, respectively. Due to the order-2 symmetry realized by $\hat{\mathcal{U}}_0$, we have

$$\hat{\mathcal{U}}_0'\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{U}}_0'^{-1} = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, t), \tag{63}$$

with $\hat{\mathcal{U}}_0' = \rho_0 \otimes \hat{\mathcal{U}}_0$. Moreover, we have

$$\hat{\mathcal{U}}_0'\hat{\mathcal{T}}' = \eta_T\hat{\mathcal{T}}'\hat{\mathcal{U}}_0', \tag{64}$$

$$\hat{\mathcal{U}}_0'\hat{\mathcal{C}}' = \eta_T\hat{\mathcal{C}}'\hat{\mathcal{U}}_0', \tag{65}$$

and $\hat{\mathcal{U}}_0'^2 = \epsilon_U$. Under the Hermitian map, unitary loops with symmetries $(\text{AI}, \hat{\mathcal{U}}^{\epsilon_U}_{0,\eta_T})$ and $(\text{AII}, \hat{\mathcal{U}}^{\epsilon_U}_{0,\eta_T})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{CI}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_T,\eta_T})$ and $(\text{DIII}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_T,\eta_T})$ in dimension $(d, d_\parallel, D+1, D_\parallel)$, respectively.

(b) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_{T/2}$. Due to the order-2 symmetry realized by $\hat{\mathcal{U}}_{T/2}$, we have

$$\hat{\mathcal{U}}_{T/2}'\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{U}}_{T/2}'^{-1} = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, T-t), \tag{66}$$

with $\hat{\mathcal{U}}_{T/2}' = \rho_x \otimes \hat{\mathcal{U}}_{T/2}$, which satisfies

$$\hat{\mathcal{U}}_{T/2}'\hat{\mathcal{T}}' = \eta_T\hat{\mathcal{T}}'\hat{\mathcal{U}}_{T/2}', \tag{67}$$

$$\hat{\mathcal{U}}_{T/2}'\hat{\mathcal{C}}' = -\eta_T\hat{\mathcal{C}}'\hat{\mathcal{U}}_{T/2}', \tag{68}$$

and $\hat{\mathcal{U}}_{T/2}'^2 = \epsilon_U$. Under the Hermitian map, unitary loops with symmetries $(\text{AI}, \hat{\mathcal{U}}^{\epsilon_U}_{T/2,\eta_T})$ and $(\text{AII}, \hat{\mathcal{U}}^{\epsilon_U}_{T/2,\eta_T})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{CI}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_T,-\eta_T})$ and $(\text{DIII}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_T,-\eta_T})$ in dimension $(d, d_\parallel, D+1, D_\parallel+1)$, respectively.

(c) $\bar{\mathcal{O}} = \bar{\mathcal{U}}_0$. Due to the order-2 antisymmetry realized by $\bar{\mathcal{U}}_0$, we have

$$\bar{\mathcal{U}}_0'\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\bar{\mathcal{U}}_0'^{-1} = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, t), \tag{69}$$

with $\bar{\mathcal{U}}_0' = \rho_x \otimes \bar{\mathcal{U}}_0$, which satisfies

$$\hat{\mathcal{U}}_0'\hat{\mathcal{T}}' = \bar{\eta}_T\hat{\mathcal{T}}'\hat{\mathcal{U}}_0', \tag{70}$$

$$\hat{\mathcal{U}}_0'\hat{\mathcal{C}}' = -\bar{\eta}_T\hat{\mathcal{C}}'\hat{\mathcal{U}}_0', \tag{71}$$

and $\bar{\mathcal{U}}_0'^2 = \bar{\epsilon}_U$. Under the Hermitian map, unitary loops with symmetries $(\text{AI}, \bar{\mathcal{U}}^{\bar{\epsilon}_U}_{0,\bar{\eta}_T})$ and $(\text{AII}, \bar{\mathcal{U}}^{\bar{\epsilon}_U}_{0,\bar{\eta}_T})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{CI}, \hat{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_T,-\bar{\eta}_T})$ and $(\text{DIII}, \hat{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_T,-\bar{\eta}_T})$ in dimension $(d, d_\parallel, D+1, D_\parallel)$, respectively.

(d) $\bar{\mathcal{O}} = \bar{\mathcal{U}}_{T/2}$. Due to the order-2 antisymmetry realized by $\bar{\mathcal{U}}_{T/2}$, we have

$$\bar{\mathcal{U}}_{T/2}'\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\bar{\mathcal{U}}_{T/2}'^{-1} = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, T-t), \tag{72}$$

with $\bar{\mathcal{U}}_{T/2}' = \rho_0 \otimes \bar{\mathcal{U}}_{T/2}$, which satisfies

$$\hat{\mathcal{U}}_0'\hat{\mathcal{T}}' = \bar{\eta}_T\hat{\mathcal{T}}'\hat{\mathcal{U}}_0', \tag{73}$$

$$\hat{\mathcal{U}}_0'\hat{\mathcal{C}}' = \bar{\eta}_T\hat{\mathcal{C}}'\hat{\mathcal{U}}_0', \tag{74}$$

and $\bar{\mathcal{U}}_{T/2}'^2 = \bar{\epsilon}_U$. Under the Hermitian map, unitary loops with symmetries $(\text{AI}, \bar{\mathcal{U}}^{\bar{\epsilon}_U}_{T/2,\bar{\eta}_T})$ and $(\text{AII}, \bar{\mathcal{U}}^{\bar{\epsilon}_U}_{T/2,\bar{\eta}_T})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{CI}, \hat{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_T,\bar{\eta}_T})$ and $(\text{DIII}, \hat{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_T,\bar{\eta}_T})$ in dimension $(d, d_\parallel, D+1, D_\parallel+1)$, respectively.

2. Classes C and D

(a) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_0$. The new Hermitian matrix $\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)$ under the Hermitian map defined by Eq. (33) acquires new time-reversal and particle-hole symmetries, realized by $\hat{\mathcal{T}}' = \rho_0 \otimes \hat{\mathcal{C}}$ and $\hat{\mathcal{C}}' = \rho_z \otimes \hat{\mathcal{T}}$, respectively. Due to the order-2 symmetry realized by $\hat{\mathcal{U}}_0$, we have

$$\hat{\mathcal{U}}_0'\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{U}}_0'^{-1} = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, t), \tag{75}$$

with $\hat{\mathcal{U}}_0' = \rho_0 \otimes \hat{\mathcal{U}}_0$, which satisfies

$$\hat{\mathcal{U}}_0'\hat{\mathcal{T}}' = \eta_C\hat{\mathcal{T}}'\hat{\mathcal{U}}_0', \tag{76}$$

$$\hat{\mathcal{U}}_0'\hat{\mathcal{C}}' = \eta_C\hat{\mathcal{C}}'\hat{\mathcal{U}}_0', \tag{77}$$

and $\hat{\mathcal{U}}_0'^2 = \epsilon_U$. Under the Hermitian map, unitary loops with symmetries $(\text{C}, \hat{\mathcal{U}}^{\epsilon_U}_{0,\eta_C})$ and $(\text{D}, \hat{\mathcal{U}}^{\epsilon_U}_{0,\eta_C})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{CII}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_C,\eta_C})$ and $(\text{BDI}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_C,\eta_C})$ in dimension $(d, d_\parallel, D+1, D_\parallel)$, respectively.

(b) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_{T/2}$. Due to the order-2 symmetry realized by $\hat{\mathcal{U}}_{T/2}$, we have

$$\hat{\mathcal{U}}_{T/2}'\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{U}}_{T/2}'^{-1} = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, T-t), \tag{78}$$

with $\hat{\mathcal{U}}_{T/2}' = \rho_x \otimes \hat{\mathcal{U}}_{T/2}$, which satisfies

$$\hat{\mathcal{U}}_{T/2}'\hat{\mathcal{T}}' = \eta_C\hat{\mathcal{T}}'\hat{\mathcal{U}}_{T/2}', \tag{79}$$

$$\hat{\mathcal{U}}_{T/2}'\hat{\mathcal{C}}' = -\eta_C\hat{\mathcal{C}}'\hat{\mathcal{U}}_{T/2}', \tag{80}$$

and $\hat{\mathcal{U}}_{T/2}'^2 = \epsilon_U$. Under the Hermitian map, unitary loops with symmetries $(\text{C}, \hat{\mathcal{U}}^{\epsilon_U}_{T/2,\eta_C})$ and $(\text{D}, \hat{\mathcal{U}}^{\epsilon_U}_{T/2,\eta_C})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\text{CII}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_C,-\eta_C})$ and $(\text{BDI}, \hat{\mathcal{U}}^{\epsilon_U}_{\eta_C,-\eta_C})$ in dimension $(d, d_\parallel, D+1, D_\parallel+1)$, respectively.

(c) $\bar{\mathcal{O}} = \bar{\mathcal{U}}_s$. Due to the order-2 antisymmetry realized by $\bar{\mathcal{U}}_s$, we have

$$\bar{\mathcal{U}}_s'\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\bar{\mathcal{U}}_s'^{-1} = \mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, t), \tag{81}$$

with $\bar{\mathcal{U}}_s' = \rho_x \otimes \bar{\mathcal{U}}_s$, which satisfies

$$\hat{\mathcal{U}}_s'\hat{\mathcal{T}}' = \bar{\eta}_C \hat{\mathcal{T}}'\hat{\mathcal{U}}_s', \tag{82}$$

$$\hat{\mathcal{U}}_s'\hat{\mathcal{C}}' = -\bar{\eta}_C \hat{\mathcal{C}}'\hat{\mathcal{U}}_s', \tag{83}$$

and $\bar{\mathcal{U}}_s'^2 = \bar{\epsilon}_U$. Hence, under the Hermitian map, unitary loops with symmetries $(\mathrm{C}, \bar{\mathcal{U}}_{s,\bar{\eta}_C}^{\bar{\epsilon}_U})$ and $(\mathrm{D}, \bar{\mathcal{U}}_{s,\bar{\eta}_C}^{\bar{\epsilon}_U})$ in dimension $(d, d_\parallel, D, D_\parallel)$ are mapped to static Hamiltonians with symmetries $(\mathrm{CII}, \hat{\mathcal{U}}_{\bar{\eta}_C,-\bar{\eta}_C}^{\bar{\epsilon}_U})$ and $(\mathrm{BDI}, \hat{\mathcal{U}}_{\bar{\eta}_C,-\bar{\eta}_C}^{\bar{\epsilon}_U})$ in dimension $(d, d_\parallel, D+1, D_\parallel)$, respectively.

3. Classes CI, CII, DIII, and BDI

In these classes, the time-reversal, particle-hole, and chiral symmetries are all present. Without loss of generality, we assume that $\hat{\mathcal{S}} = \hat{\mathcal{T}}\hat{\mathcal{C}}$ and $\hat{\mathcal{S}}^2 = 1$. The Hermitian matrix $\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)$ defined according to Eq. (32) has either time-reversal or particle-hole symmetry realized by

$$(\epsilon_C\epsilon_T)\hat{\mathcal{C}}\mathcal{H}(-\boldsymbol{k}, \boldsymbol{r}, t) = \mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{C}}, \tag{84}$$

depending on whether $\epsilon_C\epsilon_T$ is 1 or -1.

(a) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_0$. Due to the order-2 symmetry realized by $\hat{\mathcal{U}}_0$, we have

$$\hat{\mathcal{U}}_0\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)\hat{\mathcal{U}}_0^{-1} = \eta_T\eta_C\mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, t), \tag{85}$$

with $\hat{\mathcal{U}}_0\hat{\mathcal{C}} = \eta_C\hat{\mathcal{C}}\hat{\mathcal{U}}_0$ and $\hat{\mathcal{U}}_0^2 = \epsilon_U$. Under the Hermitian map, unitary loops in dimension $(d, d_\parallel, D, D_\parallel)$ with a given symmetry are mapped to static Hamiltonians in dimension $(d, d_\parallel, D+1, D_\parallel)$ with another symmetry according to

$$(\mathrm{X}, \hat{\mathcal{U}}_{0,\eta_T,\eta_C}^{\epsilon_U}) \to \begin{cases} (\mathrm{Y}, \hat{\mathcal{U}}_{\eta_C}^{\epsilon_U}), & \eta_T\eta_C = 1 \\ (\mathrm{Y}, \bar{\mathcal{U}}_{\eta_C}^{\epsilon_U}), & \eta_T\eta_C = -1, \end{cases} \tag{86}$$

with X = CI, CII, DIII, BDI and Y = C, AII, D, AI.

(b) $\hat{\mathcal{O}} = \hat{\mathcal{U}}_{T/2}$. Due to the order-2 symmetry realized by $\hat{\mathcal{U}}_{T/2}$, we have

$$\begin{aligned}(\hat{\mathcal{S}}\hat{\mathcal{U}}_{T/2})&\mathcal{H}(\boldsymbol{k}, \boldsymbol{r}, t)(\hat{\mathcal{S}}\hat{\mathcal{U}}_{T/2})^{-1} \\ &= \eta_T\eta_C\mathcal{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp, T-t).\end{aligned} \tag{87}$$

Moreover, we have $(\hat{\mathcal{S}}\hat{\mathcal{U}}_{T/2})\hat{\mathcal{C}} = \eta_C\epsilon_C\epsilon_T\hat{\mathcal{C}}(\hat{\mathcal{S}}\hat{\mathcal{U}}_{T/2})$ and $(\hat{\mathcal{S}}\hat{\mathcal{U}}_{T/2})^2 = \eta_T\eta_C\epsilon_U$. Under the Hermitian map, unitary loops in dimension $(d, d_\parallel, D, D_\parallel)$ with a given symmetry are mapped to static Hamiltonians in dimension $(d, d_\parallel, D+1, D_\parallel+1)$ with another symmetry according to

$$\begin{aligned}&(\mathrm{X}, \hat{\mathcal{U}}_{T/2,\eta_T,\eta_C}^{\epsilon_U}) \\ &\to \begin{cases} (\mathrm{Y}, \hat{\mathcal{U}}_{\eta_C\epsilon_C\epsilon_T}^{\epsilon_U}), & \eta_T\eta_C = 1 \\ (\mathrm{Y}, \bar{\mathcal{U}}_{\eta_C\epsilon_C\epsilon_T}^{-\epsilon_U}) = (\mathrm{Y}, \bar{\mathcal{U}}_{-\eta_C\epsilon_C\epsilon_T}^{\epsilon_U}), & \eta_T\eta_C = -1, \end{cases}\end{aligned} \tag{88}$$

with X = CI, CII, DIII, BDI and Y = C, AII, D, AI.

VI. K GROUPS IN THE PRESENCE OF ORDER-2 SYMMETRY

Using the Hermitian map introduced in the previous sections, the unitary loops with an order-2 space-time symmetry/antisymmetry are successfully mapped to static Hamiltonians with an order-2 crystalline symmetry/antisymmetry, whose classification has already been worked out in Ref. [11]. Thus, the latter result can be directly applied to the classification of unitary loops.

