*New Eigenfunction Expansions and Alternative Representations for the Reduced Wave Equation**

DONALD S. COHEN

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1. Introduction. It has long been known that certain integral transforms and Fourier-type series can be used methodically for the resolution of certain kinds of classical boundary and initial value problems in separable coordinate systems. More recently, it has been shown that these classical transforms and series are spectral representations associated with an ordinary differential system which results on applying separation of variables to the given boundary value problem. This has been the basis for recent work concerned with systematically generating the proper spectral representation needed to solve a given problem. See [1] and [2] for a list of references.

We shall consider the problem of solving boundary value problems in polar and cylindrical cooredinates involving the wave equation in which the spatial terms $r^2u_{rr} + ru_r + k^2r^2u$ occur. We seek "radial" expansions of the solutions. Problems can be divided into four classes, namely, those in which the radial variable r is restricted to the intervals $[0, \infty]$, $[a, \infty]$, $[0, a]$ or $0 < a \leqq r \leqq b < \infty$. Problems where $0 \leqq r \leqq \infty$ can be treated by the Kantorovich–Lebedev transform (see [3] and [4], for example, where this is used). Problems where $a \leqq r \leqq \infty$ have been considered by Cohen [2] where it is shown that due to the non-self adjoint nature of the problem the radial eigenfunctions are not complete but that nevertheless, solutions in terms of them can be found but with restricted regions of validity. In this paper we shall consider the cases where $0 \leqq r \leqq a$ and $0 < a \leqq r \leqq b < \infty$.

D. Naylor [5] has recently considered the case where $0 \leqq r \leqq a$ and obtained results similar to the integral in (11), but involving an unspecified path L. If L is taken to be a vertical line to the right of all the zeros of the denominator, his

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integral can be converted into the form (11). His method depends upon finding one representation of the solution of a boundary value problem and converting it into a different form. This is neither desirable nor necessary. It has been shown [1], [2], [6]–[8] that one can systematically generate the proper spectral representation by considering certain naturally occurring ordinary differential systems. We do this in section 2.

In Section 3 we apply our representations to two typical boundary value problems. For one of the problems we derive a new representation for the harmonically forced motion of a circular membrane whose edge is clamped, free, or elastically bound. In another application we solve a similar problem for a pie shaped membrane.

2. The spectral representations. We shall need the spectral representations (in $L^2[0, a]$) associated with the following second order ordinary differential systems:

$$u'' + \frac{1}{r}u' + \left(k^2 - \frac{\lambda}{r^2}\right)u = 0, \qquad 0 \leqq r \leqq a, \tag{1}$$

$$\Omega u(a) = 0, \tag{2}$$

where the operator Ω, which is determined by the boundary conditions, is defined as follows:

$$\begin{aligned} &\text{Case I:} && \Omega(f)x = f(x),\\ &\text{Case II:} && \Omega f(x) = f'(x),\\ &\text{Case III:} && \Omega f(x) = f'(x) + Zf(x). \end{aligned} \tag{3}$$

Here k and Z are given constants. Note that since the equation is in the limit-point case at $r = 0$, no boundary condition at the origin is required.

By using the methods of [8], we shall first give a formal derivation of the spectral representation associated with the system (1), (2). This consists of showing that the integral

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{C_R} G(r, \rho, \lambda)\, d\lambda \tag{4}$$

converges to the Dirac delta function $\delta(r - \rho)$. Here $G(r, \rho, \lambda)$ is the Green's function for system (1), (2) given by

$$G(r, \rho, \lambda) = \frac{\pi}{2i\rho} J_{\sqrt{\lambda}}(kr_<)\left[H^{(1)}_{\sqrt{\lambda}}(kr_>) - \frac{\Omega H^{(1)}_{\sqrt{\lambda}}(ka)}{\Omega J_{\sqrt{\lambda}}(ka)} J_{\sqrt{\lambda}}(kr_>)\right], \tag{5}$$

where $r_<$ and $r_>$ are, respectively, the lesser and greater of r and ρ, and C_R is a circle of radius R about the origin in the complex λ-plane. Because of the regularity condition at the origin we choose Re $\{\sqrt{\lambda}\} > 0$. This implies that $-\pi < \arg(\lambda) < \pi$; and thus in (3) there is a branch cut along the negative real axis.

