*Positive Solutions of a Class of Nonlinear Eigenvalue Problems**

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1. Introduction. In this paper we shall add to the theory of the nonlinear eigenvalue problems recently studied by H. B. Keller and the present author [1]. Just as in [1] we treat equations of the form

$$Lu = \lambda f(x, u), \qquad x \in D, \tag{1.1}$$

where $x \equiv (x_1, \cdots, x_m)$ and L is the uniformly elliptic, self-adjoint, second order operator

$$Lu \equiv -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u. \tag{1.2}$$

The coefficients $a_{ij}(x) = a_{ji}(x)$ are continuously differentiable, $a_0(x) \geqq 0$, and for all unit vectors $\xi \equiv (\xi_i, \cdots, \xi_m)$,

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geqq a > 0, \qquad x \in D. \tag{1.3}$$

The boundary conditions will be taken as

$$Bu \equiv \alpha(x)u(x) + \beta(x)\frac{\partial u(x)}{\partial \nu} = 0, \qquad x \in \partial D, \tag{1.4}$$

$$\alpha(x) \geqq 0, \not\equiv 0; \qquad \beta(x) \geqq 0.$$

Here $\partial/\partial\nu$ is the conormal derivative:

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{m} n_i(x)a_{ij}(x)\frac{\partial u}{\partial x_j}, \tag{1.5}$$

where $n(x) \equiv (n_1(x), \cdots, n_m(x))$ is the outer unit normal to ∂D at x. The functions $\alpha(x)$ and $\beta(x)$ are assumed piecewise continuous on ∂D; in fact, we require

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$\alpha(x) \equiv 1$, $\beta(x) \equiv 0$ on ∂D_1 and $\beta(x)$ on ∂D_2 where $\partial D_1 + \partial D_2 = \partial D$ and the measure of ∂D_1 is positive.

In [1], for very general classes of nonlinearities $f(x, u)$, we characterized those values of λ for which the problem (1.1), (1.4) has positive solutions, we studied their dependence on λ and their "stability." (The physical motivation for considering such problems is also discussed in [1].) The main results of the present paper yield a more complete characterization of solutions of (1.1), (1.4) for one class of nonlinearity investigated in [1].

More precisely, we shall show that for functions $f(x, u)$ which are strictly concave in u, we can introduce iteration procedures, defined by solutions of *linear* equations, which "pinch" the positive solution in the sense that one sequence will converge monotonically to the unique solution from above while another sequence will converge monotonically to this solution from below. Thus, in a given problem, for example, we can obtain pointwise upper and lower bounds on the solution with the assurance that the bounds become more accurate with each iterate. In addition, we obtain bounds on the spectrum (*i.e.*, those values of λ for which positive solutions exist). These bounds were obtained in [1], but for the strictly concave nonlinearities treated here the present derivation is simpler. Our results are derived in Section 3 following the brief summary in Section 2 where we review the pertinent previously established results of Keller and Cohen [1]. Our procedure is based on combining the standard Picard iteration scheme with the quasilinearization technique introduced by R. Bellman and used so extensively by Bellman and R. Kalaba (see [2] and [3] for references) and more recently by B. Wendroff [4].

2. Summary of previously established results. In order to establish the results of the present paper, we shall need certain previously established facts. Rather than rederive these results we shall simply state them and refer the reader to the paper of H. B. Keller and D. S. Cohen where they are proved.

Under certain conditions to be imposed on $f(x, u)$ we seek those values of λ for which the boundary value problem

$$Lu = \lambda f(x, u), \qquad x \in D,$$
$$Bu = 0, \qquad x \in \partial D, \tag{2.1}$$

has positive solutions, $u(x) > 0$, $x \in D$. We call the set $\{\lambda\}$ of real values of λ for which positive solutions of (2.1) exist the spectrum of the problem (2.1), and the least upper bound of the spectrum is denoted by λ^*. The conditions to be imposed on f will be the following:

H-0. *$f(x, \phi)$ is continuous on the $m + 1$ dimensional half-cylinder $x \in D$, $\phi \geqq 0$.*

H-1. *$f(x, 0) \equiv f_0(x) > 0$ on D.*

H-2. *$f_\phi(x, \phi) > 0$ and continuous on D for $\phi > 0$.*

H-3. *$f(x, \phi)$ is strictly concave and twice differentiable in ϕ on D for $\phi > 0$.*

Note that the strict concavity condition H-3 implies the following two important properties:

$$f_u(x, \phi) < f_u(x, \psi) \quad \text{on} \quad D \quad \text{if} \quad \phi > \psi \geqq 0. \tag{2.2}$$

$$f_{uu}(x, \phi) < 0 \quad \text{on} \quad D. \tag{2.3}$$

For most of the results in [1] we imposed considerably milder restrictions on f; however, the present restrictions seem to be necessary to establish the results of Section 3.