We first summarize the K-theory-based method used for classifying static Hamiltonians and then finish the classification of unitary loops. Let us consider static Hamiltonians defined on a base space of momentum $\boldsymbol{k} \in T^d$ and real space coordinate $\boldsymbol{r} \in S^D$. For the classification of strong topological phases, one can instead simply use S^{d+D} as the base space [5,7]. To classify these Hamiltonians, we will use the notion of stable homotopy equivalence as we defined for unitaries in Sec. II, by identifying Hamiltonians which are continuously deformable into each other up to adding extra trivial bands, while preserving an energy gap at the chemical potential. These equivalence classes can be formally added and they form an Abelian group.

For a given AZ symmetry class s, the classification of static Hamiltonians is given by the set of stable equivalence classes of maps $\mathcal{H}(\boldsymbol{k}, \boldsymbol{r})$, from the base space $(\boldsymbol{k}, \boldsymbol{r}) \in S^{d+D}$ to the classifying space, denoted by $\mathcal{C}_s$ or $\mathcal{R}_s$, for complex and real symmetry classes, as listed in Table II. The Abelian group structure inherited from the equivalence classes leads to the group structure in this set of maps, which is called the K group, or classification group.

For static topological insulators and superconductors of dimension (d, D) in an AZ class s without additional spatial symmetries, the K groups are denoted by $K_{\mathbb{C}}(s; d, D)$ and $K_{\mathbb{R}}(s; d, D)$ for complex and real symmetry classes, respectively. Note that for complex symmetry classes, we have $s = 0, 1 \bmod 2$, whereas for real symmetry classes $s = 0, 1, \ldots, 7 \bmod 8$.

These K groups have the properties

$$K_{\mathbb{C}}(s; d, D) = K_{\mathbb{C}}(s-d+D; 0, 0) = \pi_0(\mathcal{C}_{s-d+D}), \tag{89}$$

$$K_{\mathbb{R}}(s; d, D) = K_{\mathbb{R}}(s-d+D; 0, 0) = \pi_0(\mathcal{R}_{s-d+D}), \tag{90}$$

known as the Bott periodicity, where π_0 denotes the zeroth homotopy group which counts the number of path connected components in a given space. In the following, we will introduce the K groups for Hamiltonians supporting an additional order-2 spatial symmetry/antisymmetry following Ref. [11]. Because of the Hermitian map, these K groups can also be associated with the unitary loops, whose classification is then obtained.

A. Complex symmetry classes with an additional order-2 unitary symmetry/antisymmetry

When a spatial or space-time symmetry/antisymmetry is considered, one needs to include the number of "flipped" coordinates for both $\boldsymbol{k}$ and $\boldsymbol{r}$ in the dimensions. For a static Hamiltonian of dimension $(d, d_\parallel, D, D_\parallel)$ in complex AZ classes with an additional order-2 unitary symmetry/antisymmetry, the K group is denoted by $K_{\mathbb{C}}^U(s, t; d, d_\parallel, D, D_\parallel)$, where the additional parameter $t = 0, 1 \bmod 2$ specifies the coexisting order-2 unitary symmetry/antisymmetry. These K groups satisfy the relation

$$\begin{aligned}K_{\mathbb{C}}^U(s, t; d, d_\parallel, D, D_\parallel) &= K_{\mathbb{C}}^U(s-\delta, t-\delta_\parallel; 0, 0, 0, 0) \\ &\equiv K_{\mathbb{C}}^U(s-\delta, t-\delta_\parallel),\end{aligned} \tag{91}$$

TABLE III. Possible types ($t = 0, 1 \bmod 2$) of order-2 additional unitary symmetry $\hat{\mathcal{U}}^{\epsilon_U}_{\eta_S}/\bar{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_S}$ in complex AZ classes ($s = 0, 1 \bmod 2$). The superscript and subscript are defined as $\epsilon_U = \hat{\mathcal{U}}^2$, $\bar{\epsilon}_U = \bar{\mathcal{U}}^2$, $\hat{\mathcal{U}}\hat{\mathcal{S}} = \eta_S\hat{\mathcal{S}}\hat{\mathcal{U}}$, and $\bar{\mathcal{U}}\hat{\mathcal{S}} = \bar{\eta}_S\hat{\mathcal{S}}\bar{\mathcal{U}}$.

s	AZ class	$t=0$	$t=1$
0	A	$\hat{\mathcal{U}}^+_0, \hat{\mathcal{U}}^+_{T/2}$	$\bar{\mathcal{U}}^+_s$
1	AIII	$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,-}$	$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,+}$

where $\delta = d - D$ and $\delta_\parallel = d_\parallel - D_\parallel$. Thus, for classification purpose, one can use the pair $(\delta, \delta_\parallel)$ instead of $(d, d_\parallel, D, D_\parallel)$ to denote the dimension of the base space, in which the static Hamiltonian is defined.

To define K groups for unitary loops, we use the fact that the K group for certain unitary loops should be the same as the one for the corresponding static Hamiltonians under the Hermitian map. The K groups for unitary loops are explicitly defined in Table III, where the two arguments s and t label the AZ class and the coexisting order-2 space-time symmetry/antisymmetry.

B. Complex symmetry classes with an additional order-2 antiunitary symmetry/antisymmetry

We now consider static Hamiltonians of dimension $(d, d_\parallel, D, D_\parallel)$, in complex AZ classes, with an order-2 antiunitary symmetry/antisymmetry, realized by $\hat{\mathcal{A}}$ or $\bar{\mathcal{A}}$. It turns out that complex AZ classes acquire real structures because of the antiunitary symmetry [11]. Indeed, effective time-reversal or particle-hole symmetry realized by $\hat{\mathcal{A}}$ or $\bar{\mathcal{A}}$ emerges if we regard $(\boldsymbol{k}_\perp, \boldsymbol{r}_\parallel)$ as momenta and $(\boldsymbol{k}_\parallel, \boldsymbol{r}_\perp)$ as spatial coordinates. Thus, a system in complex AZ classes with an antiunitary symmetry can be mapped to a real AZ class without additional spatial symmetries.

The K groups for these Hamiltonians are denoted by $K^A_{\mathbb{C}}(s; d, d_\parallel, D, D_\parallel)$, which satisfies

$$K^A_{\mathbb{C}}(s; d, d_\parallel, D, D_\parallel) = K^A_{\mathbb{C}}(s - \delta + 2\delta_\parallel; 0, 0, 0, 0) \equiv K^A_{\mathbb{C}}(s - \delta + 2\delta_\parallel). \quad (92)$$

Similar to the previous case, the unitary loops with an antiunitary space-time symmetry/antisymmetry can also be associated with these K groups.

If we group these antiunitary symmetries and antisymmetries in terms of the index $s = 0, \ldots, 7 \bmod 8$, according to Table IV, then $K^A_{\mathbb{C}}(s)$ can further be reduced to $K_{\mathbb{R}}(s) \equiv K_{\mathbb{R}}(s; 0, 0)$.

C. Real symmetry classes with an additional order-2 symmetry

In real symmetry classes, there are equivalence relations between order-2 unitary and antiunitary symmetries/antisymmetries, as discussed previously. Thus, one can focus on unitary symmetries/antisymmetries only. The existence of an additional order-2 unitary symmetry divides each class into four families ($t = 0, \ldots, 3 \bmod 4$), as summarized in Table V, where we have used the equivalence of K groups for static

TABLE IV. Possible types ($s = 0, \ldots, 7 \bmod 8$) of order-2 additional antiunitary symmetry $\hat{\mathcal{A}}^{\epsilon_A}_{\eta_S}/\bar{\mathcal{A}}^{\bar{\epsilon}_A}_{\bar{\eta}_S}$ in complex AZ classes. The superscript and subscript are defined as $\epsilon_A = \hat{\mathcal{A}}^2$, $\bar{\epsilon}_A = \bar{\mathcal{A}}^2$, $\hat{\mathcal{A}}\hat{\mathcal{S}} = \eta_S\hat{\mathcal{S}}\hat{\mathcal{A}}$, and $\bar{\mathcal{A}}\hat{\mathcal{S}} = \bar{\eta}_S\hat{\mathcal{S}}\bar{\mathcal{A}}$.

s	AZ class	Coexisting symmetry	Mapped AZ class
0	A	$\hat{\mathcal{A}}^+_s$	AI
1	AIII	$\hat{\mathcal{A}}^+_{0,+}, \hat{\mathcal{A}}^+_{T/2,-}$	BDI
2	A	$\bar{\mathcal{A}}^+_0, \bar{\mathcal{A}}^-_{T/2}$	D
3	AIII	$\hat{\mathcal{A}}^-_{0,-}, \hat{\mathcal{A}}^-_{T/2,+}$	DIII
4	A	$\hat{\mathcal{A}}^-_s$	AII
5	AIII	$\hat{\mathcal{A}}^-_{0,+}, \hat{\mathcal{A}}^-_{T/2,-}$	CII
6	A	$\bar{\mathcal{A}}^-_0, \bar{\mathcal{A}}^+_{T/2}$	C
7	AIII	$\hat{\mathcal{A}}^+_{0,-}, \hat{\mathcal{A}}^+_{T/2,+}$	CI

Hamiltonians and unitary loops in terms of the Hermitian map.

We denote the K group for unitary loops in real AZ classes ($s = 0, \ldots, 7 \bmod 8$) with an additional order-2 unitary symmetry/antisymmetry ($t = 0, \ldots, 3 \bmod 4$) by $K^U_{\mathbb{R}}(s, t; d, d_\parallel, D, D_\parallel)$, which satisfies

$$K^U_{\mathbb{R}}(s, t; d, d_\parallel, D, D_\parallel) = K^U_{\mathbb{R}}(s - \delta, t - \delta_\parallel; 0, 0, 0, 0) \equiv K^U_{\mathbb{R}}(s - \delta, t - \delta_\parallel). \quad (93)$$

D. Nontrivial space-time vs static spatial symmetries/antisymmetries

The classification of unitary loops with an order-2 space-time symmetry/antisymmetry is given by the K group $K^U_{\mathbb{C}}(s, t)$, $K^A_{\mathbb{C}}(s)$, or $K^U_{\mathbb{R}}(s, t)$. As can be seen in Tables III–V, for every order-2 space-time (anti)unitary symmetry/antisymmetry that is nontrivial, namely, the half-period time translation is involved, there always exists a unique static spatial (anti)unitary symmetry/antisymmetry such that both symmetries/antisymmetries give rise to the same K group. It is worth mentioning that when looking at the static symmetries/antisymmetries alone, the corresponding K groups for unitary loops are defined in the same way as the ones for Hamiltonians introduced in Ref. [11], as expected.

The explicit relations between the two types of symmetries/antisymmetries (nontrivial space-time vs static) with the same K group can be summarized as follows. Recall that we use η_S ($\bar{\eta}_S$), η_T ($\bar{\eta}_T$), and η_C ($\bar{\eta}_C$) to characterize the commutation relations between the order-2 symmetry (antisymmetry) operator and the nonspatial symmetry operators. For two unitary order-2 symmetries giving rise to the same K group, the η_S and η_C for the two symmetries take opposite signs, whereas the η_T are the same. For two antiunitary order-2 symmetries, we have η_S take opposite signs. For two unitary antisymmetries, the $\bar{\eta}_T$ have opposite signs. Finally, for class A, the antiunitary space-time antisymmetry operator $\bar{\mathcal{A}}^\pm_{T/2}$ has the same K group as the one for $\bar{\mathcal{A}}^\mp_0$. These relations are summarized in Table I and can be better understood after we introduce the frequency-domain formulation of the Floquet problem in Sec. VIII B.

TABLE V. Possible types ($t = 0, \ldots, 3 \bmod 4$) of order-2 additional symmetry $\hat{\mathcal{U}}^{\epsilon_U}_{\eta_S}/\bar{\mathcal{U}}^{\bar{\epsilon}_U}_{\bar{\eta}_S}$ in real AZ classes. The superscript and subscript are defined as $\epsilon_U = \hat{\mathcal{U}}^2$, $\bar{\epsilon}_U = \bar{\mathcal{U}}^2$, $\hat{\mathcal{U}}\hat{\mathcal{S}} = \eta_S \hat{\mathcal{S}}\hat{\mathcal{U}}$, and $\bar{\mathcal{U}}\hat{\mathcal{S}} = \bar{\eta}_S \hat{\mathcal{S}}\bar{\mathcal{U}}$. We set $\epsilon_U = \bar{\epsilon}_U = 1$.

s	AZ Class	$t = 0$	$t = 1$	$t = 2$	$t = 3$
0	AI	$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,+}$	$\bar{\mathcal{U}}^+_{0,-}, \bar{\mathcal{U}}^+_{T/2,+}$	$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,-}$	$\bar{\mathcal{U}}^+_{0,+}, \bar{\mathcal{U}}^+_{T/2,-}$
1	BDI	$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$
2	D	$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,-}$	$\bar{\mathcal{U}}^+_{s,+}$	$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,+}$	$\bar{\mathcal{U}}^+_{s,-}$
3	DIII	$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{0,-+}$	$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$
4	AII	$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,+}$	$\bar{\mathcal{U}}^+_{0,-}, \bar{\mathcal{U}}^+_{T/2,+}$	$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,-}$	$\bar{\mathcal{U}}^+_{0,+}, \bar{\mathcal{U}}^+_{T/2,-}$
5	CII	$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	$\hat{\mathcal{U}}^+_{0++,-+}, \hat{\mathcal{U}}^+_{T/2,--}$
6	C	$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,-}$	$\bar{\mathcal{U}}^+_{s,+}$	$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,+}$	$\bar{\mathcal{U}}^+_{s,-}$
7	CI	$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$

E. Periodic table

From the K groups introduced previously, we see that in addition to the mod 2 or mod 8 Bott periodicity in δ, there also exists a periodic structure in flipped dimensions $\delta_\parallel$, because of the twofold or fourfold periodicity in t, which accounts for the additional order-2 symmetry/antisymmetry. In particular, for complex symmetry classes with an order-2 unitary symmetry/antisymmetry, the classification has a twofold periodicity in $\delta_\parallel$, whereas for complex symmetry classes with an order-2 antiunitary symmetry/antisymmetry and for real symmetry classes with an order-2 unitary/antisymmetry, the periodicity in $\delta_\parallel$ is fourfold. These periodic features are the same as the ones obtained in Ref. [11] for static Hamiltonians with an order-2 crystalline symmetry/antisymmetry. We summarize the periodic tables for the four ($\delta_\parallel = 0, \ldots, 3 \bmod 4$) different families in the Supplemental Material [44]. Note that in obtaining the classification tables, we made use of the K groups in their zero-dimensional forms defined in Eqs. (91)–(93), as well as the relations

$$
\begin{aligned}
K^U_{\mathbb{C}}(s, t = 0) &= \pi_0(\mathcal{C}_s \times \mathcal{C}_s) = \pi_0(\mathcal{C}_s) \oplus \pi_0(\mathcal{C}_s),\\
K^U_{\mathbb{C}}(s, t = 1) &= \pi_0(\mathcal{C}_{s+1}),\\
K^A_{\mathbb{C}}(s) &= \pi_0(\mathcal{R}_s),\\
K^U_{\mathbb{R}}(s, t = 0) &= \pi_0(\mathcal{R}_s \times \mathcal{R}_s) = \pi_0(\mathcal{R}_s) \oplus \pi_0(\mathcal{R}_s), \qquad (94)\\
K^U_{\mathbb{R}}(s, t = 1) &= \pi_0(\mathcal{R}_{s+7}),\\
K^U_{\mathbb{R}}(s, t = 2) &= \pi_0(\mathcal{C}_s),\\
K^U_{\mathbb{R}}(s, t = 3) &= \pi_0(\mathcal{R}_{s+1}),
\end{aligned}
$$

where $\mathcal{C}_s$ ($s = 0, 1 \bmod 2$) and $\mathcal{R}_s$ ($s = 0, \ldots, 7 \bmod 8$) represent the classifying space of complex and real AZ classes, respectively (see Table II).