It will be convenient to let $\sqrt{\lambda} = \nu$, $-\pi < \arg(\lambda) < \pi$. Then, (4) transforms to

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\Gamma_R} \frac{\pi\nu}{i\rho} J_\nu(kr_<)\left[H_\nu^{(1)}(kr_>) - \frac{\Omega H_\nu^{(1)}(ka)}{\Omega J_\nu(ka)} J_\nu(kr_>)\right] d\nu, \tag{6}$$

where Γ_R is a closed contour in the complex ν-plane consisting of a semicircle of radius R in the right half-plane together with the imaginary axis from $-iR$ to iR. As a function of the complex variable ν the integrand in (6) has singularities only at the zeroes of $\Omega J_\nu(ka)$. It has been shown [9] that for all three definitions of Ω the zeroes ν_n $(n = 1, 2, 3, \cdots)$ of $\Omega J_\nu(ka)$, for fixed real ka, are real and simple and lie in the interval $-\infty < \nu < ka$. Furthermore, as $n \to \infty$, the zeroes ν_n are asymptotic to the negative integers, and only a finite number of the ν_n lie in the interval $0 < \nu < ka$.

Now, we assume as in [8] that the integral along the circular part of the Γ_R countour tends to $-\delta(r - \rho)$ as $R \to \infty$. Then, by the residue theorem

$$\begin{aligned}\delta(r-\rho) = &-\sum_{\nu_n>0} \frac{\pi i\nu_n}{\rho} \frac{\Omega H_{\nu_n}^{(1)}(ka)}{\left[\frac{\partial}{\partial\nu}\Omega J_\nu(ka)\right]_{\nu=\nu_n}} J_{\nu_n}(k\rho)J_{\nu_n}(kr) \\ &- \int_{-i\infty}^{i\infty} \frac{\nu}{2\rho} J_\nu(kr_<)\left[H_\nu^{(1)}(kr_>) - \frac{\Omega H_\nu^{(1)}(ka)}{\Omega J_\nu(ka)} J_\nu(kr_>)\right] d\nu,\end{aligned} \tag{7}$$

where the sum in (7) is a *finite* sum taken over the positive zeroes ν_n of $\Omega J_\nu(ka)$. We shall manipulate this into a more useful form. Write the integral in (7) as

$$\int_{-i\infty}^{i\infty} = \int_{-i\infty}^{0} + \int_{0}^{i\infty},$$

and let $\nu = -\mu$ in the first integral on the right. Then,

$$\begin{aligned}\int_{-i\infty}^{i\infty} \frac{\nu}{2\rho} J_\nu(kr_<)&\left[H_\nu^{(1)}(kr_>) - \frac{\Omega H_\nu^{(1)}(ka)}{\Omega J_\nu(ka)} J_\nu(kr_>)\right] d\nu \\ &= -\int_0^{i\infty} \frac{\nu}{2\rho} J_{-\nu}(kr_<)\left[H_{-\nu}^{(1)}(kr_>) - \frac{\Omega H_{-\nu}^{(1)}(ka)}{\Omega J_{-\nu}(ka)} J_{-\nu}(kr_>)\right] d\nu \\ &\quad + \int_0^{i\infty} \frac{\nu}{2\rho} J_\nu(kr_<)\left[H_\nu^{(1)}(kr_>) - \frac{\Omega H_\nu^{(1)}(ka)}{\Omega J_\nu(ka)} J_\nu(kr_>)\right] d\nu.\end{aligned}$$

Now, we use the facts that

$$J_{-\nu}(x) = i\sin\nu\pi H_\nu^{(1)}(x) + e^{-i\nu\pi}J_\nu(x), \qquad H_{-\nu}^{(1)}(x) = e^{i\nu\pi}H_\nu^{(1)}(x).$$

Then, after some lengthy but straightforward algebraic manipulation, we obtain that

$$\int_{-i\infty}^{i\infty} \frac{\nu}{2\rho} J_\nu(kr_<)\left[H_\nu^{(1)}(kr_<) - \frac{\Omega H_\nu^{(1)}(ka)}{\Omega J_\nu(ka)} J_\nu(kr_>)\right] d\nu = -\int_0^{i\infty} \frac{i\nu \sin \nu\pi}{2\rho\Omega J_\nu(ka)\,\Omega J_{-\nu}(ka)} s(\nu, r)s(\nu, \rho)\, d\nu, \tag{8}$$

where

$$s(\nu, x) = H_\nu^{(1)}(kx)\,\Omega J_\nu(ka) - \Omega H_\nu^{(1)}(ka) J_\nu(kx). \tag{9}$$

