EXISTENCE, UNIQUENESS, AND SPECTRAL PROPERTIES OF NONLINEAR EIGENVALUE PROBLEMS[1]

DONALD S. COHEN

We consider the following nonlinear eigenvalue problem:

$$(1) \qquad (p(x)u')' + \lambda f(x, u) = 0, \quad 0 \leqq x \leqq 1,$$

$$(2) \qquad a_0u(0) - a_1u'(0) = 0, \quad |a_0| + |a_1| \neq 0,$$

$$(3) \qquad b_0u(1) + b_1u'(1) = 0, \quad |b_0| + |b_1| \neq 0.$$

We suppose that $p(x)>0$ and $p'(x)$ is continuous on $0 \leqq x \leqq 1$ and that $f(x, u)$ satisfies the following conditions:

H-1: $f(x, u)$ is continuously differentiable in D:

$$0 \leqq x \leqq 1, \quad -\infty < u < \infty.$$

H-2: $0 < f_u(x, u) < \rho(x)$ on D, where $\rho(x) > 0$ in $0 \leqq x \leqq 1$,

H-3: $f(x,0) \not\equiv 0$ on $0 \leqq x \leqq 1$.

Our main result is the

THEOREM. *Let $f(x, u)$ satisfy* H-1, 2, 3, *and let the constants a_i, b_i satisfy*

$$a_i \geqq 0, \quad b_i \geqq 0, \qquad (i = 0, 1), \ a_0 + b_0 > 0.$$

Then, there exists a unique solution of (1), (2), (3) *for all λ in $0<\lambda<\mu_1\{\rho\}$, where $\mu_1\{\rho\}$ is the principal* (*i.e., least*) *eigenvalue of*

$$(4) \qquad (p(x)u')' + \mu\rho(x)u = 0, \qquad 0 \leqq x \leqq 1,$$

$$(5) \qquad a_0u(0) - a_1u'(0) = 0,$$

$$(6) \qquad b_0u(1) + b_1u'(1) = 0.$$

PROOF. We outline the proof which is based on a technique used recently by H. B. Keller [1]. The initial value problem

$$(p(x)y')' + \lambda f(x, y) = 0,$$

$$a_0y(0) - a_1y'(0) = 0,$$

$$c_0y(0) - c_1y'(0) = s, \qquad a_1c_0 - a_0c_1 = 1,$$

has the unique solution $y(s; x)$. The problem (1), (2), (3) has as many solutions as there are real roots, s^*, of

Received by the editors October 7, 1966.

[1] This work was supported by the National Science Foundation under Grant No. GP-4597 at the California Institute of Technology.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$\phi(s) \equiv b_0 y(s; 1) + b_1 y'(s; 1) = 0.$$

We shall show that $\phi'(s)$ is positive and bounded away from zero, from which it follows that $\phi(s)=0$ always has one and only one root.

Since $y(s; x)$ is continuously differentiable with respect to s, the derivative $w(x) \equiv \partial y(s; x)/\partial s$ satisfies the variational problem

$$(p(x)w')' + \lambda f_u(x, y)w = 0, \tag{7}$$

$$a_0 w(0) - a_1 w'(0) = 0, \tag{8}$$

$$c_0 w(0) - c_1 w'(0) = 1. \tag{9}$$

Clearly we must show that $\phi'(s) \equiv b_0 w(1) + b_1 w'(1)$ is positive and bounded away from zero. To do this we consider the linear problem

$$(p(x)v')' + \lambda \rho(x) v = 0, \tag{10}$$

$$a_0 v(0) - a_1 v'(0) = 0, \tag{11}$$

$$c_0 v(0) - c_1 v'(0) = 1. \tag{12}$$

For a fixed $\lambda \equiv \lambda_1$, say, let l be the first value of $x>0$ at which $b_0 v(l) + b_1 v'(l) = 0$. (That such an l exists will be clear from the formulation of problem (13), (14), (15).) Then, the unique solution $v_1(x)$ of (10), (11), (12) also satisfies

$$(p(x)v_1')' + \lambda_1 \rho(x) v_1 = 0, \tag{13}$$

$$a_0 v_1(0) - a_1 v_1'(0) = 0, \tag{14}$$

$$b_0 v_1(l) + b_1 v_1'(l) = 0. \tag{15}$$

where $\lambda_1 = \lambda_1(l)$ is the principal eigenvalue of (13), (14), (15) and $v_1(x)$ is the corresponding eigenfunction normalized so that it satisfies (12).