VII. FLOQUET HIGHER-ORDER TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS

In the previous sections, we obtained a complete classification of the anomalous Floquet TIs and SCs using K theory, where the K groups for the unitary loops were defined as the same ones for the static Hamiltonians, according to the Hermitian map. Noticeably, the classification obtained in this way is a bulk classification, since only bulk unitary evolution operators were considered. These bulk K groups include the information of topological classification at any order. For static tenfold-way TIs and SCs, in which the topological property is determined from the nonspatial symmetries, there is a bulk-boundary correspondence which essentially says that the nontrivial topological bulk indicates protected gapless boundary modes existing in one dimension lower. This boundary mode is irrespective of boundary orientation and lattice termination. The same is true for tenfold-way Floquet TIs and SCs with only nonspatial symmetries. In this situation, since only first-order topological phases are allowed, this bulk K group is enough to understand the existence of gapless boundary modes. However, when an additional crystalline symmetry/antisymmetry is taken into account, the existence of gapless boundary modes due to nontrivial topological bulk is not guaranteed unless the boundary is invariant under the nonlocal transformation of the symmetry/antisymmetry [9,12].

A more intriguing fact regarding crystalline symmetries/antisymmetries is that they can give rise to boundary modes with codimension higher than 1, such as corners of two-dimensional (2D) or 3D systems as well as hinges of 3D systems [17–25]. Such systems are known as HOTIs and SCs, in which the existence of the high-codimension gapless boundary modes is guaranteed when the boundaries are compatible with the crystalline symmetry/antisymmetry, i.e., a group of boundaries with different orientations are mapped onto each other under the nonlocal transformation of a particular crystalline symmetry/antisymmetry. For example, to have a HOTI and a SC protected by inversion, one needs to create boundaries in pairs related by inversion [24,25].

An additional requirement for these corner or hinge modes is that they should be intrinsic, namely, their existence should not depend on lattice termination; otherwise such high-codimension boundary modes can be thought of as a (codimension-1) boundary mode in the low-dimensional system, which is then glued to the original boundary. In other words, nth-order TIs and SCs have codimension-n boundary modes which cannot be destroyed through modifications of lattice terminations at the boundaries while preserving the bulk gap and the symmetries. According to this definition, the tenfold-way TIs and SCs are indeed intrinsic first-order TIs and SCs.

TABLE VI. Subgroup series $K^{(d)} \subseteq \cdots \subseteq K' \subseteq K$ for zero- ($d = 0$), one- ($d = 1$), and two-dimensional ($d = 2$) anomalous Floquet HOTIs and SCs with a unitary order-2 space-time symmetry/antisymmetry in complex classes. The number of flipped dimensions for the symmetry/antisymmetry is denoted by $d_\parallel$.

Symmetry	Class	$d=0$ $d_\parallel=0$	$d=1$ $d_\parallel=0$	$d=1$ $d_\parallel=1$	$d=2$ $d_\parallel=0$	$d=2$ $d_\parallel=1$	$d=2$ $d_\parallel=2$
$\hat{\mathcal{U}}_0^+, \hat{\mathcal{U}}_{T/2}^+$	A	$\mathbb{Z}^2$	$0 \subseteq 0$	$\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0$	$\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$
$\hat{\mathcal{U}}_{0,+}^+, \hat{\mathcal{U}}_{T/2,-}^+$	AIII	0	$0 \subseteq \mathbb{Z}^2$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}_s^+$	A	0	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}_{0,-}^+, \hat{\mathcal{U}}_{T/2,+}^+$	AIII	$\mathbb{Z}$	$0 \subseteq 0$	$\mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$

In Ref. [26], a complete classification of these intrinsic corner or hinge modes was derived and a higher-order bulk-boundary correspondence between these high-codimension boundary modes and the topological bulk was obtained. These were accomplished by considering a K subgroup series for a d-dimensional crystal,

$$K^{(d)} \subseteq \cdots \subseteq K'' \subseteq K' \subseteq K, \tag{95}$$

where $K \equiv K^{(0)}$ is the K group which classifies the bulk band structure of Hamiltonians with coexisting order-2 symmetry/antisymmetry, defined in the preceding section. Here $K^{(n)} \subseteq K$ is a subgroup excluding topological phases of order n or lower, for any crystalline-symmetry compatible boundaries. For example, K' classifies the purely crystalline phases [23,26], which exclude the tenfold-way topological phases, which are first-order topological phases protected by nonspatial symmetries alone and have gapless modes at any codimension-1 boundaries. These purely crystalline phases can have gapless modes only when the boundary preserves the crystalline symmetry, and the gapless modes will be gapped when the crystalline symmetry is broken.

From a boundary perspective, one can define the boundary K group $\mathcal{K}'$, which classifies the tenfold-way topological phases with gapless codimension-1 boundary modes irrespectively of boundary orientations, as long as the crystal shape and lattice termination are compatible crystalline symmetries. According to the above definitions, $\mathcal{K}'$ can be identified as the quotient group

$$\mathcal{K}' = K/K'. \tag{96}$$

Generalizing this idea, a series of boundary K groups denoted by $\mathcal{K}^{(n)}$ can be defined which classify the intrinsic nth-order TIs and SCs with intrinsic gapless codimension-n boundary modes, when the crystal has crystalline-symmetry-compatible shape and lattice termination. In Ref. [26], the authors proved the relation

$$\mathcal{K}^{(n+1)} = K^{(n)}/K^{(n+1)}, \tag{97}$$

known as the higher-order bulk-boundary correspondence: An intrinsic higher-order topological phase is uniquely associated with a topologically nontrivial bulk. Moreover, the above equation provides a systematic way of obtaining the complete classification of intrinsic HOTIs and SCs from K subgroup series, which were computed for crystals up to three dimensions with order-2 crystalline symmetries/antisymmetries.

We can generalize these results to anomalous Floquet HOTIs/SCs, by considering unitary loops $U(\boldsymbol{k}, t)$ in d dimensions without topological defect. To define a K subgroup series for unitary loops with an order-2 space-time symmetry/antisymmetry, one can exploit the Hermitian map and introduce the K groups according to their corresponding Hamiltonians with an order-2 crystalline symmetry/antisymmetry. One obtains that the K subgroup series for each nontrivial space-time symmetry/antisymmetry are the same as the ones for a corresponding static order-2 crystalline symmetry/antisymmetry, according to the substitution rules summarized in Sec. VI D and Table I. On the other hand, the K groups are the same for unitary loops and Hamiltonians when static order-2 symmetries/antisymmetries are considered. Using the results from Ref. [26], we present the K subgroup series for unitary loops with an order-2 space-time symmetry/antisymmetry in Tables VI–XI, for systems up to three dimensions. In these tables we use G^2 to denote $G \oplus G$, with $G = \mathbb{Z}, 2\mathbb{Z}, \mathbb{Z}_2$. Note that the largest K group

TABLE VII. Same as Table VI but for antiunitary symmetries/antisymmetries.

Symmetry	Class	$d=0$ $d_\parallel=0$	$d=1$ $d_\parallel=0$	$d=1$ $d_\parallel=1$	$d=2$ $d_\parallel=0$	$d=2$ $d_\parallel=1$	$d=2$ $d_\parallel=2$
$\hat{\mathcal{A}}_s^+$	A	$\mathbb{Z}$	$0 \subseteq 0$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{A}}_{0,+}^+, \hat{\mathcal{A}}_{T/2,-}^+$	AIII	$\mathbb{Z}_2$	$0 \subseteq \mathbb{Z}$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{A}}_0^+, \bar{\mathcal{A}}_{T/2}^-$	A	$\mathbb{Z}_2$	$0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 2\mathbb{Z}$
$\hat{\mathcal{A}}_{0,-}^-, \hat{\mathcal{A}}_{T/2,+}^-$	AIII	0	$0 \subseteq \mathbb{Z}_2$	$0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{A}}_s^-$	A	$2\mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{A}}_{0,+}^-, \hat{\mathcal{A}}_{T/2,-}^-$	AIII	0	$0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{A}}_0^-, \bar{\mathcal{A}}_{T/2}^+$	A	0	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}$
$\hat{\mathcal{A}}_{0,-}^+, \hat{\mathcal{A}}_{T/2,+}^+$	AIII	0	$0 \subseteq 0$	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$

TABLE VIII. Subgroup series $K^{(d)} \subseteq \cdots \subseteq K' \subseteq K$ for zero- ($d = 0$), one- ($d = 1$), and two-dimensional ($d = 2$) anomalous Floquet HOTIs and SCs with a unitary order-2 space-time symmetry/antisymmetry in real classes. The number of flipped dimensions for the symmetry/antisymmetry is denoted by $d_\parallel$.

Symmetry	Class	$d=0$, $d_\parallel=0$	$d=1$, $d_\parallel=0$	$d=1$, $d_\parallel=1$	$d=2$, $d_\parallel=0$	$d=2$, $d_\parallel=1$	$d=2$, $d_\parallel=2$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,+}$	AI	$\mathbb{Z}^2$	$0 \subseteq 0$	$\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	BDI	$\mathbb{Z}_2^2$	$0 \subseteq \mathbb{Z}^2$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,-}$	D	$\mathbb{Z}_2^2$	$0 \subseteq \mathbb{Z}_2^2$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq \mathbb{Z}^2$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	DIII	0	$0 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2^2$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,+}$	AII	$2\mathbb{Z}^2$	$0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	CII	0	$0 \subseteq 2\mathbb{Z}^2$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,-}$	C	0	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	CI	0	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}^+_{0,-}, \bar{\mathcal{U}}^+_{T/2,+}$	AI	0	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	BDI	$\mathbb{Z}$	$0 \subseteq 0$	$\mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$
$\bar{\mathcal{U}}^+_{s,+}$	D	$\mathbb{Z}_2$	$0 \subseteq \mathbb{Z}$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	DIII	$\mathbb{Z}_2$	$0 \subseteq \mathbb{Z}_2$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\bar{\mathcal{U}}^+_{0,-}, \bar{\mathcal{U}}^+_{T/2,+}$	AII	0	$0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	CII	$2\mathbb{Z}$	$0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z}^2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$	$4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$
$\bar{\mathcal{U}}^+_{s,+}$	C	0	$0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	CI	0	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,-}$	AI	$\mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	BDI	0	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,+}$	D	$\mathbb{Z}$	$0 \subseteq 0$	$2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	DIII	0	$0 \subseteq \mathbb{Z}$	$0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,-}$	AII	$\mathbb{Z}$	$0 \subseteq 0$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2$	$\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	CII	0	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,+}$	C	$\mathbb{Z}$	$0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}^2$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	CI	0	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}^+_{0,+}, \bar{\mathcal{U}}^+_{T/2,-}$	AI	$\mathbb{Z}_2$	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	BDI	$\mathbb{Z}_2$	$0 \subseteq \mathbb{Z}_2$	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}^+_{s,-}$	D	0	$0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	DIII	$2\mathbb{Z}$	$0 \subseteq 0$	$2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0$	$4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$
$\bar{\mathcal{U}}^+_{0,+}, \bar{\mathcal{U}}^+_{T/2,-}$	AII	0	$0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	CII	0	$0 \subseteq 0$	$0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\bar{\mathcal{U}}^+_{s,-}$	C	0	$0 \subseteq 0$	$0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	CI	$\mathbb{Z}$	$0 \subseteq 0$	$\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$

$K^{(0)}$ in the series is actually the one shown in the tables in the Supplemental Material [44]. The classification of intrinsic codimension-n anomalous Floquet boundary modes is then given by the quotient $\mathcal{K}^{(n)} = K^{(n-1)}/K^{(n)}$.

VIII. FLOQUET HOTIS AND SCS IN THE FREQUENCY DOMAIN

In this section we take an alternative route to connect a Floquet HOTI/SC with a nontrivial space-time symmetry/antisymmetry to a static HOTI/SC with a corresponding crystalline symmetry/antisymmetry. This connection is based on the frequency-domain formulation of the Floquet problem [42], which provides a more intuitive perspective to the results obtained by K theory.

A. Frequency-domain formulation

In the frequency-domain formulation of the Floquet problem, the quasienergies are obtained by diagonalizing the

TABLE IX. Subgroup series $K^{(d)} \subseteq \cdots \subseteq K' \subseteq K$ for three-dimensional ($d = 3$) anomalous Floquet HOTIs and SCs with a unitary order-2 space-time symmetry/antisymmetry in complex classes. The number of flipped dimensions for the symmetry/antisymmetry is denoted by $d_\parallel$.

Symmetry	Class	$d_\parallel = 0$	$d_\parallel = 1$	$d_\parallel = 2$	$d_\parallel = 3$
$\hat{\mathcal{U}}_0^+, \hat{\mathcal{U}}_{T/2}^+$	A	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}_{0,+}^+, \hat{\mathcal{U}}_{T/2,-}^+$	AIII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}_s^+$	A	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}_{0,-}^+, \hat{\mathcal{U}}_{T/2,+}^+$	AIII	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$

enlarged Hamiltonian

$$\mathscr{H}(\boldsymbol{k}, \boldsymbol{r}) = \begin{pmatrix} \ddots & & & & \\ & h_0 + \omega & h_1 & h_2 & \\ & h_1^\dagger & h_0 & h_1 & \\ & h_2^\dagger & h_1^\dagger & h_0 - \omega & \\ & & & & \ddots \end{pmatrix}, \tag{98}$$

where the matrix blocks are given by

$$h_n(\boldsymbol{k}, \boldsymbol{r}) = \frac{1}{T} \int_0^T dt\, H(\boldsymbol{k}, t) e^{-in\omega t}. \tag{99}$$

Here the appearance of the infinite-dimensional matrix $\mathscr{H}$ can be subtle and should be defined more carefully. Since later we would like to discuss the gap at $\epsilon_{\text{gap}} = \omega/2$, we will assume that the infinite-dimensional matrix $\mathscr{H}$ should be obtained by taking the limit $n \to \infty$ of a finite-dimensional matrix whose diagonal blocks are given from $h_0 + n\omega$ to $h_0 - (n-1)\omega$, with n a positive integer. With this definition, $\omega/2$ will be the particle-hole/chiral symmetric energy whenever the system has particle-hole/chiral symmetries.