Finally, we substitute (8) into (7), multiply the delta function by an "arbitrary" function $f(\rho)$, and integrate with respect to ρ from 0 to a to obtain the spectral representation associated with the system (1), (2) as

$$f(r) = -\pi i \sum_{\nu_n > 0} \frac{\nu_n \Omega H_{\nu_n}^{(1)}(ka) J_{\nu_n}(kr)}{\left[\frac{\partial}{\partial\nu} \Omega J_\nu(ka)\right]_{\nu=\nu_n}} \int_0^a \frac{1}{\rho} f(\rho) J_{\nu_n}(k\rho)\, d\rho + \frac{i}{2}\int_0^{i\infty} \frac{\nu \sin \nu\pi}{\Omega J_\nu(ka)\,\Omega J_{-\nu}(ka)} s(\nu, r) \int_0^a \frac{1}{\rho} f(\rho) s(\nu, \rho)\, d\rho\, d\nu, \tag{10}$$

where $s(\nu, x)$ is given by (9).

The representation (10) has been derived only formally. The justification of essentially this method can be affected by the methods in [10] (in particular, see paragraph 4.14). That is, by using Hilbert space terminology, this problem can easily be phrased as one of finding the spectral representation for an unbounded self-adjoint differential operator in the limit-point case at the origin. For this type of operator the validity of the spectral representation is proved in general [10], and we conclude that for $f(r) \in L^2[0, a]$ our representation (10) is valid. Note that the operator of system (1), (2) has both a continuous spectrum $-\infty < \lambda \leqq 0$ and a discrete spectrum consisting of the finite number of positive zeros ν_n of $\Omega J_\nu(ka)$.

We shall find it convenient to write representation (10) as a "transform pair" in the following way:

$$f(r) = \sum_{\nu_n > 0} \beta_n J_{\nu_n}(kr) + \frac{i}{2}\int_0^{i\infty} \delta(\nu) \frac{\nu \sin \nu\pi}{\Omega J_\nu(ka)\,\Omega J_{-\nu}(ka)} s(\nu, r)\, d\nu, \tag{11}$$

where

$$\beta_n = \frac{1}{N_n}\int_0^a \frac{1}{\rho} f(\rho) J_{\nu_n}(k\rho)\, d\rho, \qquad \delta(\nu) = \int_0^a \frac{1}{\rho} f(\rho) s(\nu, \rho)\, d\rho. \tag{12}$$

Here $s(\nu, x)$ is given by (9) and N_n is the normalization constant for the eigenfunctions $J_{\nu_n}(kr)$. N_n is given by

$$N_n = \frac{-\left[\frac{\partial}{\partial\nu} \Omega J_\nu(ka)\right]_{\nu=\nu_n}}{i\pi\nu_n \Omega H_{\nu_n}^{(1)}(ka)}. \tag{13}$$

It is easy to show by the standard method that the eigenfunctions $J_{\nu_n}(kr)$ form a mutually orthogonal set on $0 \leqq r \leqq a$ with weight function $1/r$; that is,

$$\int_0^a \frac{1}{r} J_{\nu_m}(kr) J_{\nu_n}(kr)\, dr = N_n \delta_{mn} ,$$

where δ_{mn} is the Kronecker delta. This completes our derivation of the spectral representation associated with the system (1), (2).

For problems where $0 < a \leqq r \leqq b < \infty$ we consider the system

$$u'' + \frac{1}{r} u' + \left(k^2 - \frac{\lambda}{r^2}\right) u = 0, \qquad a \leqq r \leqq b,$$

$$\Omega u(a) = 0,$$

$$\Omega u(b) = 0.$$

This constitutes a non-singular self-adjoint eigenvalue problem on a finite interval for which the whole spectral theory is valid. Therefore, all the classical Sturm–Liouville type techniques and results apply. These are adequately treated in many places; accordingly we shall consider this problem no longer other than to point out that the "radial" expansion of solutions of $\Delta u + k^2 u = F$ where $0 < a \leqq r \leqq b < \infty$ can be found with as much ease as the more common angular expansions.

