Note on Decay of Homogeneous Turbulence

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The assumption of self-similarity and the existence of an exact invariant are combined to predict the decay rate of homogeneous turbulence.

ON the assumptions that turbulence remains self-similar during decay and that the "Loitsianskii integral"

$$u^2 \int_0^\infty r^4 f(r)\, dr$$

remains invariant, Kolmogorov[1] predicted for isotropic turbulence that $u^2 \propto t^{-10/7}$ and $L \propto t^{2/7}$ during decay. Here, $u = \langle u_i^2 \rangle^{\frac{1}{2}}$ is the root-mean-square velocity, $f(r)$ is the longitudinal correlation function[2], and L is an integral length scale. An alternative deriviation of the Kolmogorov decay law based on equivalent assumptions has been given recently by Comte-Bellot and Corrsin[3], who have also discussed extensively the comparison with experiment.

Now, it has been known for some time[4,5] that the Loitsianskii integral is not invariant, so that the Kolmogorov decay law is of particularly doubtful significance unless it can be shown that the change in the Loitsianskii integral is slow compared with the energy decay. However, a more important objection is that recent work by the author[6] has confirmed the speculation by Birkhoff[7] that the Loitsianskii integral is in general divergent, and that it is only for a restricted type of isotropic turbulence that the Loitsianskii integral exists.[8]

On the other hand, for general homogeneous turbulence it was found that another invariant exists, namely

$$\int_0^\infty r^2 R(r)\, dr = C, \qquad (1)$$

where

$$R(r) = \frac{1}{8\pi r^2} \int_{|\mathbf{r}|=r} R_{ii}(\mathbf{r})\, dA(\mathbf{r});$$

$R_{ij}(\mathbf{r})$ is the velocity covariance tensor, and $dA(\mathbf{r})$ is the element of area on a sphere of radius r. The equivalent statement to (1) for the energy spectrum function is that $E(k) \sim (2C/\pi)\, k^2$ as $k \to 0$, where C is a constant which will not in general be zero. For isotropic turbulence, $R(r) = \frac{1}{6}\, u^2\, (3f + rf')$; and the condition for the Loitsianskii integral to exist is $C = 0$, since from (1) $f(r) \sim (6C/u^2) r^{-3}$ as $r \to \infty$.

If we now follow Kolmogorov[1], or Comte-Bellot and Corrsin[3], but replace the invariance of the Loitsianskii integral by the invariance of (1) or the equivalent condition on $E(k)$, we find for the decay rate

$$u^2 = KC^{2/5} t^{-6/5}, \qquad L = K'C^{1/5} t^{2/5}, \qquad (2)$$

where K, K' are constants that depend upon the structure of the turbulence. A simple way of deriving these results is to write $R(r) = u^2 \psi(r/L)$ from the assumption of self-similarity and $du^2/dt = -Au^3/L$ from the further assumption of Reynolds number independence (this is basically equivalent to the assumption[1,3] that an inertial subrange exists). The results (2) follow immediately with K, K' related to A and $\int_0^\infty \rho^2\, \psi(\rho) d\rho$.

Notice that there is no need to assume that the turbulence is isotropic, but the assumption of self-similarity is of course crucial. Comparison with the experimental data[3] shows that the results (2) fit the measurements for the initial period of decay at least as well and probably better than the Kolmogorov decay law. Indeed, the agreement is much closer than the nature of the assumptions would entitle one to expect.

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[1] A. N. Kolmogorov, C. R. Akad. Sci. SSSR **30**, 301 (1941).
[2] G. K. Batchelor, *Homogeneous Turbulence* (Cambridge University Press, London, 1953).
[3] G. Comte-Bellot and S. Corrsin, J. Fluid Mech. **25**, 657 (1966).
[4] I. Proudman and W. H. Reid, Phil. Trans. Roy. Soc. **A247**, 163 (1954).
[5] G. K. Batchelor and I. Proudman, Phil. Trans. Roy. Soc. **A248**, 369 (1956).
[6] P. G. Saffman, J. Fluid Mech. **27**, 581 (1967).
[7] G. Birkhoff, Commun. Pure Appl. Math. **7**, 19 (1954).
[8] Similar conclusions have been reached independently by Professor O. M. Phillips.

Turbulent Flow Measurements Utilizing the Doppler Shift of Scattered Laser Radiation

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The probability function for the turbulent velocity in a duct flow is determined from the frequency shift of laser illumination scattered by small particles contained in the flow. From this, the mean turbulent velocity and the intensity of turbulence are obtained.

THE feasibility of measuring steady fluid velocities from the Doppler shift of scattered laser radiation was first demonstrated by Yeh and Cummins.[1]