As a static Hamiltonian, $\mathscr{H}(\boldsymbol{k}, \boldsymbol{r})$ has the same nonspatial symmetries as the original $H(\boldsymbol{k}, \boldsymbol{r}, t)$. Indeed, one can define the effective time-reversal $\mathscr{T}$, particle-hole $\mathscr{C}$, and chiral $\mathscr{S}$ symmetries for the enlarged Hamiltonian $\mathscr{H}(\boldsymbol{k}, \boldsymbol{r})$ as

$$\mathscr{T} = \begin{pmatrix} \ddots & & & \\ & \hat{\mathcal{T}} & & \\ & & \hat{\mathcal{T}} & \\ & & & \hat{\mathcal{T}} \\ & & & & \ddots \end{pmatrix}, \tag{100}$$

$$\mathscr{C} = \begin{pmatrix} & & & \iddots \\ & & \hat{\mathcal{C}} & \\ & \hat{\mathcal{C}} & & \\ \hat{\mathcal{C}} & & & \\ \iddots & & & \end{pmatrix}, \tag{101}$$

$$\mathscr{S} = \begin{pmatrix} & & & \iddots \\ & & \hat{\mathcal{S}} & \\ & \hat{\mathcal{S}} & & \\ \hat{\mathcal{S}} & & & \\ \iddots & & & \end{pmatrix}. \tag{102}$$

On the other hand, when the original $H(\boldsymbol{k}, \boldsymbol{r}, t)$ has a nontrivial space-time symmetry/antisymmetry, the enlarged Hamiltonian $\mathscr{H}(\boldsymbol{k}, \boldsymbol{r})$ will acquire the spatial (crystalline) symmetry/antisymmetry inherited from the spatial part of the space-time symmetry/antisymmetry.

Let us first consider $\hat{\mathcal{U}}_{T/2}$ defined in Eq. (20) for $s = T/2$, which is an unitary operation together with a half-period time

TABLE X. Same as Table IX but for antiunitary symmetries/antisymmetries.

Symmetry	Class	$d_\parallel = 0$	$d_\parallel = 1$	$d_\parallel = 2$	$d_\parallel = 3$
$\hat{\mathcal{A}}_s^+$	A	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{A}}_{0,+}^+, \hat{\mathcal{A}}_{T/2,-}^+$	AIII	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 2\mathbb{Z}$
$\bar{\mathcal{A}}_0^+, \bar{\mathcal{A}}_{T/2}^-$	A	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{A}}_{0,-}^-, \hat{\mathcal{A}}_{T/2,+}^-$	AIII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{A}}_s^-$	A	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{A}}_{0,+}^-, \hat{\mathcal{A}}_{T/2,-}^-$	AIII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$
$\bar{\mathcal{A}}_0^-, \bar{\mathcal{A}}_{T/2}^+$	A	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{A}}_{0,-}^+, \hat{\mathcal{A}}_{T/2,+}^+$	AIII	$0 \subseteq 0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$

TABLE XI. Subgroup series $K^{(d)} \subseteq \cdots \subseteq K' \subseteq K$ for three-dimensional ($d = 3$) anomalous Floquet HOTIs and SCs with a unitary order-2 space-time symmetry/antisymmetry in real classes. The number of flipped dimensions for the symmetry/antisymmetry is denoted by $d_\parallel$.

Symmetry	Class	$d_\parallel = 0$	$d_\parallel = 1$	$d_\parallel = 2$	$d_\parallel = 3$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,+}$	AI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	BDI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq \mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,-}$	D	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	DIII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,+}$	AII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$4\mathbb{Z} \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	CII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{U}}^+_{0,+}, \hat{\mathcal{U}}^+_{T/2,-}$	C	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{U}}^+_{0,++}, \hat{\mathcal{U}}^+_{T/2,+-}$	CI	$0 \subseteq 0 \subseteq 0 \subseteq 2\mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}^+_{0,-}, \bar{\mathcal{U}}^+_{T/2,+}$	AI	$0 \subseteq 0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	BDI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$
$\bar{\mathcal{U}}^+_{s,+}$	D	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	DIII	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$
$\bar{\mathcal{U}}^+_{0,-}, \bar{\mathcal{U}}^+_{T/2,+}$	AII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	CII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$
$\bar{\mathcal{U}}^+_{s,+}$	C	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	CI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,-}$	AI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	BDI	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,+}$	D	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	DIII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,-}$	AII	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	CII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-}, \hat{\mathcal{U}}^+_{T/2,+}$	C	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$4\mathbb{Z} \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$
$\hat{\mathcal{U}}^+_{0,--}, \hat{\mathcal{U}}^+_{T/2,-+}$	CI	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}^2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}^+_{0,+}, \bar{\mathcal{U}}^+_{T/2,-}$	AI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	BDI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\bar{\mathcal{U}}^+_{s,-}$	D	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	DIII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$
$\bar{\mathcal{U}}^+_{0,+}, \bar{\mathcal{U}}^+_{T/2,-}$	AII	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$
$\hat{\mathcal{U}}^+_{0,-+}, \hat{\mathcal{U}}^+_{T/2,--}$	CII	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 2\mathbb{Z} \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}_2$	$0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2^2$
$\bar{\mathcal{U}}^+_{s,-}$	C	$0 \subseteq 0 \subseteq 0 \subseteq 2\mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$	$0 \subseteq 0 \subseteq 0 \subseteq 0$
$\hat{\mathcal{U}}^+_{0,+-}, \hat{\mathcal{U}}^+_{T/2,++}$	CI	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$0 \subseteq 0 \subseteq 0 \subseteq \mathbb{Z}$	$0 \subseteq 0 \subseteq 0 \subseteq 0$	$2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq 2\mathbb{Z}^2$

translation. Since

$$\hat{\mathcal{U}}_{T/2} h_n(\boldsymbol{k}, \boldsymbol{r}) \hat{\mathcal{U}}_{T/2}^{-1} = (-1)^n h_n(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp), \tag{103}$$

the enlarged Hamiltonian thus respects a unitary spatial symmetry defined by

$$\mathscr{U} \mathscr{H}(\boldsymbol{k}, \boldsymbol{r}) \mathscr{U}^{-1} = \mathscr{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp), \tag{104}$$

where the unitary operator

$$\mathscr{U} = \begin{pmatrix} \ddots & & & \\ & \hat{\mathcal{U}}_{T/2} & & \\ & & -\hat{\mathcal{U}}_{T/2} & \\ & & & \hat{\mathcal{U}}_{T/2} \\ & & & & \ddots \end{pmatrix} \tag{105}$$

is inherited from $\hat{\mathcal{U}}_{T/2}$.

Next we consider $\bar{\mathcal{A}}_{T/2}$. Since

$$\bar{\mathcal{A}}_{T/2} h_n(\boldsymbol{k}, \boldsymbol{r}) \bar{\mathcal{A}}_{T/2}^{-1} = -(-1)^n h_{-n}(\boldsymbol{k}_\parallel, -\boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp), \tag{106}$$

we can define

$$\bar{\mathscr{A}} = \begin{pmatrix} & & & & \cdot^{\cdot^{\cdot}} \\ & & & -i\bar{\mathcal{A}}_{T/2} & \\ & & i\bar{\mathcal{A}}_{T/2} & & \\ & -i\bar{\mathcal{A}}_{T/2} & & & \\ \cdot^{\cdot^{\cdot}} & & & & \end{pmatrix} \tag{107}$$

such that

$$\bar{\mathscr{A}} \mathscr{H}(\boldsymbol{k}, \boldsymbol{r}) \bar{\mathscr{A}}^{-1} = -\mathscr{H}(\boldsymbol{k}_\parallel, -\boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp). \tag{108}$$

We now consider symmetry operators $\hat{\mathcal{A}}_{T/2}$ and $\bar{\mathcal{U}}_{T/2}$, for symmetry classes other than A, C, and D. For $\hat{\mathcal{A}}_{T/2}$ we have

$$\hat{\mathcal{A}}_{T/2} h_n(\boldsymbol{k},\boldsymbol{r})\hat{\mathcal{A}}_{T/2}^{-1} = (-1)^n h_n(\boldsymbol{k}_\parallel, -\boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp). \quad (109)$$

Thus, the enlarged Hamiltonian $\mathscr{H}(\boldsymbol{k},\boldsymbol{r})$ also has an antiunitary spatial symmetry inherited from $\hat{\mathcal{A}}_{T/2}$, given by

$$\mathscr{A}\mathscr{H}(\boldsymbol{k},\boldsymbol{r})\mathscr{A}^{-1} = \mathscr{H}(\boldsymbol{k}_\parallel, -\boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp), \quad (110)$$

where the antiunitary operator

$$\mathscr{A} = \begin{pmatrix} \ddots & & & & \\ & \hat{\mathcal{A}}_{T/2} & & & \\ & & -\hat{\mathcal{A}}_{T/2} & & \\ & & & \hat{\mathcal{A}}_{T/2} & \\ & & & & \ddots \end{pmatrix}. \quad (111)$$

Finally, for $\bar{\mathcal{U}}_{T/2}$, the enlarged Hamiltonian satisfies

$$\bar{\mathcal{U}}_{T/2} h_n(\boldsymbol{k},\boldsymbol{r})\bar{\mathcal{U}}_{T/2}^{-1} = -(-1)^n h_{-n}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp). \quad (112)$$

Hence, if we define

$$\bar{\mathscr{U}} = \begin{pmatrix} & & & \iddots \\ & & -i\bar{\mathcal{U}}_{T/2} & \\ & i\bar{\mathcal{U}}_{T/2} & & \\ -i\bar{\mathcal{U}}_{T/2} & & & \\ \iddots & & & \end{pmatrix}, \quad (113)$$

the enlarged Hamiltonian will satisfy

$$\bar{\mathscr{U}}\mathscr{H}(\boldsymbol{k},\boldsymbol{r})\bar{\mathscr{U}}^{-1} = -\mathscr{H}(-\boldsymbol{k}_\parallel, \boldsymbol{k}_\perp, -\boldsymbol{r}_\parallel, \boldsymbol{r}_\perp). \quad (114)$$

B. Harmonically driven systems

To simplify the discussion, it is helpful to restrict ourselves to a specific class of periodically driven systems, the harmonically driven ones, whose Hamiltonians have the form

$$H(\boldsymbol{k},t) = h_0(\boldsymbol{k}) + h_1(\boldsymbol{k})e^{i\omega t} + h_1^\dagger(\boldsymbol{k})e^{-i\omega t}. \quad (115)$$

To discuss the band topology at $\epsilon_{\text{gap}} = \omega/2$, we can further truncate the enlarged Hamiltonian $\mathscr{H}$ to the 2×2 block, containing two Floquet zones with energy difference ω, namely,

$$\mathscr{H}(\boldsymbol{k}) = \begin{pmatrix} h_0(\boldsymbol{k}) + \frac{\omega}{2} & h_1(\boldsymbol{k}) \\ h_1^\dagger(\boldsymbol{k}) & h_0(\boldsymbol{k}) - \frac{\omega}{2} \end{pmatrix} + \frac{\omega}{2}\rho_0, \quad (116)$$

where ρ_0 is the identity in the two-Floquet-zone basis. For later convenience, we use $\rho_{x,y,z}$ to denote the Pauli matrices of this basis. Since the last term in Eq. (116) is a shift in energy by $\omega/2$, we have a Floquet HOTI/SC at $\epsilon_{\text{gap}} = \omega/2$ if and only if the first term in Eq. (116) is a static HOTI/SC.

When restricted to the two-Floquet-zone basis, the nonspatial symmetries can be conveniently written as

$$\mathscr{T} = \rho_0\hat{\mathcal{T}}, \quad \mathscr{C} = \rho_x\hat{\mathcal{C}}, \quad \mathscr{S} = \rho_x\hat{\mathcal{S}}. \quad (117)$$

The spatial symmetries/antisymmetries for $\mathscr{H}$, which are inherited from the space-time symmetries/antisymmetries, can also be written simply as

$$\mathscr{U} = \rho_z\hat{\mathcal{U}}_{T/2}, \quad \bar{\mathscr{A}} = \rho_y\bar{\mathcal{A}}_{T/2},$$
$$\mathscr{A} = \rho_z\hat{\mathcal{A}}_{T/2}, \quad \bar{\mathscr{U}} = \rho_y\bar{\mathcal{U}}_{T/2}. \quad (118)$$

From these relations, we arrive at the same results as the ones from K theory in the previous sections. When a spatial symmetry $\mathscr{O}$, with $\mathscr{O} = \mathscr{U}, \mathscr{A}$, coexists with the particle-hole or/and chiral symmetry the operators $\mathscr{C}$, $\mathscr{S}$, and $\mathscr{O}$ will commute or anticommute with $\mathscr{C}$ or/and $\mathscr{S}$. Let us write

$$\mathscr{O}\mathscr{C} = \chi_C\mathscr{C}\mathscr{O}, \quad (119)$$

$$\mathscr{O}\mathscr{S} = \chi_S\mathscr{S}\mathscr{O}, \quad (120)$$

with $\chi_C, \chi_S = \pm1$. Because of the additional Pauli matrices $\rho_{x,y,z}$ in Eqs. (117) and (118), we have $\eta_C = -\chi_C$ and $\eta_S = -\chi_S$.

For $\mathscr{O}$, the commutation relation with respect to the time-reversal symmetry does not vary, whereas for a spatial antisymmetry $\bar{\mathscr{O}}$, with $\bar{\mathscr{O}} = \bar{\mathscr{U}}, \bar{\mathscr{A}}$, coexisting with the time-reversal symmetry, the commutation relation with respect to the latter does get switched. If we write

$$\bar{\mathscr{O}}\mathscr{T} = \chi_T\mathscr{T}\bar{\mathscr{O}}, \quad (121)$$

with $\chi_T = \pm1$, then we would have

$$\eta_T = -\chi_T, \quad (122)$$

because ρ_y is imaginary. Because of this, we can also obtain $\bar{\mathscr{A}}^2 = -\bar{\mathcal{A}}_{T/2}^2$.

IX. MODEL HAMILTONIANS FOR FLOQUET HOTIS AND SCS

In this section we introduce model Hamiltonians, which are simple but still sufficiently general, for Floquet HOTIs and SCs in all symmetry classes. In particular, we consider harmonically driven Floquet HOTIs and SCs Hamiltonians with a given nontrivial space-time symmetry/antisymmetry, realized by $\hat{\mathcal{U}}_{T/2}$, $\bar{\mathcal{A}}_{T/2}$, $\hat{\mathcal{A}}_{T/2}$, or $\bar{\mathcal{U}}_{T/2}$. One should notice that the latter two symmetries/antisymmetries are only available when the system is not in class A, C, or D, because in these classes the symmetries with $s = 0$ and $T/2$ are the same up to redefining the origin of the time coordinate.

