ON THE SPECTRUM OF ASYMPTOTIC ENTROPIES OF RANDOM WALKS

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Abstract. Given a random walk on a free group, we study the random walks it induces on the group's quotients. We show that the spectrum of asymptotic entropies of the induced random walks has no isolated points, except perhaps its maximum.

1. Introduction

Let G be a finitely generated group, and let μ be a probability measure on G. The μ-random walk on G is a time homogeneous Markov chain $g_1, g_2, \ldots$ on the state space G whose steps are distributed i.i.d. μ: for $g, h \in G$ the transition probability from g to h is $\mu(g^{-1}h)$. An important statistic of a random walk is its *Avez Asymptotic Entropy* [4]

$$h_{\mathrm{RW}}(G, \mu) := \lim_{n\to\infty} \frac{1}{n} H\left(g_n\right),$$

where $H(\cdot)$ is the Shannon entropy. The asymptotic entropy vanishes if and only if every bounded μ-harmonic function is constant; that is, if the μ-random walk has a trivial Poisson boundary [4, 30]. The asymptotic entropy is the limit of the mutual information $I(g_1; g_n)$ between the first step of the random walk and its position in later time periods, and hence quantifies the extent by which the random walk fails to have the Liouville property.

Given G and μ, and given a quotient $\Gamma = G/N$, the induced random walk $g_1N, g_2N, \ldots$ on Γ has step distribution μ_Γ, which is the push-forward of μ under the quotient map; we will simply write μ instead of μ_Γ whenever this is unambiguous. A natural question is, for a given G and μ, what values can be realized as the asymptotic random walk entropies of such quotients. This is particularly interesting when G has many quotients, and we indeed focus on the case that of $\mathbf{F}_d$, the free groups with $d \geq 2$ generators. In this case, the question can be restated as follows: for a uniformly chosen μ, what asymptotic random walk entropy values are realizable in groups with d generators?

A closely related question—and, to our knowledge, a much better studied one—is that of the spectrum of *Furstenberg entropies*. Let (X, ν) be a standard Borel space, equipped with a probability measure, and on which G acts

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by measure class preserving transformations. We say that (X,ν) is a (G,μ)-*space*, if the measure ν is μ-stationary, that is $\mu * \nu = \nu$. The Furstenberg entropy of a (G,μ)-space (X,ν) is a numerical invariant defined in [19] as

$$h_\mu(X,\nu) := \sum_{g \in G} \mu(g) \int_X -\log \frac{dg^{-1}\nu}{\nu} d\nu.$$

The *Furstenberg entropy realization problem* asks given (G,μ), what is the spectrum of the Furstenberg entropy $h_\mu(X,\nu)$, as (X,ν) varies over all ergodic μ-stationary actions of G. Kaimonovich and Vershik [30] showed that $h_\mu(X,\nu) \le h_{\mathrm{RW}}(G,\mu)$, with equality when (X,ν) is the Poisson boundary. The Poisson boundary of an induced random walk on a quotient is a (G,μ)-space, whose Furstenberg entropy is equal to the random walk's asymptotic entropy, and so every realizable random walk entropy value is also a realizable Furstenberg entropy value. One of the results due to Nevo and Zimmer [38] implies that if G is a connected semisimple Lie group with finite center and $\mathbb{R}$-rank at least 2, then the Furstenberg entropy of a stationary action of G satisfying a certain mixing condition (called P-mixing) can only take value in a finite set. This is a higher rank rigidity phenomenon: in [38] it is shown that for $\mathrm{PSL}(2,\mathbb{R})$ there exists an infinite sequence of stationary actions satisfying the P-mixing condition such that the Furstenberg entropies are all distinct. Nevo [37] shows that whenever G has property (T) then there is an $\epsilon > 0$ such that whenever $h_\mu(X,\nu) < \epsilon$ then it in fact vanishes.

In [12], Bowen showed that for the free group $\mathbf{F}_d$, $d \ge 2$, and μ uniform on the symmetric generating set $\mathbf{S} \cup \mathbf{S}^{-1}$, all values in $[0, h_{\mathrm{RW}}(\mathbf{F}_d,\mu)]$ can be realized as Furstenberg entropy of an ergodic stationary action of $\mathbf{F}_d$. The approach in [12] is to take an ergodic invariant random subgroup of G and construct an ergodic stationary system (which can be referred to as a Poisson bundle, using the terminology introduced in [29]). The Furstenberg entropy of this stationary system is then studied by considering random walk entropies on the coset spaces associated with the invariant random subgroups. Recall that an IRS is a Borel probability measure η on the Chabauty space $\mathrm{Sub}(G)$ of subgroups of G, which is invariant under conjugation by G. This term was coined in [2]. For further work on the Furstenberg entropy realization problem using the IRS-Poisson bundle approach, see [26, 27] and references therein.

One may consider restricted versions of this problem, namely the range of Furstenberg entropy restricted to smaller classes of ergodic μ-stationary systems. Of particular interest are the following two subclasses. The first is the above mentioned spectrum of random walk entropies:

$$\mathfrak{H}^{\mathrm{rw}}_\mu(G) := \{h_{\mathrm{RW}}(\Gamma,\bar{\mu}) : \Gamma \text{ is a quotient group of } G,\ \bar{\mu} \text{ is pushforward of } \mu\}.$$

The second is the spectrum of Furstenberg entropies of (G,μ)-*boundaries;* recall that these are the G-factors of the Poisson boundary of (G,μ):

$$\mathfrak{H}_\mu^{\mathrm{bnd}}(G) := \{h(X,\nu) : \ (X,\nu) \text{ is a } (G,\mu)\text{-boundary}\}.$$

One can ask the following

- **Quotient group random walk asymptotic entropy realization**: describe the set $\mathfrak{H}_\mu^{\mathrm{rw}}(G)$.
- (G,μ)**-boundary Furstenberg entropy realization:** describe the set $\mathfrak{H}_\mu^{\mathrm{bnd}}(G)$.

As we discussed above, since the Poisson boundary of (Γ,μ), where Γ is a quotient group of G, is a (G,μ)-boundary, by definitions we have $\mathfrak{H}_\mu^{\mathrm{rw}}(G) \subseteq \mathfrak{H}_\mu^{\mathrm{bnd}}(G)$. Note that if an ergodic IRS measure η is not a δ-mass supported at a normal subgroup, then the Poisson bundle constructed using the IRS η is not a quotient of the Poisson boundary of (G,μ) because of the measure-preserving factor $(\mathrm{Sub}(G),\eta)$. Hence Bowen's results do not resolve the question for Furstenberg entropies of $(\mathbf{F}_d,\mu)$-boundaries, or for asymptotic random walk entropies.