Under the stated hypotheses H-0, 1, 2, 3 the following propositions have all been established in [1]:

P-1. *Only positive λ can be in the spectrum of* (2.1).

P-2. *Problem* (2.1) *has a unique positive solution for all λ in $0 < \lambda < \lambda^*$.*

P-3. *For any $\lambda > 0$ define the sequence $\{u_n(\lambda; x)\}$ by*

$$\begin{aligned} u_0(x) &\equiv 0, \\ Lu_n(x) &= \lambda f(x, u_{n-1}(x)), \qquad x \in D, \quad n = 1, 2, 3, \cdots, \\ Bu_n(x) &= 0, \qquad x \in \partial D. \end{aligned} \tag{2.4}$$

Then, this sequence is monotone increasing for $\lambda > 0$; that is

$$u_{n+1}(\lambda; x) > u_n(\lambda; x), \qquad x \in D, \quad n = 0, 1, 2, \cdots.$$

Furthermore, $\lambda > 0$ is in the spectrum of (2.1) *if and only if the sequence $\{u_n(\lambda; x)\}$ converges, and when this sequence does converge, its limit, say*

$$\lim_{n\to\infty} [u_n(\lambda; x)] = \mathbf{u}(\lambda; x),$$

is the unique positive solution of (2.1).

P-4. *Let $\mu_1(\lambda) \equiv \mu_1\{f_u(x, \mathbf{u}(\lambda; x))\}$ denote the principal (i.e., least) eigenvalue of*

$$\begin{aligned} L\psi - \mu f_u(x, \mathbf{u}(\lambda; x))\psi &= 0, \qquad x \in D, \\ B\psi &= 0, \qquad x \in \partial D. \end{aligned} \tag{2.5}$$

Note that $\mu_1(0) = \mu_1\{f_u(x, 0)\}$ since $\mathbf{u}(0; x) \equiv 0$ is the unique solution of (2.1) *for $\lambda = 0$. Then, the spectrum of* (2.1) *contains the interval $0 < \lambda < \mu_1(0)$, and $\lambda^* > \mu_1(\lambda) \geqq \mu_1(0)$.*

3. Characterization of positive solutions. We have seen that the sequence $\{u_n(\lambda; x)\}$ defined by (2.4) converges monotonically to the solution, $\mathbf{u}(\lambda; x)$, of (2.1) from below. We shall now introduce another scheme (also given by solutions of *linear* equations) and show that it defines a sequence, say $\{v_n(\lambda; x)\}$, which converges monotonically to the solution from above. Thus, by using both sequences, we can "pinch" $\mathbf{u}(\lambda; x)$ by solutions of linear equations as follows:

$$u_1 < u_2 < \cdots < u_n < u_{n+1} \tag{3.1}$$
$$< \cdots < \mathbf{u} \leqq \cdots \leqq v_{n+1} \leqq v_n \leqq \cdots \leqq v_2 \leqq v_1 .$$

In addition, the sequence $\{v_n(\lambda; x)\}$ has the property that, in a sense to be made more precise below, it converges to $\mathbf{u}(\lambda; x)$ faster than the sequence $\{u_n(\lambda; x)\}$.

We shall require the following basic positivity lemma which is proved in the previously mentioned work of H. B. Keller and the author [1]:

Positivity Lemma. *Let $\rho(x)$ be positive and continuous on D and let $\phi(x)$ be twice continuously differentiable and satisfy*

$$L\phi - \lambda\rho(x)\phi > 0, \qquad x \in D, \tag{3.2}$$
$$B\phi = 0, \qquad x \in \partial D.$$

Then, $\phi(x) > 0$ on D if and only if $\lambda < \mu_1$, where μ_1 is the principal eigenvalue of

$$L\psi - \mu\rho(x)\psi = 0, \qquad x \in D,$$
$$B\psi = 0, \qquad x \in \partial D.$$

Remark. In the proof of Lemma 3 below we shall need a slightly different form of the Positivity Lemma which states that if $L\phi - \lambda\rho(x)\phi \geqq 0$ on D with $B\phi = 0$ on ∂D, then $\phi(x) \geqq 0$ on D if $\lambda < \mu_1$. This somewhat weaker form of the Positivity Lemma easily follows from the proof given by Keller and Cohen.