A. Hamiltonians

The harmonically driven Floquet HOTIs and SCs in d dimensions to be constructed have Bloch Hamiltonians of the general form

$$H(\boldsymbol{k},t,m) = d_0(\boldsymbol{k},m)\Gamma_0 + \sum_{j=1}^{d} d_j(\boldsymbol{k})\Gamma_j\cos(\omega t), \quad (123)$$

where

$$d_0(\boldsymbol{k},m) = m + \sum_{j=1}^{d}(1-\cos k_j) + \cdots, \quad (124)$$

$$d_j(\boldsymbol{k}) = \sin k_j, \quad j = 1,\ldots d, \quad (125)$$

and $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}\mathbb{I}$, with $\mathbb{I}$ the identity matrix. Here the ellipsis represents $\boldsymbol{k}$-independent symmetry allowed perturbations that will in general gap out unprotected gapless modes.

We can further choose a representation of these Γ_j such that

$$\Gamma_0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} = \tau_z, \quad \Gamma_j = \begin{pmatrix} 0 & \gamma_j \\ \gamma_j^\dagger & 0 \end{pmatrix} \tag{126}$$

for $j = 1, \ldots, d$. By the transformation properties of the symmetry/antisymmetry operators, we have, in this representation, that $\hat{\mathcal{T}}$, $\hat{\mathcal{U}}_{T/2}$, and $\hat{\mathcal{A}}_{T/2}$ are block diagonal, namely, they act independently on the two subspaces with $\tau_z = \pm 1$, whereas the operators $\hat{\mathcal{C}}$, $\hat{\mathcal{S}}$, $\bar{\mathcal{U}}_{T/2}$, and $\bar{\mathcal{A}}_{T/2}$ are block off-diagonal, which couples the two subspaces.

In this representation, the enlarged Hamiltonian $\mathscr{H}(\boldsymbol{k})$ truncated to two Floquet zones, up to the constant shift $\omega/2$, can be decoupled into two sectors with $\rho_z\tau_z = \pm 1$. Hence, one can write it as a direct sum

$$\mathscr{H}(\boldsymbol{k}) = h(\boldsymbol{k}, m + \omega/2) \oplus h(\boldsymbol{k}, m - \omega/2), \tag{127}$$

with

$$h(\boldsymbol{k}, m) = d_0(\boldsymbol{k}, m)\tilde{\Gamma}_0 + \sum_{j=1}^{d} d_j(\boldsymbol{k})\tilde{\Gamma}_j. \tag{128}$$

Here the matrices $\tilde{\Gamma}_j$ have a 2×2 block structure when restricted to the $\rho_z\tau_z = \pm 1$ sectors of $\mathscr{H}(\boldsymbol{k})$. If we abuse the notation by still using $\tau_{x,y,z}$ for this 2×2 degree of freedom, we can identify $\tilde{\Gamma}_j = \Gamma_j$ for $j = 0, \ldots, d$.

It is straightforward to verify that the static Hamiltonian $h(\boldsymbol{k}, m)$ respects the same nonspatial symmetries as the harmonically driven Hamiltonian $H(\boldsymbol{k}, t, m)$, with the same symmetry operators. Moreover, if $H(\boldsymbol{k}, t, m)$ respects a nontrivial space-time symmetry, realized by $\hat{\mathcal{U}}_{T/2}$ or $\hat{\mathcal{A}}_{T/2}$, then $h(\boldsymbol{k}, m)$ will respect a spatial symmetry, realized by $\Gamma_0\hat{\mathcal{U}}_{T/2}$ or $\Gamma_0\hat{\mathcal{A}}_{T/2}$, respectively. However, if $H(\boldsymbol{k}, t, m)$ respects a nontrivial space-time antisymmetry, realized by $\bar{\mathcal{U}}_{T/2}$ or $\bar{\mathcal{A}}_{T/2}$, then $h(\boldsymbol{k}, m)$ will respect a spatial antisymmetry, realized by $-i\Gamma_0\hat{\mathcal{U}}_{T/2}$ or $-i\Gamma_0\hat{\mathcal{A}}_{T/2}$, respectively. These relations can be worked out by using the block diagonal or off-diagonal properties of the operators of space-time symmetries/antisymmetries, as well as the relations in Eq. (118).

Thus, we have established a mapping between harmonically driven Hamiltonians $H(\boldsymbol{k}, t, m)$ and static Hamiltonians $h(\boldsymbol{k}, m)$, as well as their transformation properties under symmetry/antisymmetry operators. On the other hand, $h(\boldsymbol{k}, m)$ given in Eq. (128) are well studied models for static HOTIs and SCs [23,26]. It is known that for $-2 < m < 0$, the Hamiltonian $h(\boldsymbol{k}, m)$ is in the topological phases (if the classification is nontrivial), whereas for $m > 0$ the Hamiltonian is in a trivial phase. A topological phase transition occurs at $m = 0$ with the band gap closing at $\boldsymbol{k} = 0$.

Since the enlarged Hamiltonian $\mathscr{H}(\boldsymbol{k})$, up to a constant $\omega/2$ shift, can be written as a direct sum of $h(\boldsymbol{k}, m \pm \omega/2)$, the static Hamiltonian $\mathscr{H}(\boldsymbol{k})$ will be in the topological phase (with chemical potential inside the gap at $\omega/2$) if $-2 < m - \omega/2 < 0$ and $m + \omega/2 > 0$. This is also the condition when $H(k, t, m)$ is in a Floquet topological phase at $\epsilon_{\text{gap}} = \omega/2$.

B. symmetry/antisymmetry-breaking mass terms

Let us consider $-2 < m - \omega/2 < 0$ and $m + \omega/2 > 0$. In this parameter regime, $h(\boldsymbol{k}, m + \omega/2)$ is always in a trivial insulating phase, whereas $h(\boldsymbol{k}, m - \omega/2)$ is in a nontrivial topological phase, if there exists no mass term M that respect the nonspatial symmetries, as well as the spatial symmetry/antisymmetry inherited from the space-time symmetry/antisymmetry of $H(\boldsymbol{k}, t, m)$. Here the mass term in addition satisfies $M^2 = 1$, $M = M^\dagger$, and $\{M, h(\boldsymbol{k}, m)\} = 0$. Such a mass term will gap out any gapless states that may appear in a finite-size system whose bulk is given by $h(\boldsymbol{k}, m - \omega/2)$. When M exists, one can define a term $M\cos(\omega t)$ respecting all nonspatial symmetries and the space-time symmetry/antisymmetry of $H(\boldsymbol{k}, m, t)$ and it will gap out any gapless Floquet boundary modes at quasienergy $\epsilon_{\text{gap}} = \omega/2$.

If no mass term M, which satisfies only the nonspatial symmetries irrespectively of the spatial symmetry/antisymmetry, exists, then $h(\boldsymbol{k}, m - \omega/2)$ $[H(\boldsymbol{k}, m, t)]$ is in the static (Floquet) tenfold-way topological phases, as it remains nontrivial even when the spatial (space-time) symmetry/antisymmetry is broken. Thus, the tenfold-way phases are always first-order topological phases. However, if such an M exists, $h(\boldsymbol{k}, m - \omega/2)$ $[H(\boldsymbol{k}, m, t)]$ describes a static (Floquet) purely crystalline topological phase, which can be a higher-order topological phase, and the topological protection relies on the spatial (space-time) symmetry/antisymmetry.

As pointed out in Ref. [26], several mutually anticommuting spatial-symmetry/antisymmetry-breaking mass terms M_l can exist for $h(\boldsymbol{k}, m - \omega/2)$, where M_l also anticommutes with h. Furthermore, if h has the minimum possible dimension for a given purely crystalline topological phase, then the mass terms M_l all anticommute (commute) with the spatial symmetry (antisymmetry) operator of $h(\boldsymbol{k}, m - \omega/2)$. In this case, one can relate the number of these mass terms M_l and the order of the topological phase [26]: When n mass terms M_l exist, with $l = 1, \ldots, n$, boundaries of codimension up to $\min(n, d_\parallel)$ are gapped and one has a topological phase of order $\min(n+1, d_\parallel + 1)$ if $\min(n + 1, d_\parallel + 1) \leqslant d$. However, if $\min(n + 1, d_\parallel + 1) > d$, the system does not support any protected boundary modes at any codimension. See Ref. [26] or Appendix C for a proof of this statement.

Hence, the order of the Floquet topological phase described by $H(\boldsymbol{k}, t, m)$ is reflected in the number of symmetry/antisymmetry-breaking mass terms M_l, due to the mapping between $H(\boldsymbol{k}, t, m)$ and $h(\boldsymbol{k}, m - \omega/2)$. In the following, we explicitly construct model Hamiltonians for Floquet HOTIs and SCs with a given space-time symmetry/antisymmetry.

C. First-order phase in the $d_\parallel = 0$ family

When $d_\parallel = 0$, the symmetries/antisymmetries are on-site. From Tables VI–XI we see that the on-site symmetries/antisymmetries only give rise to first-order TIs and SCs, since only the $K^{(0)}$ in the subgroup series can be nonzero. This can also be understood from the fact that $\min(n + 1, d_\parallel + 1) = 1$ in this case. We will in the following provide two examples in which we have anomalous Floquet boundary modes of codimension 1 which are protected by the unitary on-site space-time symmetry.

1. The 2D system in class AII with $\hat{\mathcal{U}}^{+}_{T/2,-}$

The simplest static topological insulator protected by unitary on-site symmetry is the quantum spin Hall insulator with additional twofold spin rotation symmetry around the z axis [11]. This system is in class AII with time-reversal symmetry $\hat{\mathcal{T}}^2 = -1$. It is known that either a static or a Floquet system of class AII in two dimensions will have a $\mathbb{Z}_2$ topological invariant [8,32]. However, with a static unitary $d_\parallel = 0$ symmetry (such as a twofold spin rotation symmetry), realized by the operator $\hat{\mathcal{U}}^{+}_{0,-}$ that squares to one and anticommutes with the time-reversal symmetry operator, a $K^{(0)} = \mathbb{Z}$ topological invariant known as the spin Chern number can be defined. In fact, such a $\mathbb{Z}$ topological invariant (see Table VIII) can also appear due to the existence of space-time symmetry realized by $\hat{\mathcal{U}}^{+}_{T/2,-}$ at quasienergy gap $\epsilon_{\text{gap}} = \omega/2$.

A lattice model that realizes a spin Chern insulator can be defined using the Bloch Hamiltonian

$$h(\boldsymbol{k}, m) = (m + 2 - \cos k_x - \cos k_y)\tau_z + (\sin k_x \tau_x s_z + \sin k_y \tau_y), \tag{129}$$

where $s_{x,y,z}$ and $\tau_{x,y,z}$ are two sets of Pauli matrices for spins and orbitals. This Hamiltonian has time-reversal symmetry realized by $\hat{\mathcal{T}} = -is_y\hat{\mathcal{K}}$ as well as the unitary symmetry realized by the operator $\hat{\mathcal{U}}^{+}_{0,-} = s_z$. When we choose an open boundary condition along x while keeping the y direction with a periodic boundary condition, there will be gapless helical edge states inside the bulk gap propagating along the x edge at $k_y = 0$ for $-2 < m < 0$. The corresponding harmonically driven Hamiltonian can be written as

$$H(\boldsymbol{k}, t, m) = (m + 2 - \cos k_x - \cos k_y)\tau_z + (\sin k_x \tau_x s_z + \sin k_y \tau_y)\cos(\omega t), \tag{130}$$

where the time-reversal and the half-period time translation on-site symmetry operators are defined as $\hat{\mathcal{T}} = -is_y\hat{\mathcal{K}}$ and $\hat{\mathcal{U}}^{+}_{T/2,-} = s_z\tau_z$ respectively.

When $-2 < m - \omega/2 < 0$ and $m + \omega/2 > 0$ are satisfied, this model supports gapless helical edge states at $k_y = 0$ inside the bulk quasienergy gap $\epsilon_{\text{gap}} = \omega/2$ when the x direction has an open boundary condition. Furthermore, such gapless Floquet edge modes persist as one introduces more perturbations that preserve the time-reversal and the $\hat{\mathcal{U}}^{+}_{T/2,-}$ symmetry.

2. The 2D system in class D with $\hat{\mathcal{U}}^{+}_{T/2,-}$

For 2D (either static or Floquet) superconductors in class D with no additional symmetries, the topological invariant is $\mathbb{Z}$ given by the Chern number of the Bogoliubov–de Gennes bands. When there exists a static unitary $d_\parallel = 0$ symmetry, realized by $\hat{\mathcal{U}}^{+}_{0,+}$ which commutes with the particle-hole symmetry operator, the topological invariant instead becomes $K^{(0)} = \mathbb{Z} \oplus \mathbb{Z}$ (see Table VIII). The same topological invariant can also be obtained from a space-time unitary symmetry realized by $\hat{\mathcal{U}}^{+}_{T/2,-}$, which anticommutes with the particle-hole symmetry operator. In the following, we construct a model Hamiltonian for such a Floquet system.

Let us start from the static 2D Hamiltonian in class D given by

$$h(\boldsymbol{k}, m) = (m + 2 - \cos k_x - \cos k_y + bs_z)\tau_z + \sin k_x s_z \tau_x + \sin k_y \tau_y, \tag{131}$$

with particle-hole symmetry and the unitary on-site symmetries realized by $\hat{\mathcal{C}} = \tau_x\hat{\mathcal{K}}$ and $\hat{\mathcal{U}}^{+}_{0,+} = s_z$, where $\tau_{x,y,z}$ are the Pauli matrices for the Nambu space. Here the unitary symmetry can be thought of as the mirror reflection with respect to the xy plane, and bs_z is the Zeeman term which breaks the time-reversal symmetry.

The $\mathbb{Z} \oplus \mathbb{Z}$ structure results from the fact that $\hat{\mathcal{U}}^{+}_{0,+}$, $\hat{\mathcal{C}}$, and $h(\boldsymbol{k}, m)$ can be simultaneously block diagonalized, according to the ± 1 eigenvalues of $\hat{\mathcal{U}}^{+}_{0,+}$. Each block is a class D system with no additional symmetries and thus has a $\mathbb{Z}$ topological invariant. Since the two blocks are independent, the topological invariant of the system should be a direct sum of the topological invariant for each block, leading to $\mathbb{Z} \oplus \mathbb{Z}$.