Our main result is the following. A measure μ on Γ has finite first moment if $\sum_{g\in\Gamma} |g|_S\, \mu(g) < \infty$, where $|\cdot|_S$ is the word length with respect to generating set S. With slight abuse of notation, given a measure μ on $\mathbf{F}_d$ and a quotient $\pi : \mathbf{F}_d \to \Gamma$, we write $h_{\mathrm{RW}}(\Gamma,\mu)$ for the asymptotic entropy of the $(\pi_*\mu)$-random walk on Γ.

Theorem 1.1. *Let μ be a non-degenerate probability measure with finite first moment on the free group $\mathbf{F}_d$, $d \geq 2$. Suppose Γ is a proper quotient of $\mathbf{F}_d$. Then for any $\epsilon > 0$, there exists a quotient group $\tilde{\Gamma}$ of $\mathbf{F}_d$ such that $\mathbf{F}_d \twoheadrightarrow \tilde{\Gamma} \twoheadrightarrow \Gamma$ and*

$$h_{\mathrm{RW}}(\Gamma,\mu) < h_{\mathrm{RW}}(\tilde{\Gamma},\mu) < h_{\mathrm{RW}}(\Gamma,\mu) + \epsilon$$

In particular, the set $\mathfrak{H}_\mu^{\mathrm{rw}}(\mathbf{F}_d)$ has no isolated points, except perhaps its maximum.

It follows from Theorem 1.1 that if $\mathfrak{H}_\mu^{\mathrm{rw}}(\mathbf{F}_d)$ is a closed subset in $\mathbb{R}$, then $\mathfrak{H}_\mu^{\mathrm{rw}}(\mathbf{F}_d)$ must be the full interval $[0, h_{\mathrm{RW}}(\mathbf{F}_d,\mu)]$. To the best of our knowledge, it is not known whether the sets $\mathfrak{H}_\mu^{\mathrm{rw}}(\mathbf{F}_d)$ or $\mathfrak{H}_\mu^{\mathrm{bnd}}(\mathbf{F}_d)$ are closed.

One next result is a similar statement—but slightly weaker—regarding Furstenberg entropies of boundaries.

Theorem 1.2. *In the setting of Theorem 1.1, suppose (X,ν) is a $(\mathbf{F}_d,\mu)$-boundary such that the action of $\mathbf{F}_d$ is not essentially free. Then for any $\epsilon > 0$, there exists a $(\mathbf{F}_d,\mu)$-boundary $\left(\tilde{X},\tilde{\nu}\right)$ such that*

$$h(X,\nu) < h(\tilde{X},\tilde{\nu}) < h(X,\nu) + \epsilon,$$

and (X,ν) is an $\mathbf{F}_d$-factor of $(\tilde{X},\tilde{\nu})$.

The key ingredient in the proofs of Theorems 1.1 and 1.2 is an explicit construction, which might be of independent interest, of a sequence of groups in the Chabauty space $\mathcal{G}_d$ of d-marked groups with the following properties.

Proposition 1.3. *Let μ be a non-degenerate probability measure on $\mathbf{F}_d$, $d \geq 2$, with finite first moment. Then there exists a sequence of marked groups $((\Gamma_k, S_k))_{k=1}^{\infty}$ in $\mathcal{G}_d$ such that:*

(i): *The sequence (Γ_k, S_k) converges to $(\mathbf{F}_d, \mathbf{S})$ as $k \to \infty$ in the Chabauty topology.*

(ii): *The sequence of asymptotic entropies $h_{\mathrm{RW}}(\Gamma_k, \mu) \to 0$ as $k \to \infty$.*

(iii): *For each $k \in \mathbb{N}$, Γ_k is non-amenable, has no nontrivial amenable normal subgroups, and has only countably many amenable subgroups.*

The moment condition on μ is used to bound the random walk asymptotic entropy. It seems to be an interesting question whether Proposition 1.3 remains true assuming only that μ has finite entropy.

Property (iii) in the statement above implies that the action of Γ_k on the Poisson boundary of (Γ_k, μ) is essentially free. This property is crucial for our purposes. Any sequence of d-marked finite groups with girth growing to infinity would satisfy properties (i) and (ii), but the Poisson boundaries are trivial for finite groups.

We construct the sequence of marked groups as stated via taking extensions of the Fabrykowski-Gupta group. In the proof, a result of Bartholdi and Erschler [7] on almost identities in weakly branch groups are applied. Necessary terminology and background are reviewed in Section 2. Provided the sequence of marked groups stated in Proposition 1.3, the proof of Theorem 1.1 is completed by taking suitable diagonal product of groups or stationary joinings of μ-boundaries, see Section 4.

The same kind of construction implies the following result on spectral radii of symmetric random walks. Recall that the spectral radius of a μ-random walk on Γ is defined as

$$\rho(\Gamma, \mu) = \limsup_{n\to\infty} \mu^{(n)}(id_\Gamma)^{\frac{1}{n}}.$$

Theorem 1.4. *Let μ be a symmetric non-degenerate probability measure on the free group $\mathbf{F}_d$, $d \geq 2$. Suppose Γ is a proper marked quotient of $\mathbf{F}_d$. Then for any $\epsilon > 0$, there exists a quotient group $\tilde{\Gamma}$ of $\mathbf{F}_d$ such that $\mathbf{F}_d \twoheadrightarrow \tilde{\Gamma} \twoheadrightarrow \Gamma$ and*

$$\rho(\Gamma, \mu) - \epsilon < \rho(\tilde{\Gamma}, \mu) < \rho(\Gamma, \mu).$$

Using a diagonal product of marked groups is similar to the construction in [31]. A result of Kassabov and Pak [32] states that the set of the spectral radii $\{\rho(\Gamma, \mu) : \Gamma \text{ is a quotient of } \mathbf{F}_d\}$ contains a subset homeomorphic to the Cantor set. The same construction shows that the set $\mathfrak{H}_\mu^{\mathrm{rw}}(\mathbf{F}_d)$ contains a subset homeomorphic to the Cantor set as well. It is not known whether this set of spectral radii is closed.

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2. Preliminaries

2.1. **(G,μ)-boundaries.** In this note we only consider countable groups. A probability measure μ on G is non-degenerate if $\operatorname{supp}\mu$ generates G as a semigroup. For a countable group G, we say a Lesbesgue space (X,ν) is a G-space, if G acts measurably on X and the probability measure ν is quasi-invariant with respect to the G-action. A G-space (X,ν) is ergodic if a G-invariant subset is either null or conull. A measurable map $\pi:(X,\nu)\to(Y,\eta)$ is called a *G-map* if it is G-equivariant and η is the pushforward of ν under π.