It is interesting to note that the functions $J_{\nu_n}(kr)$ where ν_n is a negative zero of $\Omega J_\nu(ka)$ can not be eigenfunctions. This is clear from the standard series definition of $J_\nu(x)$ which shows that if ν is negative but not an integer, then $J_\nu(x)$ is singular at $x = 0$. Thus, the functions $J_{\nu_n}(kr)$ for $\nu_n < 0$ do not satisfy the limit-point boundary condition at $r = 0$.

3. Solutions of some boundary value problems. The transforms (11), (12) are suited to solving boundary value problems involving the reduced wave equation $\Delta u + k^2 u = F$ in regions bounded by the natural coordinates of a polar or cylindrical coordinate system where the radial variable r is restricted to the interval $0 \leqq r \leqq a < \infty$; we shall solve two typical problems. We should bear in mind that even in cases where solutions are already known, our method yields alternative representations which are often more rapidly convergent and from which asymptotic expansions of solutions with respect to parameters can often be found [2].

Problem I. We seek a solution $u(r, \theta)$ of the following problem:

$$\Delta u + k^2 u = 0, \qquad 0 \leqq r \leqq a, \qquad 0 \leqq \theta \leqq \alpha < 2\pi, \tag{14}$$

$$\Omega u(a, \theta) = 0, \qquad 0 \leqq \theta \leqq \alpha, \tag{15}$$

$$u(r, \alpha) = 0, \qquad 0 < r < a, \tag{16}$$

$$u(r, 0) = f(r), \qquad 0 < r < a. \tag{17}$$

Here $f(r)$ is a prescribed function, and the boundary condition (15), where the operator Ω acts on functions of two variables, is one of the following:

$$u(a, \theta) = 0, \qquad u_r(a, \theta) = 0, \qquad u_r(a, \theta) + Zu(a, \theta) = 0. \tag{18}$$

From (11), (12) we assume that

$$u(r, \theta) = \sum_{\nu_n > 0} \beta_{1n}(\theta) J_{\nu_n}(kr) + \frac{i}{2} \int_0^{i\infty} \delta_1(\nu, \theta) \frac{\nu \sin \nu\pi}{\Omega J_\nu(ka) \Omega J_{-\nu}(ka)} s(\nu, r)\, d\nu, \tag{19}$$

$$f(r) = \sum_{\nu_n > 0} \beta_{2n} J_{\nu_n}(kr) + \frac{i}{2} \int_0^{i\infty} \delta_2(\nu) \frac{\nu \sin \nu\pi}{\Omega J_\nu(ka) \Omega J_{-\nu}(ka)} s(\nu, r)\, d\nu, \tag{20}$$

where

$$\beta_{1n}(\theta) = \frac{1}{N_n} \int_0^a \frac{1}{\rho} u(\rho, \theta) J_{\nu_n}(k\rho)\, d\rho, \qquad \delta_1(\nu, \theta) = \int_0^a \frac{1}{\rho} u(\rho, \theta) s(\nu, \rho)\, d\rho,$$

$$\beta_{2n} = \frac{1}{N_n} \int_0^a \frac{1}{\rho} f(\rho) J_{\nu_n}(k\rho)\, d\rho, \qquad \delta_2(\nu) = \int_0^a \frac{1}{\rho} f(\rho) s(\nu, \rho)\, d\rho.$$

Since $f(r)$ is a given function, the β_{2n} and $\delta_2(\nu)$ are known. It remains to determine the $\beta_{1n}(\theta)$ and $\delta_1(\nu, \theta)$. First, we find the $\beta_{1n}(\theta)$. Write (14) as

$$r^2 u_{rr} + r u_r + u_{\theta\theta} + k^2 r^2 u = 0. \tag{21}$$

We multiply (21) by $1/rJ_{\nu_n}(kr)$ and integrate with respect to r from 0 to a to obtain

$$\int_0^a r u_{rr}(r, \theta) J_{\nu_n}(kr)\, dr + \int_0^a u_r(r, \theta) J_{\nu_n}(kr)\, dr + \int_0^a \frac{1}{r} u_{\theta\theta}(r, \theta) J_{\nu_n}(kr)\, dr + \int_0^a k^2 r u(r, \theta) J_{\nu_n}(kr)\, dr = 0. \tag{22}$$