For any $\lambda > 0$ define the sequence $\{v_n(\lambda; x)\}$ by

$$Lv_1 = \lambda[f(x, 0) + f_u(x, 0)v_1], \qquad x \in D,$$
$$Bv_1 = 0, \qquad x \in \partial D, \tag{3.3}$$
$$Lv_n = \lambda[f(x, v_{n-1}) + f_u(x, v_{n-1})(v_n - v_{n-1})], \qquad x \in D, \quad n = 2, 3, 4, \cdots,$$
$$Bv_n = 0, \qquad x \in \partial D.$$

Lemma 2. *Let $f(x, \phi)$ satisfy H-0, 1, 2, 3. Then, $v_n(\lambda; x) > 0$ on D, $n \geqq 1$, for all λ in $0 < \lambda < \mu_1\{f_u(x, 0)\}$, where $\mu_1\{f_u(x, 0)\}$ is the principal eigenvalue of*

$$L\psi - \mu f_u(x, 0)\psi = 0, \qquad x \in D, \tag{3.4}$$
$$B\psi = 0, \qquad x \in \partial D.$$

Proof. The proof is by induction. From (3.3) and H-1 we have

$$Lv_1 - \lambda f_u(x, 0)v_1 = \lambda f(x, 0) > 0 \quad \text{on} \quad D,$$
$$Bv_1 = 0 \quad \text{on} \quad \partial D.$$

Hence by the Positivity Lemma it follows that $v_1(\lambda; x) > 0$ on D since $0 < \lambda < \mu_1\{f_u(x, 0)\}$. Assume $v_\nu(\lambda; x) > 0$ on D for all $\nu \leqq n - 1$. Then,

$$Lv_n - \lambda f_u(x, v_{n-1})v_n = \lambda[f(x, v_{n-1}) - f_u(x, v_{n-1})v_{n-1}] \tag{3.5}$$
$$= \lambda[f(x, 0) - \tfrac{1}{2}f_{uu}(\xi, v_{n-1})v_{n-1}^2], \qquad 0 \leqq \xi \leqq v_{n-1},$$

where we have clearly used Taylor's Theorem. We now use (2.3) in (3.5) to to conclude that

$$Lv_n - \lambda f_u(x, v_{n-1})v_n > 0 \quad \text{on} \quad D,$$
$$Bv_n = 0 \quad \text{on} \quad \partial D.$$

Thus, by the Positivity Lemma it follows that $v_n(\lambda; x) > 0$ on D if $0 < \lambda < \mu_1\{f_u(x, v_{n-1})\}$, where $\mu_1\{f_u(x, v_{n-1})\}$ is the principal eigenvalue of

$$L\psi - \mu f_u(x, v_{n-1})\psi = 0, \qquad x \in D, \tag{3.6}$$
$$B\psi = 0, \qquad x \in \partial D.$$

To conclude the proof it remains to show that $\mu_1\{f_u(x, 0)\} < \mu_1\{f_u(x, v_n)\}$ for all $n \geqq 1$. We do this by using the variational characterization of the principal eigenvalue of problems (3.4) and (3.6). Thus,

$$\mu_1\{f_u(x, v_n)\} = \min_{\psi(x) \in \mathfrak{M}} \left[\frac{(\psi, L\psi)}{(\psi, f_u(x, v_n)\psi)}\right] \tag{3.7}$$

where the obvious inner product used is

$$(\psi, \phi) = \int_D \psi(x)\phi(x)\, dx,$$

and the class $\mathfrak{M}$ of admissible functions can be taken as

$$\mathfrak{M} = \{\psi(x) \mid \psi(x) > 0 \text{ on } D, \quad \psi(x) \in C(\bar{D}) \cap C'(D), \quad \psi(x) = 0 \text{ on } \partial D_1\}.$$

Here we recall that ∂D_1 is that portion of ∂D on which $\beta(x) \equiv 0$ and $\alpha(x) \equiv 1$. Now, (2.2) implies that $f_u(x, v_n) < f_u(x, 0)$ on D for all $n \geqq 1$. Thus, it follows from (3.7) that

$$\mu_1\{f_u(x, 0)\} < \mu_1\{f_u(x, v_n)\}, \qquad n \geqq 1,$$

which concludes the proof. Q.E.D.

Lemma 3. *Let $f(x, \phi)$ satisfy H-0, 1, 2, 3. Then, for all λ in $0 < \lambda < \mu_1\{f_u(x, 0)\}$ the sequence $\{v_n(\lambda; x)\}$ defined by* (3.3) *is monotone nonincreasing; that is,*

$$v_{n+1}(\lambda; x) \leqq v_n(\lambda; x), \qquad x \in D, \qquad n = 1, 2, 3, \cdots.$$

Proof. The strict concavity property of $f(x, u)$ implies that

$$f(x, v_n) \leqq f(x, v_{n-1}) + f_u(x, v_{n-1})(v_n - v_{n-1}).$$

This follows easily, geometrically as a consequence of the fact that f lies below its tangent, or analytically from (2.3) and a Taylor's series expansion of $f(x, v_n)$ about $v = v_{n-1}$. Thus,

$$Lv_n = \lambda[f(x, v_{n-1}) + f_u(x, v_{n-1})(v_n - v_{n-1})] \geqq \lambda f(x, v_n). \tag{3.8}$$