Given a probability measure μ on G, let $\Omega=G^{\mathbb{N}}$ be the path space, $\mathbb{P}_\mu$ be the law of the μ-random walk starting at id, and $\mathcal{I}$ be the σ-field on Ω that is invariant under time shifts. The Poisson boundary of (G,μ) is denoted by the measure space $(B,\mathcal{F},\nu_B)$ together with a G-map $\mathbf{b}:(\Omega,\mathcal{I},\mathbb{P}_\mu)\to(B,\mathcal{F},\nu_B)$, where $\mathbf{b}^{-1}\mathcal{F}=\mathcal{I}$ up to null sets with respect to $\mathbb{P}_\mu$, and the σ-algebra $\mathcal{F}$ is countably generated and separating points. The existence and uniqueness up to isomorphism of the Poisson boundary of (G,μ) was shown by Furstenberg [19, 20, 21]. The G-action on the Poisson boundary (B,ν_B) is ergodic, and in fact doubly ergodic, by Kaimanovich [28].

We use the notation (B,ν_B) to denote a compact model of the Poisson boundary of (G,μ), which exists by the Mackey realization [35]. A (G,μ)-boundary (X,ν) is defined to be a G-factor of (B,ν_B). Moreover, the factor map $(B,\nu_B)\to(X,\nu)$ is essentially unique, see [5, Theorem 2.14], and we will denote it by $\boldsymbol{\beta}_X$.

Denote by $P(X)$ the space of Borel probability measures on the compact space X. A factor map $\pi:(Y,\eta)\to(X,\nu)$ gives a unique disintegration map $D_\pi:X\to P(Y)$ such that for ν-a.e. $x\in X$, $D_\pi(x)$ is supported on the fiber of x and $\int_X D_\pi(x)d\nu(x)=\eta$. We say (Y,η) is a *relatively measure preserving extension* of X if D_π is G-equivariant, that is $D_\pi(g\cdot x)=g\cdot D_\pi(x)$.

We will need the following properties regarding Furstenberg entropy and relatively measure preserving extensions.

Proposition 2.1 ([38, Proposition 1.9])**.** *Let $\pi:(Y,\eta)\to(X,\nu)$ be a G-factor map. Suppose $h(X,\nu)<\infty$ and $h(Y,\eta)=h(X,\nu)$. Then (Y,η) is a relative measure preserving extension of (X,ν).*

Lemma 2.2 ([5, Corollary 2.20])**.** *Let $\pi:(Y,\eta)\to(X,\nu)$ be a relatively measure-preserving extension of two (G,μ)-boundaries. Then $(Y,\eta)=(X,\nu)$.*

2.2. **Chabauty space of marked groups and convergence to the free group.** Denote by $\mathcal{G}_d$ the space of d-generated groups (G,S), where $S=(s_1,\ldots,s_d)$ is a generating tuple, equipped with the *Chabauty-Grigorchuk topology.* We refer to the pair (G,S) as a marked group and $\mathcal{G}_d$ the Chabauty space of d-marked groups. Recall that in this topology, two marked groups

(G_1, S_1) and (G_2, S_2) are close if marked balls of large radius in the Cayley graphs of (G_1, S_1) and (G_2, S_2) around the identities are isomorphic.

Following [7], for G a finitely generated group and $(H, T) \in \mathcal{G}_d$, we say G *preforms* (H, T) if there exists a sequence of d-markings $(S_n)_{n\in\mathbb{N}}$ such that (G, S_n) converges to (H, T) in the Chabauty-Grigorchuk topology.

We will use groups that preforms the free group $(\mathbf{F}_d, \mathbf{S})$, where $\mathbf{S} = (\mathbf{s}_1, \ldots, \mathbf{s}_d)$ consists of the free generators. Let G be a d-generated group. Following the definition in Akhmedov [3] and Olshanskii-Sapir [39], we say a non-trivial word $w(x_1, \ldots, x_d)$ is a d-almost-identity for G, if the identity $w(g_1, \ldots, g_d) = 1$ is satisfied for any d-generating tuple $(g_1, \ldots, g_d)$. By [39, Theorem 9], G preforms $(\mathbf{F}_d, \mathbf{S})$ if G is d-generated and satisfies no d-almost identity.

In [1], Abért gives a general criterion for a group to satisfy no identity. Suppose $G \curvearrowright X$ by permutations. We say G separates X, if for every finite subset Y of X, the pointwise fixator $G_Y = \{g \in G : y \cdot g = y \text{ for all } y \in Y\}$ has no fixed point outside Y. Abért shows that if G separate X then G satisfies no identity. Bartholdi and Erschler [7] provides a criterion for absence of almost-identities: under the additional assumption that the Frattini subgroup $\Phi(G)$ has finite index in G, the condition in Abért's criterion implies that G satisfies no almost-identity.

Weakly branch groups provide examples of groups satisfying Abért's criterion. The notion of weakly branch group is introduced by Grigorchuk in [24]. Let T be a rooted spherical symmetric tree. For a vertex $u \in \mathsf{T}$, let C_u be the set of infinite rays with prefix u. We say a group G acting by automorphisms on T is weakly branching if the rigid stabilizer $\mathrm{Rist}_G(C_u)$ is nontrivial, where $\mathrm{Rist}_G(C_u) = \{g \in G : x \cdot g = x \text{ for all } x \notin C_u\}$. By definition it is clear that if G is weakly branching, then G separates ∂T.

Recall that the Frattini subgroup of G is the intersection of maximal subgroups of G. Pervova [40] proves for a torsion Grigorchuk-Gupta-Sidki (GGS) group G, all of its maximal subgroups are normal, hence the Frattini subgroup contains the derived subgroup $[G, G]$ and is of finite index in G. Thus results in [40] and [7] imply that a torsion, weakly branching GGS group preforms $(\mathbf{F}_2, \mathbf{S})$.