In the first term of (22) we integrate by parts twice and in the second term we integrate by parts once. Then, (22) becomes

$$\int_0^a [r^2 J''_{\nu_n}(kr) + rJ'_{\nu_n}(kr) + k^2 r^2 J_{\nu_n}(kr)] \frac{1}{r} u(r, \theta)\, dr + N_n \beta''_{1n}(\theta) = 0. \tag{23}$$

Finally, we use the differential equation which the eigenfunctions $J_{\nu_n}(kr)$ satisfy to conclude that the terms in square brackets in (23) are equal to $\nu_n^2 J_{\nu_n}(kr)$. Then, (23) becomes

$$\beta''_{1n}(\theta) + \nu_n^2 \beta_{1n}(\theta) = 0, \qquad 0 \leqq \theta \leqq \alpha,$$

which we must solve subject to the boundary conditions that

$$\beta_{1n}(\alpha) = 0, \qquad \beta_{1n}(0) = \beta_{2n}.$$

We find that

$$\beta_{1n}(\theta) = \frac{\beta_{2n}}{\sin \nu_n\alpha} \sin \nu_n(\alpha - \theta). \tag{24}$$

We now determine $\delta_1(\nu, \theta)$. We multiply (21) by $1/rs(\nu, r)$, integrate with respect to r from 0 to a, and proceed just as we did above in the determination of the $\beta_{1n}(\theta)$ to obtain

$$\frac{d^2\delta_1(\nu, \theta)}{d\theta^2} + \nu^2\delta_1(\nu, \theta) = 0, \qquad 0 \leqq \theta \leqq \alpha,$$

which we shall solve subject to the boundary conditions that

$$\delta_1(\nu, \alpha) = 0, \qquad \delta_1(\nu, 0) = \delta_2(\nu).$$

In the usual manner we find that

$$\delta_1(\nu, \theta) = \frac{\delta_2(\nu)}{\sin \nu\alpha} \sin \nu(\alpha - \theta). \tag{25}$$

Substituting (24) and (25) into (19), we obtain finally that

$$\begin{aligned} u(r, \theta) = &\sum_{\nu_n > 0} \frac{\sin \nu_n(\alpha - \theta)}{N_n \sin \nu_n\alpha} J_{\nu_n}(kr) \int_0^a \frac{1}{\rho} f(\rho) J_{\nu_n}(k\rho)\, d\rho \\ &+ \frac{i}{2} \int_0^{i\infty} \frac{\nu s(\nu, r) \sin \nu\pi \sin \nu(\alpha - \theta)}{\sin \nu\alpha \Omega J_\nu(ka)\, \Omega J_{-\nu}(ka)} \int_0^a \frac{1}{\rho} f(\rho) s(\nu, \rho)\, d\rho\, d\nu. \end{aligned} \tag{26}$$

The series in (26) is essentially a manifestation of the fact that the spectrum of the operator of system (1), (2) has a discrete part. Solutions like (26) for boundary value problems arising in diffraction theory have been given by Felsen [7] and Marcuvitz [6] who was one of the first to use these methods to systematically generate representations which were useful for large values of ka; *i.e.*, for high frequencies. For the sake of completeness we now simply give the more standard angular expansion of the solution of the problem (14)–(17). This is

$$u(r, \theta) = \frac{-\pi^2 k}{i\alpha^2} \sum_{n=1}^{\infty} n \sin \frac{n\pi\theta}{\alpha} \int_0^a \frac{f(\rho)}{\rho} T_n(r, \rho)\, d\rho,$$

where

$$T_n(r, \rho) = J_{n\pi/\alpha}(kr_<)\left[H^{(1)}_{n\pi/\alpha}(kr_>) - \frac{\Omega H^{(1)}_{n\pi/\alpha}(ka)}{\Omega J_{n\pi/\alpha}(ka)} J_{n\pi/\alpha}(kr_>)\right].$$