The harmonically driven Hamiltonian with a unitary space-time on-site symmetry realized by $\hat{\mathcal{U}}^{+}_{T/2,-} = s_z\tau_z$ can be written as

$$H(\boldsymbol{k}, t, m) = (m + 2 - \cos k_x - \cos k_y + bs_z)\tau_z + (\sin k_x s_z \tau_x - \sin k_y \tau_y)\cos(\omega t). \tag{132}$$

The particle-hole symmetry operator for this Hamiltonian is $\hat{\mathcal{C}} = \tau_x\hat{\mathcal{K}}$.

D. Second-order phase in the $d_\parallel = 1$ family

When a $d_\parallel = 1$ space-time symmetry/antisymmetry is present, the system can be at most a second-order topological phase, since the order is given by $\min(n + 1, d_\parallel + 1) \leqslant 2$. Note that the unitary symmetry in this case is the so-called time-glide symmetry, which has already been discussed thoroughly in Refs. [38,41]. We will in the following construct models for second-order topological phases with antiunitary symmetries, as well as models with unitary antisymmetries.

1. The 2D system in class AIII with $\hat{\mathcal{A}}^{+}_{T/2,-}$

For 2D systems in class AIII without any additional symmetries, the topological classification is trivial, since the chiral symmetry will set the Chern number of the occupied bands to zero. However, in Table VII we see that when the 2D system has an antiunitary symmetry realized by either $\hat{\mathcal{A}}^{+}_{0,+}$ or $\hat{\mathcal{A}}^{+}_{T/2,-}$, the K subgroup series is $0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$.

Let us first understand the $K^{(0)} = \mathbb{Z}_2$ classification in the case of $\hat{\mathcal{A}}^{+}_{0,+}$ in a static system with Hamiltonian $h(k_x, k_y)$. Let us assume that $\hat{\mathcal{A}}^{+}_{0,+}$ corresponds to the antiunitary reflection about the x axis; then we have

$$\hat{\mathcal{A}}^{+}_{0,+} h(k_x, k_y)(\hat{\mathcal{A}}^{+}_{0,+})^{-1} = h(k_x, -k_y). \tag{133}$$

On the other hand, the chiral symmetry imposes the condition

$$\hat{\mathcal{S}} h(k_x, k_y)\hat{\mathcal{S}}^{-1} = -h(k_x, k_y). \tag{134}$$

Thus, if we regard $k_x \in S^1$ as a cyclic parameter, then at every k_x, $h(k_x, k_y)$ as a function of the Bloch momentum k_y is actually a 1D system in class BDI. Thus, the topological classification in this case is the same as the one for a topological pumping for a 1D system in class BDI described by a Hamiltonian $h'(k, t)$, with momentum k and periodic time t. This gives rise to a $\mathbb{Z}_2$ topological invariant, corresponding to whether of not the fermion parity has changed after an adiabatic cycle [7], when the 1D system has an open boundary condition. Since the bulk is gapped at any t, such a fermion

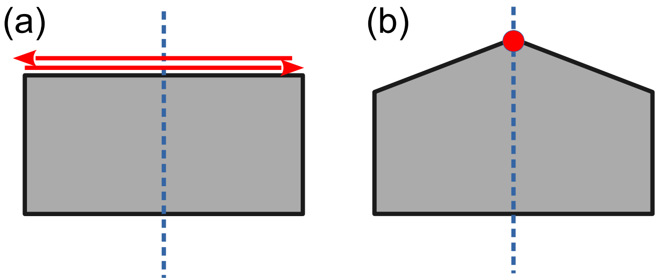

FIG. 2. (a) Gapless modes at a reflection invariant edge. (b) Corner modes at a reflection invariant corner. The dashed line indicates the reflection (time-glide) plane.

parity switch is allowed only when the boundary becomes gapless at some intermediate time t. Since our original Hamiltonian $h(k_x, k_y)$ is related to $h'(k, t)$ by replacing $k \leftrightarrow k_y$ and $t \leftrightarrow k_x$, a nontrivial phase for $h(k_x, k_y)$ implies the existence of a counterpropagating edge mode on the x edge when we choose an open boundary condition along y.

Let us explain the pure crystalline classification $K' = \mathbb{Z}_2$. One can consider the edge Hamiltonian for a pair of counterpropagating gapless modes on the edge parallel to x as $H_{\text{edge}} = k_x\sigma_z$, with $\hat{\mathcal{S}} = \sigma_x$ and $\hat{\mathcal{A}}^+_{0,+} = \hat{\mathcal{K}}$. This pair of gapless modes cannot be gapped by any mass term. However, if there exist two pairs of gapless modes, whose Hamiltonian can be written as $H_{\text{edge}} = k_x\tau_0\sigma_z$, one can then add a mass term $m\tau_y\sigma_y$ to H_{edge} to gap it out. On the other hand, if the edge does not preserve the antiunitary symmetry given by $\hat{\mathcal{A}}^+_{0,+}$, then a mass term $m\sigma_y$ can be added to gap out a single pair of gapless mode, which implies that there is no intrinsic codimension-1 boundary mode. Thus, $K' = \mathbb{Z}_2$ and $\mathcal{K}' = 0$.

Instead of intrinsic codimension-1 boundary modes, the system supports intrinsic codimension-2 boundary modes, which are referred to as a second-order TI. If one creates a corner that is invariant under the reflection $x \to -x$, this corner will support a codimension-two zero mode, with a $\mathcal{K}'' = K'/K'' = \mathbb{Z}_2$ classification.

An explicit Hamiltonian that realizes these phases can have the form

$$h(\boldsymbol{k}, m) = (m + 2 - \cos k_x \cos k_y)\tau_z + \sin k_x\tau_x\sigma_x + \sin k_y\tau_y + b\tau_z\sigma_z, \qquad (135)$$

where $\tau_{x,y,z}$ and $\sigma_{x,y,z}$ are two sets of Pauli matrices and the parameter b, which gaps out the y edge, is numerically small. One can show that this Hamiltonian has the desired chiral and antiunitary reflection symmetries given by $\hat{\mathcal{S}} = \tau_x\sigma_z$ and $\hat{\mathcal{A}}^+_{0,+} = \hat{\mathcal{K}}$, respectively. When $-2 < m < -0$, there are counterpropagating edge modes on each x edge at momentum $k_x = 0$. On the other hand, a corner, which is invariant under reflection $x \to -x$, will bound a zero mode. These two different boundary conditions are illustrated in Fig. 2.

The corresponding harmonically driven system has the Hamiltonian

$$H(\boldsymbol{k}, t, m) = (m + 2 - \cos k_x - \cos k_y + b\sigma_z)\tau_z + (\sin k_x\tau_x\sigma_x - \sin k_y\tau_y)\cos(\omega t), \qquad (136)$$

which has chiral and antiunitary time-glide (antiunitary reflection together with half-period time translation) symmetries, realized by $\hat{\mathcal{S}} = \tau_x\sigma_z$ and $\hat{\mathcal{A}}^+_{T/2,-} = \tau_z\hat{\mathcal{K}}$. With appropriately chosen boundary conditions, one can have either a counterpropagating anomalous Floquet gapless mode at the reflection symmetric edge [Fig. 2(a)] or a corner mode at $\epsilon_{\text{gap}} = \omega/2$ at the reflection symmetry corner [Fig. 2(b)].

2. The 2D system in class AI with $\bar{\mathcal{U}}^+_{T/2,-}$

For 2D systems in class AI, with only spinless time-reversal symmetry $\hat{\mathcal{T}}^2 = 1$, the topological classification is trivial. However, with a unitary (either static or space-time) $d_\parallel = 1$ antisymmetry realized by $\bar{\mathcal{U}}^+_{0,+}$ or $\bar{\mathcal{U}}^+_{T/2,-}$, the K group subseries is $0 \subseteq \mathbb{Z} \subseteq \mathbb{Z}$, as given in Table VIII.

Let us start by considering a Hamiltonian $h(k_x, k_y)$ with a static $d_\parallel = 1$ antisymmetry, given by

$$\bar{\mathcal{U}}^+_{0,+}h(k_x, k_y)(\bar{\mathcal{U}}^+_{0,+})^{-1} = -h(-k_x, k_y), \qquad (137)$$

in addition to the spinless time-reversal symmetry. At the reflection symmetric momenta $k_x = 0, \pi$, the Hamiltonian as a function of k_y reduces to a 1D Hamiltonian in class BDI, which has a $\mathbb{Z}$ winding number topological invariant.

One can also understand the topological classification from the edge perspective. At the reflection invariant edge, the x edge in this case, multiple pairs of counterpropagating edge modes can exist. One can write the edge Hamiltonian as $H_{\text{edge}} = k_x\Gamma_x + m\Gamma_m$, with a possible mass term of magnitude m. Here the matrices Γ_x and Γ_m anticommute with each other and square to the identity. Since the edge is reflection invariant, we have $[\Gamma_x, \bar{\mathcal{U}}^+_{0,+}] = 0$ and $\{\Gamma_m, \bar{\mathcal{U}}^+_{0,+}\} = 0$. Hence we can simultaneously block diagonalize Γ_x and $\bar{\mathcal{U}}^+_{0,+}$ and label the pair of gapless modes in terms of the eigenvalues ± 1 of $\bar{\mathcal{U}}^+_{0,+}$. If we denote the number of pairs of gapless modes with opposite $\bar{\mathcal{U}}^+_{0,+}$ parity by $n_\pm$, then only $(n_+ - n_-) \in \mathbb{Z}$ pairs of gapless modes are stable because the mass $m\Gamma_m$ gaps out gapless modes with opposite eigenvalues of $\bar{\mathcal{U}}^+_{0,+}$.

These gapless modes are purely protected by the $d_\parallel = 1$ antisymmetry and will be completely gapped when the edge is not invariant under reflection, which implies that $K' = K^{(0)} = \mathbb{Z}$. Indeed, we can assume that there are $(n_+ - n_-)$ pairs of gapless modes which have positive parity under $\bar{\mathcal{U}}^+_{0,+}$. The time-reversal operator can be chosen as $\hat{\mathcal{T}} = \hat{\mathcal{K}}$, because $[\hat{\mathcal{T}}, \bar{\mathcal{U}}^+_{0,+}] = 0$. We will write $\Gamma_x = \mathbb{I}_{(n_+ - n_-)} \otimes \sigma_y$, where $\mathbb{I}_n$ denotes the identity matrix of dimension n. When the edge is deformed symmetrically around a corner at $x = 0$, mass terms $m_1(x)\sigma_x + m_2(x)\sigma_z$, with $m_i(x) = -m_i(-x)$, $i = 1, 2$, can be generated. This gives rise to $(n_+ - n_-)$ zero-energy corner modes, corresponding to $\mathcal{K}' = K'/K'' = \mathbb{Z}$.

An explicit Hamiltonian for $h(k_x, k_y)$ can have the form

$$h(\boldsymbol{k}, m) = (m + 2 - \cos k_x - \cos k_y)\tau_z + \sin k_x\tau_x\sigma_y + \sin k_y\tau_y + b\tau_z\sigma_z \qquad (138)$$

with $\hat{\mathcal{T}} = \hat{\mathcal{K}}, \bar{\mathcal{U}}^+_{0,+} = \tau_x$, and numerically small b. When $-2 < m < 0$, there exist counterpropagating gapless modes on the x edges when the system has an open boundary condition in the y direction.

The corresponding harmonically driven Hamiltonian with a unitary space-time antisymmetry has the form

$$H(\boldsymbol{k},t,m) = (m+2-\cos k_x - \cos_y + b\sigma_z)\tau_z + (\sin k_x\tau_x\sigma_y - \sin k_y\tau_y)\cos(\omega t), \quad (139)$$

where the time-reversal symmetry and the unitary space-time antisymmetry are realized by $\hat{\mathcal{T}} = \hat{\mathcal{K}}$ and $\bar{\mathcal{U}}^{+}_{T/2,-} = \tau_y$, respectively. Gapless Floquet edge modes, or Floquet corner modes at $\epsilon_{\text{gap}} = \omega/2$, can be created, with appropriately chosen boundary conditions, when both $-2 < m - \omega/2 < 0$ and $m + \omega/2 > 0$ are satisfied.

E. Third-order phase in the $d_\parallel = 2$ family

When a Floquet system respects a $d_\parallel = 2$ space-time symmetry/antisymmetry, it can be at most a third-order topological phase, because $\min(n+1, d_\parallel + 1) \leqslant 3$. In the following, we construct a model Hamiltonian for a third-order TI representing such systems.

The 3D system in class AIII with $\hat{\mathcal{A}}^{+}_{T/2,-}$

It is known that for a 3D system in class AIII without any additional spatial symmetries, the topological classification is $\mathbb{Z}$ [8], which counts the number of surface Dirac cones at the boundary of the 3D insulating bulk. When there exists an antiunitary twofold rotation symmetry, either $\hat{\mathcal{A}}^{+}_{0,+}$ or $\hat{\mathcal{A}}^{+}_{T/2,-}$, the topological invariants are given by the K subgroup series $0 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2 \subseteq \mathbb{Z}_2$ in Table X.

Indeed, because of the additional symmetry realized by $\hat{\mathcal{A}}^{+}_{0,+}$ or $\hat{\mathcal{A}}^{+}_{T/2,-}$, the symmetry invariant boundary surface is able to support gapless Dirac cone pairs. As will be shown in the following, it turns out that the maximum number of such pairs is one, which gives rise to the $K^{(0)} = \mathbb{Z}_2$ topological invariant.

Let us first look at the static antiunitary twofold rotation symmetry, realized by $\hat{\mathcal{A}}^{+}_{0,+}$, which transforms a static Bloch Hamiltonian as

$$\hat{\mathcal{A}}^{+}_{0,+} h(k_x,k_y,k_z)(\hat{\mathcal{A}}^{+}_{0,+})^{-1} = h(k_x,k_y,-k_z). \quad (140)$$

With an appropriate basis, one can write $\hat{\mathcal{S}} = \tau_z$ and $\hat{\mathcal{A}}^{+}_{0,+} = \hat{\mathcal{K}}$. At the symmetry invariant boundary surface perpendicular to z, while keeping the periodic boundary condition in both the x and y directions, a single Dirac cone pair with a dispersion $h_{\text{surf}} = \tau_x(\sigma_x k_x + \sigma_z k_y)$ can exist. This Dirac cone pair cannot be gapped by an additional mass term preserving the $\hat{\mathcal{A}}^{+}_{0,+}$ symmetry, which requires the mass term to be real. However, there are two pairs of Dirac cones, described by the surface Hamiltonian $h_{\text{surf}} = \mu_0\tau_x(\sigma_x k_x + \sigma_z k_y)$, with μ_0 a 2×2 identity matrix for another spinor degree of freedom, for which we also introduce a new set of Pauli matrices $\mu_{x,y,z}$. Noticeably, a mass term which couples the two pairs of Dirac cones and gaps them out can be chosen as $\mu_y\sigma_x\tau_y$, which preserves the antiunitary twofold symmetry. Hence, we have $K^{(0)} = \mathbb{Z}_2$.