3. A sequence of marked groups

This section is devoted to the proof of Proposition 1.3. The starting point is a 2-generated group of intermediate volume growth which preforms the free group $\mathbf{F}_2$. To fix ideas, we take G to be the Fabrykowski-Gupta group introduced in [15]. Recall that the Fabrykowski-Gupta group acts on the ternary rooted tree T. It is generated by the root permutation a which permutes the 3 subtrees of the root cyclically and a directed permutation b which fixes the right most ray 2^∞ and is defined recursively by

$$b = (a, id, b).$$

In other words we have for any ray $w \in \{0,1,2\}^\infty$,

$$0w \cdot a = 1w,\ 1w \cdot a = 2w,\ 2w \cdot a = 0w;$$
$$0w \cdot b = 0(w \cdot a),\ 1w \cdot b = 1w,\ 2w \cdot b = 2(w \cdot b).$$

For more background on groups acting on rooted trees and the notation of wreath recursion see the reference .The group $G = \langle a, b\rangle$ is called the Fabrykowski-Gupta group. It is an example of non-torsion Grigorchuk-Gupta-Sidki (GGS) groups.

The group $G = \langle a, b\rangle$ satisfies:

1. ([8]) G is a just infinite branch group which is regularly branching over its commutator group $[G, G]$.
2. ([16, 9]) G is of intermediate growth.
3. ([17]) G has the congruence subgroup property: every finite index subgroup of G contains some level stabilizer $\mathrm{St}_G(n)$.
4. ([18]) Maximal subgroups of G are normal and of finite index.

By Bartholdi-Erschler [7], properties 1 and 4 above implies that G preforms the free group $(\mathbf{F}_2, \mathbf{S})$.

Recall that T_n denotes the level n vertices of the rooted tree T and $\mathrm{St}_G(n)$ denotes the level n stabilizer. Let G_n be the quotient group $G/\mathrm{St}_G(n)$, which acts faithfully and transitively on T_n. Consider the level n Schreier graph $\mathcal{S}_n$ with vertex set T_n and edge set $E = \{(x, x\cdot a), (x, x \cdot b) :\ x \in \mathsf{T}_n\}$. It is a finite graph on 3^n vertices. Consider the permutation wreath product W_n of the free product $\mathbf{A} = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ and G_n over the set T_n, that is,

$$W_n = \mathbf{A} \wr_{\mathsf{T}_n} G_n = (\oplus_{\mathsf{T}_n}\mathbf{A}) \rtimes G_n,$$

where G_n acts on $\oplus_{\mathsf{T}_n}\mathbf{A}$ by permuting the coordinates. We write element of W_n as pairs (φ, g), where $\varphi \in \oplus_{\mathsf{T}_n}\mathbf{A}$ and $g \in G_n$.

Denote by s and t the two standard generators of $\mathbf{A}$, $\mathbf{A} = \left\langle s, t | s^3 = t^3 = 1\right\rangle$. Consider the subgroup Γ_n of W_n generated by

$$a_n = (id, \bar{a}),\ b_n = \left(\delta^s_{2^{n-1}0} + \delta^{id}_{2^{n-1}1} + \delta^t_{2^n}, \bar{b}\right), \tag{3.1}$$

where in the direct sum $\oplus_{\mathsf{T}_n}\mathbf{A}$, δ^γ_x denotes the element that is γ at x and identity elsewhere and we use additive notation $\delta^{\gamma_1}_x + \delta^{\gamma_2}_y$, $x \neq y$, for the element that is γ_1 at x, γ_2 at y and identity elsewhere.

The choice of (a_n, b_n) guarantees:

Lemma 3.1. *The sequence $(\Gamma_n, (a_n, b_n))$ converges to $(G, (a, b))$ in the Chabauty topology as $n \to \infty$. Indeed, $(\Gamma_{n+1}, (a_{n+1}, b_{n+1}))$ is a marked quotient of $(\Gamma_n, (a_n, b_n))$, and the ball of radius 2^{n-2} around id in the Cayley graph of $(\Gamma_n, \{\tilde{a}, \tilde{b}\})$ coincide with the ball of radius 2^{n-2} around id in $(G, \{a, b\})$.*

Proof. The Fabrykowski-Gupta group belongs to the class of bounded automaton groups. Schreier graphs of bounded automaton groups are studied systematically in Bondarenko's dissertation [11]. In particular, on the finite Schreier graph $\mathcal{S}_n$, we have that the graph distance between the points

$2^{n-1}0, 2^n$ satisfy $d_{\mathcal{S}_n}(2^{n-1}0, 2^n) = 2^{n-1}$. For more details see [11, Chapter VI].

Note that G embeds as a subgroup of $G \wr_{\mathsf{T}_n} G_n$, where the embedding is given by the wreath recursion

$$a \mapsto (id, \bar{a}),\ b \mapsto \left(\delta^{a}_{2^{n-1}0} + \delta^{id}_{2^{n-1}1} + \delta^{b}_{2^n}, \bar{b}\right).$$

Now consider a word $w = w_1 \dots w_\ell$, where $w_j \in \{a^{\pm 1}, b^{\pm 1}\}$ and evaluate this word in $G \wr_{\mathsf{T}_n} G_n$ by the embedding above. For the configuration $\phi_w \in \oplus_{\mathsf{T}_n} G$, we have that

$$\phi_w(x) = \prod_{i=1}^{n} \phi_{w_i}(x \cdot w_1 \dots w_{i-1}).$$

It follows from the triangle inequality that if $\ell \le 2^{n-2}$, then the trajectory $\{x, x \cdot w_1, \dots, x \cdot w_1 \dots w_{\ell-1}\}$ can visit at most one point in the set $\{2^{n-1}0, 2^n\}$. In particular, $\phi_w(x)$ is an element in either $\langle a \rangle$ or $\langle b \rangle$. It follows that if we evaluate the same word w in Γ_n under $a \mapsto a_n$ and $b \mapsto b_n$, the resulting element $\left(\tilde{\phi}_w, \bar{w}\right)$ can be identified with $(\phi_w, \bar{w})$ in $G \wr_{\mathsf{T}_n} G_n$. Namely, ϕ_w is obtained from $\tilde{\phi}_w$ by replacing s with a and t with b and vice versa.

The quotient map from $(\Gamma_n, (a_n, b_n))$ to $(\Gamma_{n+1}, (a_{n+1}, b_{n+1}))$ is given as follows. Note that $\mathbf{A} \wr_{\mathsf{T}_{n+1}} G_{n+1} = \left(\mathbf{A} \wr_{\{0,1,2\}} \langle a \rangle\right) \wr_{\mathsf{T}_n} G_n$. Let $\tau : \mathbf{A} \to \mathbf{A} \wr_{\{0,1,2\}} \langle a \rangle$ be the group homomorphism determined by $\tau(s) = (id, a)$ and $\tau(t) = \left(\delta^s_0 + \delta^{id}_1 + \delta^t_2, id\right)$. The homomorphism τ extends to $\oplus_{\mathsf{T}_n} \mathbf{A} \to \oplus_{\mathsf{T}_n} \left(\mathbf{A} \wr_{\{0,1,2\}} \langle a \rangle\right)$ coordinate-wise, that is $\tau(\phi)(x) = \tau(\phi(x))$, $x \in \mathsf{T}_n$. Then it is clear by the wreath recursion formula in G that the map

$$\begin{aligned} \Gamma_n &\to \Gamma_{n+1} \\ (\phi, g) &\mapsto (\tau(\phi), g) \end{aligned}$$

is an epimorphism which sends a_n to a_{n+1} and b_n to b_{n+1}.