We now show that $b_0 w(x) + b_1 w'(x) > 0$ on $0 < x < l$. We do this by contradiction. If $b_0 w(\kappa) + b_1 w'(\kappa) = 0$ for some κ in $0 < \kappa < l$, then $w(x)$ would satisfy

$$(p(x)w')' + \lambda_1 f_u(x, y)w = 0, \tag{16}$$

$$a_0 w(0) - a_1 w'(0) = 0, \tag{17}$$

$$b_0 w(\kappa) + b_1 w'(\kappa) = 0. \tag{18}$$

Now, from the usual variational characterization [2] of the principal eigenvalue of problems of the form of (13), (14), (15), we know that as the coefficient $\rho(x)$ varies in one sense, the eigenvalue λ_1 varies in the opposite sense, and as the length of the interval varies in one sense, the eigenvalue λ_1 varies in the opposite sense. Thus, for fixed $\lambda = \lambda_1$, since $f_u(x, y) < \rho(x)$, equation (18) can not hold for $\kappa < l$. Hence, we conclude that $b_0 w(x) + b_1 w'(x) > 0$ on $0 < x < l$.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

Finally, by once again using the fact that $\lambda_1(l)$ varies in the opposite sense from l, we conclude that if $\lambda < \lambda_1 \equiv \mu_1\{\rho\}$, then $l > 1$. Therefore, $\phi'(s) \equiv b_0 w(1) + b_1 w'(1) > 0$. Q.E.D.

REMARK. Actually, condition H-3 is not necessary for our proof. However, if $f(x, 0) \equiv 0$, the unique solution will be the trivial one. If $f(x, 0) = 0$, then the problem is closely related to one treated thoroughly by G. H. Pimbley [3]. Pimbley's Theorem 1 gives uniqueness in the same range of λ. The extension to the case $f(x, 0) \neq 0$ is by no means trivial, however, and the consequences of this condition are pointed out in some detail in [4].

Recently, H. B. Keller and the present author [4] studied eigenvalue problems of a more general nature than (1), (2), (3) with regard to finding those values of λ for which the problem has positive solutions, $u(x) > 0$. Such problems arise in the theory of nonlinear heat condition where the function $f(x, u)$ is always positive for $u \geqq 0$. We defined the set $\{\lambda\}$ of real values of λ for which positive solutions exist as the spectrum of the problem, and the least upper bound of the spectrum was denoted by λ^*. For the problem (1), (2), (3) with f positive and concave we proved that a unique positive solution exists for all λ in $0 < \lambda < \lambda^*$, bounds were given on λ^*, and in the case that $\lim_{\phi \to \infty} [f_u(x, \phi)] = \rho(x) > 0$, we showed that $\lambda^* = \mu_1\{\rho\}$. Exact solutions of several simple problems involving convex nonlinearities show that more than one positive solution exists for all λ in $0 < \lambda < \lambda^*$. Upon combining the results in [4] with the theorem of the present paper we can show that *for positive convex nonlinearities* $f(x, u)$ *satisfying* $\lim_{\phi \to \infty} [f_u(x, \phi)] = \rho(x) > 0$, *a unique positive solution of* (1), (2), (3) *exists for all* λ *in* $0 < \lambda < \mu_1\{\rho\} \leqq \lambda^*$. Thus, for f convex a necessary condition for nonuniqueness is that $f_u(x, u)$ be unbounded in u.

REFERENCES

1. H. B. Keller, *Existence theory for two point boundary value problems*, Bull. Amer. Math. Soc. **72** (1966), 728–731.

2. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. I, Interscience, New York, 1953.

3. G. H. Pimbley, *A sublinear Sturm Liouville problem*, J. Math. Mech. **11** (1962), 121–138.

4. H. B. Keller and D. S. Cohen, *Some positone problems in nonlinear heat conduction*, J. Math. Mech. **16** (1967), 1361–1376.

CALIFORNIA INSTITUTE OF TECHNOLOGY

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use