Problem II. We shall now derive a new representation for the solution of the problem of the harmonically forced motion of a circular membrane whose edge is clamped, free, or elastically bound. Thus, we seek the solution of the following problem:

$$\Delta u + k^2 u = F(r, \theta), \qquad 0 \leqq r \leqq a, \qquad 0 \leqq \theta \leqq 2\pi, \tag{27}$$

$$\Omega u(a, \theta) = 0, \qquad 0 \leqq \theta \leqq 2\pi. \tag{28}$$

From (11), (12) we assume that

$$u(r, \theta) = \sum_{\nu_n > 0} \beta_{1n}(\theta) J_{\nu_n}(kr) + \frac{i}{2} \int_0^{i\infty} \delta_1(\nu, \theta) \frac{\nu \sin \nu\pi}{\Omega J_\nu(ka) \Omega J_{-\nu}(ka)} s(\nu, r)\, d\nu, \tag{29}$$

where

$$\beta_{1n}(\theta) = \frac{1}{N_n} \int_0^a \frac{1}{\rho} u(\rho, \theta) J_{\nu_n}(k\rho)\, d\rho, \qquad \delta_1(\nu, \theta) = \int_0^a \frac{1}{\rho} u(\rho, \theta) s(\nu, \rho)\, d\rho. \tag{30}$$

A similar representation is assumed for $r^2 F(r, \theta)$ where functions $\beta_{2n}(\theta)$ and $\delta_2(\nu, \theta)$ replace $\beta_{1n}(\theta)$ and $\delta_1(\nu, \theta)$ and are given by (30) with $u(\rho, \theta)$ replaced by $\rho^2 F(\rho, \theta)$. Since $F(r, \theta)$ is a given function, the $\beta_{2n}(\theta)$ and $\delta_2(\nu, \theta)$ are known, and it remains to determine the $\beta_{1n}(\theta)$ and $\delta_1(\nu, \theta)$. We proceed just as we did in the preceding problem (14)–(17) to find that the $\beta_{1n}(\theta)$ satisfy

$$\begin{aligned} \beta_{1n}''(\theta) + \nu_n^2 \beta_{1n}(\theta) &= \beta_{2n}(\theta), \qquad 0 \leqq \theta \leqq 2\pi, \\ \beta_{1n}(0) &= \beta_{1n}(2\pi), \\ \beta_{1n}'(0) &= \beta_{1n}'(2\pi). \end{aligned}$$

By using standard techniques, we find that

$$\beta_{1n}(\theta) = \int_0^{2\pi} G(\theta, \phi) \beta_{2n}(\phi)\, d\phi, \tag{31}$$

where

$$G(\theta, \phi) = \frac{\cos \nu_n(\theta_< - \theta_> + \pi)}{2\nu_n \sin \nu_n \pi}.$$

Similarly, we find that

$$\delta_1(\nu, \theta) = \int_0^{2\pi} \Gamma(\theta, \phi)\, \delta_2(\nu, \phi)\, d\phi, \tag{32}$$

where

$$\Gamma(\theta, \phi) = \frac{\cos \nu(\theta_< - \theta_> + \pi)}{2\nu \sin \nu\pi}.$$

Substituting (31) and (32) into (29), we obtain finally that

$$\begin{aligned} u(r, \theta) = {} & \sum_{\nu_n > 0} \frac{J_{\nu_n}(kr)}{2N_n \nu_n \sin \nu_n \pi} \int_0^{2\pi} \int_0^a \rho F(\rho, \phi) J_{\nu_n}(k\rho) \cos \nu_n(\theta_< - \theta_> + \pi)\, d\rho\, d\phi \\ & + \frac{i}{4} \int_0^{i\infty} \frac{s(\nu, r)}{\Omega J_\nu(ka) \Omega J_{-\nu}(ka)} \int_0^{2\pi} \int_0^a \rho F(\rho, \phi) s(\nu, \rho) \cos \nu(\theta_< - \theta_> + \pi)\, d\rho\, d\phi\, d\nu. \end{aligned}$$

The standard angular expansion of the solution is

$$u(r, \theta) = \frac{-k}{4i} \sum_{n=-\infty}^{\infty} e^{in\theta} \int_0^{2\pi} \int_0^a \rho F(\rho, \phi) e^{-in\phi} U_n(r, \rho)\, d\rho\, d\phi,$$

where

$$U_n(r, \rho) = J_n(kr_<)\left[H_n^{(1)}(kr_>) - \frac{\Omega H_n^{(1)}(ka)}{\Omega J_n(ka)} J_n(kr_>)\right].$$

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Rennselaer Polytechnic Institute
Troy, New York