$\square$

Note that Γ_n is virtually a direct product of free groups:

Lemma 3.2. *The group Γ_n contains $\oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}]$ as a finite index normal subgroup.*

Proof. We proceed by induction on n.

As in the proof of Lemma 3.1, let $\tau : \mathbf{A} \to \mathbf{A} \wr_{\{0,1,2\}} \langle a \rangle$ be the group homomorphism determined by $\tau(s) = (id, a)$ and $\tau(t) = \left(\delta^s_0 + \delta^{id}_1 + \delta^t_2, id\right)$, where a is the 3-cycle $(0, 1, 2)$. When $n = 1$, by definition Γ_1 is generated by $a_1 = \tau(s)$ and $b_1 = \tau(t)$. Since $a_1^{-1} b_1 a_1 = (\delta^{id}_0 + \delta^t_1 + \delta^s_2, id)$, it follows that the projection of $\Gamma_1 \cap \oplus_{\mathsf{T}_1} \mathbf{A}$ to the component over vertex 2 is $\mathbf{A}$. Direct calculation shows that $\left[b_1 a_1^{-1} b_1 a_1, a_1 b_1 a_1^{-1} b_1\right] = \left(\delta_2^{sts^{-1}t^{-1}}, id\right)$. It follows that $[\Gamma_1, \Gamma_1] \cap \oplus_{\mathsf{T}_1} \mathbf{A}$ contains $\left\{(\delta^\gamma_2, id) :\ \gamma \in \left\langle sts^{-1}t^{-1} \right\rangle^{\mathbf{A}}\right\}$, while the normal closure $\left\langle sts^{-1}t^{-1} \right\rangle^{\mathbf{A}}$ is exactly the commutator subgroup $[\mathbf{A}, \mathbf{A}]$. Since $\tilde{a}$ acts as a 3-cycle permuting T_1, it follows that $[\Gamma_1, \Gamma_1] \cap \oplus_{\mathsf{T}_1} \mathbf{A} > \oplus_{\mathsf{T}_1}[\mathbf{A}, \mathbf{A}]$. The quotient group $\Gamma_1 / \oplus_{\mathsf{T}_1} [\mathbf{A}, \mathbf{A}]$ is a subgroup of $(\mathbf{A}/[\mathbf{A}, \mathbf{A}]) \wr_{\mathsf{T}_1} \langle a \rangle$, which is finite.

We have shown that $\tau([\mathbf{A}, \mathbf{A}])$ contains $\oplus_{\mathsf{T}_1}[\mathbf{A}, \mathbf{A}]$ as finite index normal subgroup, which reflects the property that G is regularly branching over its commutator subgroup.

Suppose the statement is true for n that Γ_n contains $\oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}]$ as a finite index normal subgroup. To prove the claim for $n+1$, it suffices to show that $\left(\delta^{\gamma}_{2^{n+1}}, id\right) \in \Gamma_{n+1}$ for any $\gamma \in [\mathbf{A}, \mathbf{A}]$. Recall the quotient map $\pi : \Gamma_n \to \Gamma_{n+1}$ explained in the proof of Lemma 3.1, where $\mathbf{A} \wr_{\mathsf{T}_{n+1}} G_{n+1}$ is identified with $\left(\mathbf{A} \wr_{\{0,1,2\}} \langle a \rangle\right) \wr_{\mathsf{T}_n} G_n$. By the induction hypothesis, $(\delta^{\sigma}_{2^n}, id) \in \Gamma_n$ for any $\sigma \in [\mathbf{A}, \mathbf{A}]$. Under the quotient map π, we have

$$\pi\left((\delta^{\sigma}_{2^n}, id)\right) = \left(\delta^{\tau(\sigma)}_{2^n}, id\right).$$

With the map τ we are back in the situation of the induction base, where we have shown that $\tau\left([\mathbf{A}, \mathbf{A}]\right)$ contains $\oplus_{\mathsf{T}_1}[\mathbf{A}, \mathbf{A}]$. In particular, for any $\gamma \in [\mathbf{A}, \mathbf{A}]$, there is an element $\sigma \in [\mathbf{A}, \mathbf{A}]$ such that $\tau(\sigma) = (\delta^{\gamma}_2, id)$. It follows that $\pi\left((\delta^{\sigma}_{2^n}, id)\right) = \left(\delta^{\tau(\sigma)}_{2^n}, id\right) = \left(\delta^{\gamma}_{2^{n+1}}, id\right)$, in particular it is an element of Γ_{n+1}.

□

Next we consider random walks on the group Γ_n, $n \geq 1$. To bound the asymptotic entropy from above, we simply use the well-known "fundamental inequality", see e.g. [10]. More precisely, let μ be a probability measure on $\mathbf{F}_d$ with finite first moment, $\pi : \mathbf{F}_d \to \Gamma$ an epimorphism. Let $S = \pi(\mathbf{S})$ be the induced marking on Γ and $\bar{\mu} = \pi \circ \mu$ be the pushforward of μ. The fundamental inequality implies that

$$h_{\mathrm{RW}}(\Gamma, \bar{\mu}) \leq v_{\Gamma,S} \cdot \ell_{\Gamma,\bar{\mu}},$$

where $v_{\Gamma,S}$ and $\ell_{\Gamma,\bar{\mu}}$ are asymptotic volume growth rate and asymptotic speed with respect to generating set S:

$$v_{\Gamma,S} = \lim_{r\to\infty} \frac{1}{r} \log V_{\Gamma,S}(r) \text{ and } \ell_{\Gamma,\bar{\mu}} = \lim_{t\to\infty} \frac{1}{t} \sum_{g\in\Gamma} |g|_S \bar{\mu}^{(t)}(g).$$

By subadditivity, we have $\ell_{\Gamma,\bar{\mu}} \leq \sum_{g\in\Gamma} |g|_S \bar{\mu}(g) \leq \sum_{g\in\mathbf{F}_d} |g|_{\mathbf{S}} \mu(g)$. Thus the asymptotic entropy can be bounded by

$$h_{\mathrm{RW}}\left(\Gamma, \bar{\mu}\right) \leq v_{\Gamma,S} \sum_{g\in\mathbf{F}_d} |g|_{\mathbf{S}} \mu(g). \tag{3.2}$$

The estimate (3.2) is the only place where the moment condition on μ is needed.