When the surface is tilted away from the rotation invariant direction, two mutually anticommuting rotation-symmetry-breaking mass terms exist and can be written as $m_1\tau_y\sigma_0 + m_2\tau_x\sigma_y$, in which $m_{1,2}$ must change sign under twofold rotation. Hence, boundaries of codimension up to $\min(n, d_\parallel) = 2$ are gapped. This leads to $\mathcal{K}'' = \mathcal{K}' = 0$, which implies that $K'' = K' = K^{(0)} = \mathbb{Z}_2$. Moreover, at the symmetry invariant corner, this mass must vanish, and thus the system can host the zero-energy corner mode.

We write the concrete model Hamiltonian with eight bands

$$h(\boldsymbol{k},m) = (m+3-\cos k_x - \cos k_y - \cos k_z)\tau_z + \sin k_x\tau_x\sigma_x + \sin k_y\tau_x\sigma_z + \sin k_z\tau_y + b_1\mu_x\tau_z\sigma_z + b_2\mu_x\tau_z\sigma_x, \quad (141)$$

where the parameters b_1 and b_2 are numerically small. Here the chiral and the antiunitary twofold rotation symmetries are realized by $\hat{\mathcal{S}} = \mu_y\tau_x\sigma_y$ and $\hat{\mathcal{A}}^{+}_{0,+} = \hat{\mathcal{K}}$, respectively. This Hamiltonian supports a single pair of Dirac cones on the boundary surfaces perpendicular to the z axis, at $k_x = k_y = 0$ for $-2 < m < 0$, as illustrated in Fig. 3(a). When the surface perpendicular to the rotation axis gets deformed from Fig. 3(a) to Fig. 3(b), the rotation invariant corner then bounds a codimension-2 boundary mode.

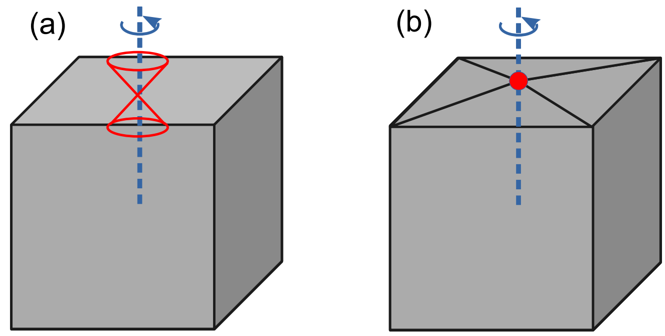

FIG. 3. (a) Gapless surface mode (Dirac cone) on the rotation invariant surface. (b) Corner mode at a rotation invariant corner. The dashed line indicates the twofold rotation (time-screw) axis.

The corresponding harmonically driven model has the Hamiltonian

$$H(\boldsymbol{k},t,m) = (m+3-\cos k_x - \cos k_y - \cos k_z + b_1\mu_x\sigma_z + b_2\mu_x\sigma_x)\tau_z + [(\sin k_x\sigma_x + \sin k_y\sigma_y)\tau_x - \sin k_z\tau_y]\cos(\omega t). \quad (142)$$

Here the chiral symmetry is realized by $\hat{\mathcal{S}} = \mu_y\tau_x\sigma_y$, while the antiunitary twofold time-screw symmetry is realized by $\hat{\mathcal{A}}^{+}_{T/2,-} = \tau_z\hat{\mathcal{K}}$. This Hamiltonian is able to support a pair of Dirac cones on the boundary surface perpendicular to the z direction inside the bulk quasienergy gap around $\epsilon_{\text{gap}} = \omega/2$ [Fig. 3(a)], as well as the codimension-2 mode with quasienergy $\omega/2$ localized at the rotation invariant corner of the system [Fig. 3(b)].

F. Higher-order topological phases in the $d_\parallel = 3$ family

Unlike the symmetries discussed previously, the $d_\parallel = 3$ symmetry (antisymmetry) operator $\hat{\mathcal{P}}$ ($\bar{\mathcal{P}}$) does not leave any point invariant in our three-dimensional world. In particular, since the surface of a 3D system naturally breaks the inversion symmetry, the topological classification of the gapless surface modes (if they exist) should be the same as the 3D tenfold classification disregarding the crystalline symmetry, in the same symmetry class. Hence, we have the boundary K group

$$\mathcal{K}' = K^{(0)}/K' = \begin{cases} K_{\mathrm{TF}} & \text{for } K_{\mathrm{TF}} \subseteq K^{(0)} \\ 0 & \text{otherwise,} \end{cases} \tag{143}$$

where K_{TF} is the corresponding K group for the tenfold-way topological phase, with only nonspatial symmetries considered.

However, inversion related pairs of boundaries with codimension larger than 1 are able to host gapless modes, which cannot be gapped out without breaking the symmetry (antisymmetry) realized by $\hat{\mathcal{P}}$ ($\bar{\mathcal{P}}$). This can be understood by simply considering the surface Hamiltonian $h(\boldsymbol{p}_\parallel, \hat{\boldsymbol{n}})$ with $\hat{\boldsymbol{n}} \in S^2$. Here $\boldsymbol{p}_\parallel$ is the momentum perpendicular to $\hat{\boldsymbol{n}}$. Let us assume that there are n spatially dependent mass terms $m_l(\hat{\boldsymbol{n}})M_l$, with $l = 1, \ldots, n$, that can gap out the surface Hamiltonian $h(\boldsymbol{p}_\parallel, \hat{\boldsymbol{n}})$. The inversion symmetry/antisymmetry restricts $m_l(\hat{\boldsymbol{n}}) = -m_l(-\hat{\boldsymbol{n}})$ (see Appendix C for details), which implies that there must exist a 1D inversion symmetric loop $S^1 \subseteq S^2$ such that $m_l(\hat{\boldsymbol{n}}) = 0$ for $\hat{\boldsymbol{n}} \in S^1$. This 1D loop for different l can be different, but they all preserve the inversion symmetry and cannot be removed. Hence, for $n = 1$, we have a 1D massless great circle, whereas for $n = 2$ we have a pair of antipodal massless points. The 1D or 0D massless region consists of irremovable topological defects which are able to host gapless modes.

Since the inversion operation maps one point to another point, the stability of the gapless modes on the massless 1D or 0D region must be protected by the nonspatial symmetries alone [24]. Hence, the codimension-k gapless modes are stable only when the $(4-k)$-dimensional system has a nontrivial tenfold classification, namely, $K_{\mathrm{TF}} \neq 0$.

Moreover, the number of these gapless modes is at most one [24]. Indeed, a system consisting of a pair of inversion symmetric systems with protected gapless modes can be deformed into a system with completely gapped boundaries without breaking the inversion symmetry. This statement can be understood by considering a pair of inversion symmetric surface Hamiltonians

$$h'(\boldsymbol{p}_\parallel, \hat{\boldsymbol{n}}) = \begin{pmatrix} h(\boldsymbol{p}_\parallel, \hat{\boldsymbol{n}}) & 0 \\ 0 & \pm h(\boldsymbol{p}_\parallel, -\hat{\boldsymbol{n}}) \end{pmatrix}, \tag{144}$$

where the $+$ $(-)$ sign is taken when we have a inversion symmetry (antisymmetry). In this situation, the $h'(\boldsymbol{p}_\parallel, \hat{\boldsymbol{n}})$ has a inversion symmetry or antisymmetry realized by

$$\hat{\mathcal{P}}' = \begin{pmatrix} 0 & \hat{\mathcal{P}} \\ \hat{\mathcal{P}} & 0 \end{pmatrix} \quad \text{or } \bar{\mathcal{P}}' = \begin{pmatrix} 0 & \bar{\mathcal{P}} \\ \bar{\mathcal{P}} & 0 \end{pmatrix}. \tag{145}$$

Now we can introduce mass terms

$$\begin{pmatrix} m_l(\hat{\boldsymbol{n}})M_l & 0 \\ 0 & -m_l(-\hat{\boldsymbol{n}})M_l \end{pmatrix}. \tag{146}$$

In this case $m_l(\hat{\boldsymbol{n}})$ can be nonzero for all $\hat{\boldsymbol{n}} \in S^2$ and therefore $h'(\boldsymbol{p}_\parallel, \hat{\boldsymbol{n}})$ can always be gapped.

Hence, we obtain the boundary K groups $\mathcal{K}^{(k)}$ which classify boundary modes of codimensions $k = 2$ and 3 as

$$\mathcal{K}^{(k)} = K^{(k-1)}/K^{(k)} \begin{cases} \mathbb{Z}_2 & \text{for } \mathbb{Z}_2 \subseteq K_{\mathrm{TF}} \text{ in } 4-k \text{ dimensions} \\ 0 & \text{otherwise.} \end{cases} \tag{147}$$

Having explained the general structure of K subgroup series, let us in the following construct model Hamiltonians for Floquet HOTIs and SCs in class DIII with a unitary space-time symmetry realized by $\hat{\mathcal{U}}^+_{T/2,++}$ ($d_\parallel = 3$), as an example. From Table XI we see that the K subgroup series is $4\mathbb{Z} \subseteq 2\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Z}^2$, which implies that we can have a first-order phase classified by $\mathcal{K}' = \mathbb{Z}^2/\mathbb{Z} = \mathbb{Z}$, a second-order phase classified by $\mathcal{K}'' = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$, and a third-order phase classified by $\mathcal{K}^{(3)} = 2\mathbb{Z}/4\mathbb{Z} = \mathbb{Z}_2$.

1. First-order topological phase

Under the operator $\hat{\mathcal{U}}^+_{T/2,++}$, no points on the surface of a 3D bulk are left invariant. Hence, the existence of codimension-1 boundary modes is due to the protection from the nonspatial symmetries alone. A tight-binding model realizing such a phase can be constructed from its static counterpart, namely, a model in class DIII with a static inversion symmetry realized by $\hat{\mathcal{U}}^+_{0,+-}$.

The static model can have the Hamiltonian

$$\begin{aligned} h_\pm(\boldsymbol{k}, m) &= (m + 3 - \cos k_x - \cos k_y - \cos k_z)\tau_z \\ &\quad \pm (\sin k_x\sigma_x + \sin k_y\sigma_y + \sin k_z\sigma_z)\tau_x, \end{aligned} \tag{148}$$

where the time-reversal, particle-hole, chiral, and inversion symmetries are realized by $\hat{\mathcal{T}} = -i\sigma_y\hat{\mathcal{K}}$, $\hat{\mathcal{C}} = \sigma_y\tau_y\hat{\mathcal{K}}$, $\hat{\mathcal{S}} = \tau_y$, and $\hat{\mathcal{U}}^+_{0,+-} = \tau_z$, respectively. When $-2 < m < 0$, this model hosts a gapless Dirac cone with chirality ±1 on any surfaces of the 3D bulk.

Hence, the Hamiltonian for the corresponding Floquet first-order topological phase with a space-time symmetry can be written as

$$\begin{aligned} H_\pm(\boldsymbol{k}, t, m) &= (m + 3 - \cos k_x - \cos k_y - \cos k_z)\tau_z \\ &\quad \pm (\sin k_x\sigma_x + \sin k_y\sigma_y + \sin k_z\sigma_z)\tau_x \cos(\omega t), \end{aligned} \tag{149}$$

where the space-time symmetry is realized by $\hat{\mathcal{U}}^+_{T/2,++} = \mathbb{I}$ and the nonspatial symmetry operators are the same as in the static model. When $-2 < m - \omega/2 < 0$ and $m + \omega/2 > 0$ are satisfied, $H_\pm(\boldsymbol{k}, t, m)$ will host a gapless Dirac cone at quasienergy $\omega/2$ with chirality ±1.

2. Second-order topological phase

Similar to the construction of the first-order phase, let us start from the corresponding static model. A static second-order phase can be obtained by $h_+(\boldsymbol{k}, m_1)$ and $h_-(\boldsymbol{k}, m_2)$. When both m_1 and m_2 are within the interval $(-2, 0)$, the topological invariant for the codimension-1 boundary modes vanishes and their exists a mass term on the surface which gaps out all boundary modes of codimension 1.

Explicitly, one can define the Hamiltonian

$$h(\boldsymbol{k}, m_1, m_2) = \begin{pmatrix} h_+(\boldsymbol{k}, m_1) & 0 \\ 0 & h_-(\boldsymbol{k}, m_2) \end{pmatrix} \tag{150}$$

and introduce a set of Pauli matrices $\mu_{x,y,z}$ for this newly introduced spinor degrees of freedom. There is only one mass term $M_l = \tau_x\mu_x$, which satisfies $\{M_1, h(\boldsymbol{k}, m_1, m_2)\} = 0$, $\{M_1, \hat{\mathcal{S}}\} = 0$, $\{M_1, \hat{\mathcal{C}}\} = 0$, and $[M_1, \hat{\mathcal{T}}] = 0$. According to the discussion on the relation between mass terms and the

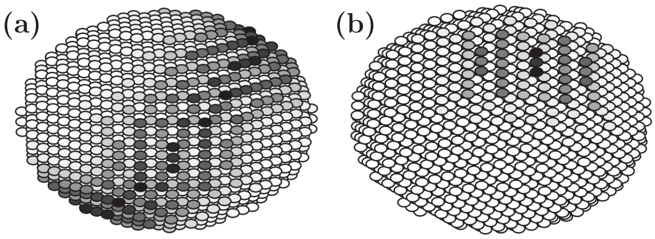

FIG. 4. Spectral weight (darkness) of the Floquet boundary mode at $\omega/2$, cut to an approximate sphere geometry with a radius of ten lattice spacing. (a) Codimension-2 boundary mode, computed with the parameters $m_1 = m_2 = 0.5$, $\omega = 3$, and $b_1^{(1)} = b_2^{(1)} = b_3^{(1)} = 0.3$. (b) Codimension-3 boundary mode, computed with $m_1 = m_2 = m_3 = m_4 = 0.5$, $\omega = 3$, and $b_1^{(1)} = b_2^{(1)} = b_3^{(1)} = b_3^{(2)} = -b_1^{(2)} = -b_2^{(2)} = 0.3$.

codimension of boundary modes in Sec. IX B, as well as Appendix C, one can add a perturbation

$$V = b_1^{(1)}\sigma_x\tau_z\mu_x + b_2^{(1)}\sigma_y\tau_z\mu_x + b_3^{(1)}\sigma_z\tau_z\mu_x \quad (151)$$

that preserves all symmetries to $h(\boldsymbol{k}, m_1, m_2)$. This perturbation gaps out all codimension-1 surfaces and leaves a codimension-2 inversion invariant loop gapless, giving rise to a second-order topological phase. The Floquet second-order topological phase can therefore be constructed by addition of the perturbation V to the Hamiltonian

$$H(\boldsymbol{k}, t, m_1, m_2) = \begin{pmatrix} H_+(\boldsymbol{k}, t, m_1) & \\ 0 & H_-(\boldsymbol{k}, t, m_2) \end{pmatrix}. \quad (152)$$

In Fig. 4(a) we show the spectral weight of the codimension-1 Floquet boundary mode at $\omega/2$, when the system is cut to an approximate sphere geometry. This boundary mode is localized on an inversion invariant loop.