Now, from (3.3) we have

$$(3.9) \qquad Lv_{n+1} = \lambda[f(x, v_n) + f_u(x, v_n)(v_{n+1} - v_n)].$$

Subtracting (3.9) from (3.8), we obtain

$$(3.10) \qquad L(v_n - v_{n+1}) - \lambda f_u(x, v_{n-1})(v_n - v_{n+1}) \geqq 0 \quad \text{on} \quad D.$$

Furthermore, it is clear that $B(v_n - v_{n+1}) = 0$ on ∂D. Hence, from the remark following the Positivity Lemma we conclude that $v_n(\lambda; x) \geqq v_{n+1}(\lambda; x)$ on D if $0 < \lambda < \mu_1\{f_u(x, v_{n-1})\}$. However, as we showed in the proof of Lemma 2, we have $\mu_1\{f_u(x, 0)\} < \mu_1\{f_u(x, v_n)\}$ for all $n \geqq 1$. Therefore, we have $v_n(\lambda; x) \geqq v_{n+1}(\lambda; x)$, $x \in D$, $n = 1, 2, 3, \cdots$, for all λ in $0 < \lambda < \mu_1\{f_u(x, 0)\}$. *Q.E.D.*

Theorem. *Let $f(x, \phi)$ satisfy H-0, 1, 2, 3. Then, the sequence $\{v_n(\lambda; x)\}$ defined by* (3.3) *converges to the unique solution, $u(\lambda; x)$ of* (2.1) *for all λ in $0 < \lambda < \mu_1\{f_u(x, 0)\}$, where $\mu_1\{f_u(x, 0)\}$ is the principal eigenvalue of* (3.4).

Proof. Having showed in Lemmas 2 and 3 that the sequence $\{v_n(\lambda; x)\}$ is monotone nonincreasing and bounded from below, we may immediately conclude that there is a limit function, say

$$\lim_{n\to\infty} [v_n(\lambda; x)] = \mathbf{u}(\lambda; x).$$

To show that $\mathbf{u}(\lambda; x)$ is a solution of (2.1), we write the iteration scheme (3.3) equivalently as

$$(3.11) \qquad v_n(\lambda; x) = \lambda \int_D G(x, \xi)[f(\xi, v_{n-1}(\lambda; \xi)) + f_u(\xi, v_{n-1}(\lambda; \xi))(v_n(\lambda; \xi) - v_{n-1}(\lambda; \xi))]\, d\xi \qquad n = 2, 3, 4, \cdots,$$

where $G(x, \xi)$ is the Green's function for L on D subject to $BG = 0$ for $x \in \partial D$. Clearly, $v_n(\lambda; x) \leqq M$ on D, for all $n \geqq 1$, for some positive number M. Thus, $f(x, v_n) \leqq f(x, M)$ on D for all $n \geqq 1$. Moreover, (2.2) implies that $f_u(x, v_n) \leqq f_u(x, 0)$ on D for all $n \geqq 1$. Therefore, the integrand in (3.11) is bounded by

$$G(x, \xi)[f(\xi, M) + f_u(\xi, 0)(2M)]$$

with

$$\int_D G(x; \xi)[f(\xi, M) + f_u(\xi, 0)(2M)]\, d\xi < \infty.$$

Thus, the Lebesgue bounded convergence theorem for Riemann integrals implies that the limit can be taken under the integral in (3.11) to conclude that

$$\mathbf{u}(\lambda; x) = \lambda \int_D G(x, \xi) f(\xi, \mathbf{u}(\lambda; \xi))\, d\xi.$$

If follows that $\mathbf{u}(\lambda; x)$ is a positive solution of (2.1). Furthermore, by property P-2 of Section 2 it is unique. *Q.E.D.*

Remark. It can be shown, just as in Bellman and Kalaba [2] that the sequence $\{v_n(\lambda; x)\}$ converges quadratically; that is for $x \varepsilon D$,

$$\max_x |v_{n+1} - v_n| \leqq k_1 \max_x |v_n - v_{n-1}|^2,$$

where k_1 is a constant independent of n, whereas the Picard iteration scheme which defines the sequence $\{u_n(\lambda; x)\}$ will in general converge only geometrically; *i.e.*,

$$|u_{n+1} - u_n| \leqq k_2 |u_n - u_{n-1}|.$$

Note that we have also given another proof (*i.e.*, different from that of [1]) that the spectrum of (2.1) contains the interval $0 < \lambda < \mu_1(0) \equiv \mu_1\{f_u(x, 0)\}$ with $\lambda^* > \mu_1(0)$.

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