By [13, Theorem 5.1], for any non-degenerate probability measure μ on a countable group Γ, a sufficient condition for the action of Γ on the Poisson boundary $\Pi(\Gamma, \mu)$ to be essentially free is that

(1): Γ has only countably many amenable subgroups,

(2): Γ does not contain any non-trivial normal amenable subgroup.

We verify that these two properties are satisfied by Γ_n in the following lemma.

Lemma 3.3. *For each n, the group Γ_n is non-amenable, has no non-trivial normal amenable subgroup, and has only countably many amenable subgroups.*

Proof. Since $[\mathbf{A}, \mathbf{A}]$ is a free group, the only amenable subgroups are the trivial group and the cyclic groups. It follows that the direct sum $\oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}]$ has only countably many amenable subgroups. The property of having only countably many amenable subgroups is clearly preserved under taking finite extensions. Thus by Lemma 3.2, Γ_n is non-amenable and has only countably many amenable subgroups.

Let N be a non-trivial normal subgroup of Γ_n. We need to show N is non-amenable. For $g \in G_n$, define

$$S_N(g) := \{\phi \in \oplus_{\mathsf{T}_n}\mathbf{A} : \ (\phi, g) \in N\}.$$

Then $S_N(g)$ is either empty or a right coset of $S_N(id_{G_n})$ in $\oplus_{\mathsf{T}_n}\mathbf{A}$. We show that $N \cap \oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}] \neq \{id_{\Gamma_n}\}$. Suppose on the contrary the intersection is $\{id_{\Gamma_n}\}$, that is $S_N(id_{G_n}) \cap \oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}] = \{id\}$. Note that in the free product $\mathbf{A}$, a subgroup with trivial intersection with $[\mathbf{A}, \mathbf{A}]$ must be finite, since $\mathbf{A}/[\mathbf{A}, \mathbf{A}]$ is finite.

We now show that $S_N(id_{G_n})$ being a finite group contradicts with the condition that N is a non-trivial normal subgroup of Γ_n. Fix a choice of coset representative $S_N(g) = S_N(id_{G_n})a_g$ for each $g \in G_n$ with $S_N(g) \neq \emptyset$. Then as a set,

$$N = \cup_{g: S_N(g) \neq \emptyset}\{(\phi a_g, g) : \ \phi \in S_N(id_{G_n})\}.$$

Since N is normal in Γ_n, it is invariant under conjugation by any $\gamma \in \oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}]$,

$$\gamma^{-1}(S_N(id_{G_n})a_g, g)\gamma = (S_N(id_{G_n})a_g, g),$$

that is

$$\gamma^{-1}S_N(id_{G_n})a_g\tau_g(\gamma) = S_N(g),$$

where τ_g acts by permuting the coordinates.

- For $g \neq id_{G_n}$, since the action of G_n on T_n is faithful, there exists a vertex $v \in \mathsf{T}_n$ such that $v \cdot g \neq v$. Then the projection of the set $\left\{\gamma^{-1}S_N(id_{G_n})a_g\tau_g(\gamma), \gamma \in \oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}]\right\}$ to the component over vertex v contains $(S_N(id_{G_n})a_g)_v[\mathbf{A}, \mathbf{A}]$ if $S_N(g) \neq \emptyset$. In particular, it contradicts with $\gamma^{-1}S_N(g)\tau_g(\gamma) = S_N(g)$, since $S_N(g)$ is a finite set. Thus $S_N(g) = \emptyset$ for all $g \neq id_{G_n}$.
- For $g = id_{G_n}$, $\gamma^{-1}S_N(id_{G_n})\gamma = S_N(id_{G_n})$ for all $\gamma \in \oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}]$ implies that elements in $S_N(id_{G_n})$ have finite conjugacy classes in $\oplus_{\mathsf{T}_n}\mathbf{A}$. Since all nontrivial conjugacy classes of $\mathbf{A}$ are infinite, it follows that $S_N(id_G) = \{id\}$.

Combine these two items we conclude that $S_N(id_G) = \{id\}$ and $S_N(g) = \emptyset$ for all $g \neq id_{G_n}$, which is equivalent to $N = \{id_{\Gamma_n}\}$ and contradicts with the condition that N is nontrivial.

We have shown that $N \cap \oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}] \neq \{id_{\Gamma_n}\}$, thus this intersection is a non-trivial normal subgroup of $\oplus_{\mathsf{T}_n}[\mathbf{A}, \mathbf{A}]$. Since $[\mathbf{A}, \mathbf{A}]$ is a non-abelian free group, all of its non-trivial normal subgroups are non-amenable. We conclude that N is non-amenable.

$\square$

We are now ready to prove Proposition 1.3 stated in the Introduction.

Completion of proof of Proposition 1.3. Let μ be a non-degenerate probability measure on $\mathbf{F}_d$ given, where $d \geq 2$ and μ is of finite first moment. Let $\Gamma_n = \langle a_n, b_n \rangle$, $n \geq 1$, be the sequence of groups defined at the beginning of this section. We are going to choose a subsequence of (Γ_n) and d-markings on them to satisfy the conditions (i),(ii).

The Fabrykowski-Gupta group G preforms the free group $\mathbf{F}_d$. Fix a sequence of d-markings $T_1, T_2, \ldots$ on G such that $(G, T_k) \to (\mathbf{F}_d, \mathbf{S})$ in the Chabauty space $\mathcal{G}_d$ as $k \to \infty$. Denote by r_k the maximum radius such that the ball around id in (G, T_k) coincide with the ball of same radius around identity in $(\mathbf{F}_d, \mathbf{S})$.

For the marking $T_k = \left(t_1^{(k)}, \ldots, t_d^{(k)}\right)$, let

$$\ell_k = \max\left\{\left|t_j^{(k)}\right|_{(a,b)} : 1 \leq j \leq d\right\}$$

be the maximum word length of these generators with respect to the original generating set (a, b) of G. For each $t_j^{(k)}$, fix a word $w_j^{(k)}$ in $\{a^{\pm 1}, b^{\pm 1}\}$ of shortest length which represents $t_j^{(k)}$. Let

$$q_k = \max\{|a|_{T_k}, |b|_{T_k}\}$$

be the maximum of length of generators a, b with respect to T_k. Similarly fix a shortest word $z_a^{(k)}$ ($z_b^{(k)}$ resp.) representing a (b resp.) in the alphabet $T_k \cup T_k^{-1}$.