3. Third-order topological phase

To construct a model for the third-order topological phase, one needs to find two anticommuting masses M_1 and M_2, which satisfy the same conditions discussed previously. This can be realized by introducing another spinor degrees of freedom, as one couples two copies of $h(\boldsymbol{k}, m_1, m_2)$. Explicitly, one can take the Hamiltonian

$$\tilde{h}(\boldsymbol{k}, m_1, m_2, m_3, m_4) = \begin{pmatrix} h(\boldsymbol{k}, m_1, m_2) & 0 \\ 0 & h(\boldsymbol{k}, m_3, m_4) \end{pmatrix} \quad (153)$$

as well as the corresponding Pauli matrices $\tilde{\mu}_{x,y,z}$ for the spinor degrees of freedom.

Thus, two anticommuting mass terms $M_1 = \tau_x\mu_x$ and $M_2 = \tau_x\mu_y\tilde{\mu}_y$ can be found. Therefore, one can introduce the symmetry preserving perturbation

$$\tilde{V} = \left(b_1^{(1)}\sigma_x + b_2^{(1)}\mu_x + b_3^{(1)}\sigma_z\right)\tau_z\mu_x + \left(b_1^{(2)}\sigma_x + b_2^{(2)}\mu_x + b_3^{(2)}\sigma_z\right)\tau_z\mu_y\tilde{\mu}_y, \quad (154)$$

which in general gaps out all boundary modes except at two antipodal points, at which codimension-3 modes can exist.

The Floquet version of such a third-order topological phase is constructed by adding the perturbation $\tilde{V}$ to the periodically driven Hamiltonian

$$\tilde{H}(\boldsymbol{k}, t, m_1, m_2, m_3, m_4) = \begin{pmatrix} H(\boldsymbol{k}, t, m_1, m_2) & 0 \\ 0 & H(\boldsymbol{k}, t, m_3, m_4) \end{pmatrix}. \quad (155)$$

In Fig. 4(b), the spectral weight of the zero-dimensional (codimension-3) Floquet modes at quasienergy $\omega/2$ is shown in a system with an approximate sphere geometry. The other zero-dimensional mode is located at the antipodal point.

X. CONCLUSION

In this work we have completed the classification of the Floquet HOTIs and SCs with an order-2 space-time symmetry/antisymmetry. By introducing a Hermitian map, we were able to map the unitary loops into Hermitian matrices and thus define bulk K groups as well as K subgroup series for unitary loops. In particular, we showed that for every order-2 nontrivial space-time (anti)unitary symmetry/antisymmetry involving a half-period time translation, there always exists a unique order-2 static spatial (anti)unitary symmetry/antisymmetry such that the two symmetries/antisymmetries share the same K group, as well as the subgroup series, and thus have the same topological classification. Further, by exploiting the frequency-domain formulation, we introduced a general recipe for constructing tight-binding model Hamiltonians for Floquet HOTIs and SCs, which provides a more intuitive way of understanding the topological classification table.

It is also worth mentioning that although in this work we only classified the Floquet HOTIs and SCs with an order-2 space-time symmetry/antisymmetry, the Hermitian map introduced here can also be used to map the classification of unitary loops involving more complicated space-time symmetry to the classification of Hamiltonians with other point group symmetries. Similarly, the frequency-domain formulation and the recipe of constructing Floquet HOTIs and SCs should also work with some modifications. In this sense, our approach can be more general than what we have shown in this work.

Finally, we comment on one possible experimental realization of Floquet HOTIs and SCs. As lattice vibrations naturally break some spatial symmetries instantaneously while preserving certain space-time symmetries, one way to engineer a Floquet HOTI/SC may involve exciting a particular phonon mode with a desired space-time symmetry, which is investigated in Ref. [45].

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APPENDIX A: EQUIVALENT CLASSIFICATION WITH SYMMETRIZED EVOLUTION OPERATORS

Let us prove the statement that the ordinary evolution operators $U_1(\boldsymbol{k},\boldsymbol{r},t)$ and $U_2(\boldsymbol{k},\boldsymbol{r},t)$ are homotopic if and only if the symmetric evolution operators $U_{\tau,1}(\boldsymbol{k},\boldsymbol{r},t)$ and $U_{\tau,2}(\boldsymbol{k},\boldsymbol{r},t)$ are homotopic.

When $U_1(\boldsymbol{k},\boldsymbol{r},t)$ and $U_2(\boldsymbol{k},\boldsymbol{r},t)$ are homotopic, there exists a continuous unitary-matrix-valued function $f(s,\boldsymbol{k},\boldsymbol{r},t)$, with $s\in[0,1]$, such that $f(0,\boldsymbol{k},\boldsymbol{r},t)=U_1(\boldsymbol{k},\boldsymbol{r},t)$ and $f(1,\boldsymbol{k},\boldsymbol{r},t)=U_2(\boldsymbol{k},\boldsymbol{r},t)$. Hence, we can define a continuous unitary-matrix-valued function $g(s,\boldsymbol{k},\boldsymbol{r},t)=f(s,\boldsymbol{k},\boldsymbol{r},\tau-t/2)f^\dagger(s,\boldsymbol{k},\boldsymbol{r},\tau+t/2)$ such that $g(0,\boldsymbol{k},\boldsymbol{r},t)=U_{\tau,1}(\boldsymbol{k},\boldsymbol{r},t)$ and $g(1,\boldsymbol{k},\boldsymbol{r},t)=U_{\tau,2}(\boldsymbol{k},\boldsymbol{r},t)$. We have that $U_{\tau,1}(\boldsymbol{k},\boldsymbol{r},t)$ and $U_{\tau,2}(\boldsymbol{k},\boldsymbol{r},t)$ are homotopic.

The other direction goes as follows. If $U_{\tau,1}(\boldsymbol{k},\boldsymbol{r},t)$ and $U_{\tau,2}(\boldsymbol{k},\boldsymbol{r},t)$ are homotopic, then there exists a continuous unitary-matrix-valued function $g(s,\boldsymbol{k},\boldsymbol{r},t)$ such that $g(0,\boldsymbol{k},\boldsymbol{r},t)=U_{\tau,1}(\boldsymbol{k},\boldsymbol{r},t)$ and $g(1,\boldsymbol{k},\boldsymbol{r},t)=U_{\tau,2}(\boldsymbol{k},\boldsymbol{r},t)$. Further, there exists another continuous unitary-matrix-valued function $f(s,\boldsymbol{k},\boldsymbol{r},t)$ such that $g(s,\boldsymbol{k},\boldsymbol{r},t)=f(s,\boldsymbol{k},\boldsymbol{r},t)f^\dagger(s,\boldsymbol{k},\boldsymbol{r},-t)$, because one requires that the symmetry property is always satisfied during the deformation when increasing s from zero to 1. Hence, we have $f(0,\boldsymbol{k},\boldsymbol{r},t)=U_1(\boldsymbol{k},\boldsymbol{r},\frac{\tau+t}{2})$ and $f(1,\boldsymbol{k},\boldsymbol{r},t)=U_2(\boldsymbol{k},\boldsymbol{r},\frac{\tau+t}{2})$. This implies that the function $f(s,\boldsymbol{k},\boldsymbol{r},2t-\tau)$ would be the continuous deformation between $U_1(\boldsymbol{k},\boldsymbol{r},t)$ and $U_2(\boldsymbol{k},\boldsymbol{r},t)$.

APPENDIX B: DECOMPOSITION OF TIME-EVOLUTION OPERATORS

In this Appendix we follow Ref. [32] to show two theorems. First, a generic time evolution can be decomposed as a unitary loop followed by a constant Hamiltonian evolution, up to homotopy. Second, $L_{\tau,1}*C_{\tau,1}\approx L_{\tau,2}*C_{\tau,2}$ if and only if $L_{\tau,1}\approx L_{\tau,2}$ and $C_{\tau,1}\approx C_{\tau,2}$, $L_{\tau,1}$ and $L_{\tau,2}$ are unitary loops, and $C_{\tau,1}$ and $C_{\tau,2}$ are constant Hamiltonian evolutions.

To prove the first theorem, let us assume that U_τ is a symmetrized time-evolution operator and H_F is its Floquet Hamiltonian. If $C_\pm(s)$ is the evolution with constant Hamiltonian $\pm sH_F$, then one can define the continuous deformation

$$f(s)=[U_\tau*C_-(s)]*C_+(s). \tag{B1}$$

We have $f(0)=U$ and $f(1)=L*C_+(1)$, which are a composition of a unitary loop followed by a constant Hamiltonian evolution.

Let us now prove the second theorem. If $L_{\tau,1}*C_{\tau,1}\approx L_{\tau,2}*C_{\tau,2}$, then there exists a continuous deformation $f(s)$ such that

$$f(0)=L_{\tau,1}*C_{\tau,1},\quad f(1)=L_{\tau,2}*C_{\tau,2}. \tag{B2}$$

If $H_F(s)$ is the corresponding Floquet Hamiltonian of the evolution $f(s)$ and $C_+(s)$ is the time-evolution operator with constant Hamiltonian $H_F(s)$, then $C_+(0)=C_{\tau,1}$ and $C_+(1)=C_{\tau,2}$, which implies $C_{\tau,1}\approx C_{\tau,2}$.

Let $g(s)=f(s)*C_-(s)$, with $C_-(s)$ the time evolution with constant Hamiltonian $-H_F(s)$; then $g(s)$ is a unitary loop for all intermediate s. Moreover, we have $g(0)=L_{\tau,1}$ and $g(1)=L_{\tau,2}$. Thus, $L_{\tau,1}\approx L_{\tau,2}$.

The proof in the opposite direction is more straightforward. If $L_{\tau,1}\approx L_{\tau,2}$ and $C_{\tau,1}\approx C_{\tau,2}$, then there exist two continuous deformations $f(s)$ and $g(s)$, which interpolate the two pairs. If we make the composition $h(s)=f(s)*g(s)$, then $h(s)$ continuously deforms $L_{\tau,1}*C_{\tau,1}$ into $L_{\tau,2}*C_{\tau,2}$.

APPENDIX C: ORDER OF HOTIS AND SCS AND SYMMETRY-BREAKING MASS TERMS

Consider static HOTIs and SCs in d dimensions described by the Hamiltonian $h(\boldsymbol{k},m)$ given in Eq. (128). Let us denote the spatial symmetry (antisymmetry) operator by $\hat{\mathcal{P}}$ ($\bar{\mathcal{P}}$) and assume that there are n mutually anticommuting M_l, with $l=1,\dots,n$, $\{M_l,h(\boldsymbol{k},m)\}=0$ and $\{M_l,\hat{\mathcal{P}}\}=0$ ($[M_l,\bar{\mathcal{P}}]=0$). We further consider a slowly position-dependent parameter $m=m(\boldsymbol{r})$, which produces a position-dependent Hamiltonian $h(\boldsymbol{k},m(\boldsymbol{r}))$. If there is a region with $m(\boldsymbol{r})<0$ and $m(\boldsymbol{r})>0$ outside this region such that the boundary defined by $m(\boldsymbol{r})=0$ is topologically the same as S^{d-1}, then there may exist gapless modes localized at the boundary. One can try to gap out the possible gapless modes, while preserving the spatial symmetry of $h(\boldsymbol{k},m(\boldsymbol{r}))$, by introducing a perturbation

$$V=i\sum_{l=1}^{n}\sum_{j=1}^{d_\parallel}b_j^{(l)}M_l\Gamma_0\Gamma_j. \tag{C1}$$

Let us focus on a point on the boundary defined by its normal unit vector $\hat{\boldsymbol{n}}$ (pointing toward the $m>0$ region). We can then define $p_\perp=\boldsymbol{k}\cdot\hat{\boldsymbol{n}}$, $\boldsymbol{p}_\parallel\cdot\hat{\boldsymbol{n}}=0$, and $x_\perp=\boldsymbol{r}\cdot\hat{\boldsymbol{n}}$. Thus, the low-energy Hamiltonian near this point at the boundary can be written as

$$h_{\text{boundary}}(\boldsymbol{p}_\parallel)=m(x_\perp)\Gamma_0+\boldsymbol{p}_\parallel\cdot\boldsymbol{\Gamma}-i(\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma})\partial_{x_\perp}, \tag{C2}$$

where $\boldsymbol{\Gamma}=(\Gamma_1,\dots,\Gamma_d)$. The wave function for a bound state of h_{boundary} can be written as

$$\psi(x_\perp,\boldsymbol{p}_\parallel)=\exp\left(-\int_0^{x_\perp}dx'm(x')\right)\tilde{\psi}(\boldsymbol{p}_\parallel). \tag{C3}$$

The gapless mode corresponds to the solution $(\Gamma_0+i\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma})\tilde{\psi}(\boldsymbol{p}_\parallel)=0$. According to this, one can define the projector into this gapless sector as

$$P(\hat{\boldsymbol{n}})=\tfrac{1}{2}[1+i(\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma})\Gamma_0]. \tag{C4}$$

Hence, we have the Hamiltonian with the additional perturbation V projected into the boundary low-energy sector

$$P(\hat{\boldsymbol{n}})[h_{\text{boundary}}(\boldsymbol{p}_\parallel)+V]P(\hat{\boldsymbol{n}})$$
$$=\boldsymbol{p}_\parallel\cdot P(\hat{\boldsymbol{n}})\boldsymbol{\Gamma}P(\hat{\boldsymbol{n}})-\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{d_\parallel}b_l^{(j)}M_l\hat{n}_j, \tag{C5}$$

where $\hat{n}_j$ is the jth component of $\hat{\boldsymbol{n}}$. Note that the second term gaps out the boundary, and we can have gapless boundary modes only at locations satisfying

$$\sum_{j=1}^{d_\parallel}b_l^{(j)}\hat{n}_j=0\,\forall\,l=1,\dots,n. \tag{C6}$$

This condition is equivalent to finding the intersection $\ker\boldsymbol{B}\cap S^{d-1}$, where $\ker\boldsymbol{B}$ denotes the kernel of matrix $\boldsymbol{B}$ whose elements are defined as $B_{ij}=b_i^{(j)}$. Since $\ker\boldsymbol{B}$ is a linear subspace of $\mathbb{R}^d$ of dimension $d-\min(n,d_\parallel)$, we find that the gapless set is given by

$$\ker \boldsymbol{B} \cap S^{d-1} = \begin{cases} S^{d-\min(n+1,d_\parallel+1)}, & \min(n+1, d_\parallel+1) \leqslant d \\ \emptyset, & \min(n+1, d_\parallel+1) > d. \end{cases} \tag{C7}$$

This means that one can have gapless boundary modes of codimension $\min(n+1, d_\parallel+1)$ if $\min(n+1, d_\parallel+1) \leqslant d$; otherwise the boundary is completely gapped.

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