By Lemma 3.1, the ball of radius 2^{n-2} around id in the Cayley graph of $(\Gamma_n, \{\tilde{a}, \tilde{b}\})$ coincide with the ball of radius 2^{n-2} around id in $(G, \{a, b\})$. With T_k given, take a sufficiently large $n_k \gg \ell_k q_k$ and consider the group Γ_{n_k}. How large n_k needs to be will be made precise shortly. Evaluating the word $w_j^{(k)}$ in Γ_{n_k} with $a \mapsto a_{n_k}$, $b \mapsto b_{n_k}$, we obtain an element $s_j^{(k)}$ in Γ_{n_k}.

Claim 3.4. *The tuple $S_k = \left(s_1^{(k)}, \ldots, s_d^{(k)}\right)$ generates Γ_{n_k}. Moreover, the ball of radius $2^{n_k-3}/\ell_k$ in (Γ_{n_k}, S_k) coincide with the ball of the same radius around id in (G, T_k).*

Proof of the claim. Recall that by Γ_{n_k} is generated by (a_{n_k}, b_{n_k}) and by Lemma 3.1 the ball of radius 2^{n_k-2} in $(\Gamma_{n_k}, (a_{n_k}, b_{n_k})))$ coincides with the ball of the same radius around id in $(G, \{a, b\})$. Since $n_k > \ell_k q_k$, it follows that the word $z_a^{(k)}$ in the alphabet $T_k \cup T_k^{-1}$ evaluated in Γ_{n_k} by $t_j^{(k)} \mapsto s_j^{(k)}$ is a_{n_k}; similarly, the word $z_b^{(k)}$ evaluates to b_{n_k}. It follows that S_k is a generating tuple. For the same reason that $n_k > \ell_k q_k$, the word $w_j^{(k)}$ in $\{a^{\pm 1}, b^{\pm 1}\}$ evaluated in Γ_{n_k} by $a \mapsto a_{n_k}$, $b \mapsto b_{n_k}$ is $s_j^{(k)}$. In particular, vertices in the ball of radius r around id in the Cayley graph (Γ_{n_k}, S_k) is contained in the vertex set of the ball of $r\ell_k$ in $(\Gamma_{n_k}, \{a, b\})$. The claim follows.

□

Recall that the Fabrykowski-Gupta group G has sub-exponential volume growth,

$$\lim_{r\to\infty} \frac{1}{r} \log V_{G,S}(r) = 0,$$

where S is any finite generating set. Fix a choice of sufficiently fast growing sequence (n_k) such that $n_k > \ell_k q_k$ and

$$\frac{1}{2^{n_k-3}/\ell_k} \log V_{G,T_k}(2^{n_k-3}/\ell_k) \le \frac{1}{k}.$$

We have completed the description of the choice of sequence of marked groups (Γ_{n_k}, S_k). It remains to verify the properties stated.

(i): The Claim above shows that (Γ_{n_k}, S_k) converges to the same limit as the sequence (G, T_k) as $k \to \infty$. The limit group is the sequence (G, T_k) is the free group $(\mathbf{F}_d, \mathbf{S})$ by the choice of (T_k).

(ii): The choice of n_k and the Claim guarantees that

$$v_{\Gamma_{n_k}, S_k} \le \frac{1}{2^{n_k-3}/\ell_k} \log V_{G,T_k}(2^{n_k-3}/\ell_k) \le \frac{1}{k}.$$

The fundamental inequality (3.2) implies that with respect to the marking $(\mathbf{F}_d, \mathbf{S}) \to (\Gamma_{n_k}, S_k)$,

$$h_{\mathrm{RW}}(\Gamma_{n_k}, \mu) \le v_{\Gamma_{n_k}, S_k} \Sigma_{g\in\mathbf{F}_d} |g|_{\mathbf{S}} \mu(g) \le \frac{1}{k} \Sigma_{g\in\mathbf{F}_d} |g|_{\mathbf{S}} \mu(g).$$

Thus the sequence of asymptotic entropies converge to 0 as $k \to \infty$.

(iii): This property is shown in Lemma 3.3.

The proof of Proposition 1.3 is complete.

□

Remark 3.5. The definition of Γ_n is inspired by the constructions in Erschler [14] and Kassabov-Pak [31]. For $d \ge 3$, in the proof of Proposition 1.3 one can use the first Grigorchuk group $G_{012} = \langle a, b, c\rangle$ introduced in [25, 23] instead of the Gupta-Sidki group. Recall that G_{012} acts on the rooted binary tree. Then one can consider the permutational wreath extension $\mathbf{B} \wr_{\mathsf{T}_n} G_n$, where $G_n = G/\mathrm{St}_G(n)$ and $\mathbf{B} = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = (\langle \mathbf{s}\rangle) * (\langle \mathbf{t}\rangle \times \langle \mathbf{u}\rangle)$. Similar to the sequence of extensions Γ_n, set

$$H_n = \langle a_n, b_n, c_n\rangle < \mathbf{B} \wr_{\mathsf{T}_{3n}} G_{3n}$$

where the generators are defined as

$$a_n = (id, \bar{a}),\ b_n = \left(\delta^{\mathbf{t}}_{1^{3n}} + \delta^{\mathbf{s}}_{1^{3n-1}0}, \bar{b}\right),\ c_n = \left(\delta^{\mathbf{u}}_{1^{3n}} + \delta^{\mathbf{s}}_{1^{3n-1}0}, \bar{c}\right).$$

Similar proof as in this section with Γ_n replaced by H_n shows that for $d \geq 3$, the statement Proposition 1.3 is true under the weaker assumption that μ has finite α_0-moment and finite entropy, where α_0 is the exponent in the growth upper bound $v_{G_{012}}(r) \lesssim e^{r^{\alpha_0}}$ from [6, 36], $\alpha_0 \approx 0.7674$. We choose to take extensions of the Fabrykowski-Gupta group here because the resulting groups are 2-generated, which allows to cover the case $d = 2$.

4. Stationary joinings and proof of the main results

Let (X,ν) and (Y,η) be two μ-stationary G-spaces. Following [22], we say a probability measure λ on $X \times Y$ is a stationary joining of ν and η if it is μ-stationary and its marginals are ν and η respectively.

In this section we focus on the situation where both stationary systems are (G,μ)-boundaries. We use notations introduced in Section 2.1. Denote by (B,ν_B) a compact model of the Poisson boundary of (G,μ). Let (X,ν) and (Y,η) be compact models of two (G,μ)-boundaries and denote by $\boldsymbol{\beta}_X$ and $\boldsymbol{\beta}_Y$ the corresponding maps from the Poisson boundary (B,ν_B) to (X,ν) and (Y,η). Consider the map

$$\begin{aligned} \boldsymbol{\beta}_X \times \boldsymbol{\beta}_Y : B &\to X \times Y \\ b &\mapsto (\boldsymbol{\beta}_X(b), \boldsymbol{\beta}_Y(b)), \end{aligned}$$

and denote by Z the range $(\boldsymbol{\beta}_X \times \boldsymbol{\beta}_Y)(B)$ and $\nu \curlywedge \eta$ the pushforward of the harmonic measure ν_B under $\boldsymbol{\beta}_X \times \boldsymbol{\beta}_Y$. Then it's clear by definition that $(Z, \nu \curlywedge \eta)$ is a G-factor of the Poisson boundary (B,ν_B), in other words, it is a (G,μ)-boundary. The G-space $(Z, \nu \curlywedge \eta)$ is the unique stationary joining of the μ-boundaries (X,ν) and (Y,η), see [22, Proposition 3.1].

On the level of groups, given two d-marked groups (G_1,S_1) and (G_2,S_2), one can take their diagonal product, denoted by $(G_1 \otimes G_2, S)$, as the subgroup of $G_1 \times G_2$ generated by

$$S = \left(\left(s_1^{(1)}, s_1^{(2)}\right), \ldots \left(s_d^{(1)}, s_d^{(2)}\right)\right),$$

where $S_i = \left(s_1^{(i)}, \ldots, s_d^{(i)}\right)$, $i = 1, 2$. This operation on two groups corresponds to taking stationary joinings of the Poisson boundaries:

Lemma 4.1. *Let μ be a probability measure on $\mathbf{F}_d$. The Poisson boundary of $(G_1 \otimes G_2, \mu)$ is the stationary joining of the Poisson boundaries of (G_1,μ) and (G_2,μ).*

Proof. Denote by (B_i,ν_i) the Poisson boundary of (G_i,μ), $i = 1, 2$ and regard them as $G_1 \otimes G_2$-spaces. Denote by $(Z, \nu_1 \curlywedge \nu_2)$ the stationary joining of (B_1,ν_1) and (B_2,ν_2) as above and $\pi_i : Z \to B_i$ the projections. We need to show $(Z, \nu_1 \curlywedge \nu_2)$ is the maximal $(G_1 \otimes G_2, \mu)$-boundary.

Let (Y, η) be a $(G_1 \otimes G_2, \mu)$-boundary. Denote by K_i the subgroup of $G_1 \otimes G_2$ which consists of elements that project to identity in G_i, that is,

$$K_i = \{(g_1, g_2) \in G_1 \times G_2 : \ (g_1, g_2) \in G_1 \otimes G_2, \ g_i = id_{G_i}\}.$$

Denote by $Y_i = Y//K_i$ the space of K_i-ergodic components of Y and η_i the pushforward of the measure η under the K_i-factor map $Y \to Y//K_i$. Since (Y, η) is an ergodic $G_1 \otimes G_2$-space and $K_1 \cap K_2 = \{id\}$, we have that Y can be viewed as a subset of $Y_2 \times Y_1$. It's easy to see that by definition of K_i that $G_1 \otimes G_2/K_2 \simeq G_1$. It follows that (Y_2, η_2) is a (G_1, μ)-boundary. Denote by β_{Y_2} the boundary map from (B_1, ν_1) to (Y_2, η_2). In the same way we have (Y_1, η_1) is a (G_2, μ)-boundary and denote by $\beta_{Y_1} : (B_2, \nu_2) \to (Y_1, \eta_1)$ the boundary map. By uniqueness of stationary joinings of μ-boundaries, we have that $(Y, \eta) = (Y_2 \times Y_1, \eta_2 \curlyvee \eta_1)$. It follows that (Y, η) is a factor of $(Z, \nu_1 \curlywedge \nu_2)$, where the boundary map is given by $z \mapsto (\beta_{Y_2} \circ \pi_1(z), \beta_{Y_1} \circ \pi_2(z))$.

□

With the sequence of marked groups provided by Proposition 1.3, we are now ready to complete the proofs of Theorem 1.1 and 1.2.

Proof of Theorem 1.1. Denote by (B, ν_B) the Poisson boundary of $(\mathbf{F}_d, \mu)$. Let $((\Gamma_k, S_k))_{k=1}^{\infty}$ be a sequence marked groups provided by Proposition 1.3. Denote by (Π_k, η_k) the Poisson boundary of (Γ_k, μ). Since (Γ_k, S_k) can be identified with a projection $\pi_k : \mathbf{F}_d \to \Gamma_k$, we regard (Π_k, η_k) as a $(\mathbf{F}_d, \mu)$-space, where the $\mathbf{F}_d$-action factors through π_k.

Since Γ is a proper quotient of $\mathbf{F}_d$, $N = \ker(\pi : \mathbf{F}_d \to \Gamma)$ is nontrivial. Fix a choice of element $g \in N$, $g \neq id$. Choose an index $k \in \mathbb{N}$ sufficiently large such that the balls of radius $2|g|_{\mathbf{S}}$ around identities in (Γ_k, S_k) and $(\mathbf{F}_d, \mathbf{S})$ coincide and $h_{\mathrm{RW}}(\Gamma_k, \mu) < \epsilon$. Take $\tilde{\Gamma}$ to be the diagonal product $(\Gamma \otimes \Gamma_k, S)$. Then

$$h_{\mathrm{RW}}(\Gamma \otimes \Gamma_k, \mu) \leq h_{\mathrm{RW}}(\Gamma, \mu) + h_{\mathrm{RW}}(\Gamma_k, \mu) < h_{\mathrm{RW}}(\Gamma, \mu) + \epsilon.$$

Since g acts trivially on the Poisson boundary of (Γ, μ) but acts freely on (Π_k, ν_k), it follows that (Π_k, ν_k) is not a $\mathbf{F}_d$-factor of the Poisson boundary of (Γ, μ). By Lemma 2.2, we conclude that $h_{\mathrm{RW}}(\Gamma \otimes \Gamma_k, \mu) > h_{\mathrm{RW}}(\Gamma, \mu)$.

□

Proof of Theorem 1.2. The proof is similar to Theorem 1.1. Since (X, ν) is assumed to be a $(\mathbf{F}_d, \mu)$-boundary where the action of $\mathbf{F}_d$ is not essentially free, we can choose an element $g \in \mathbf{F}_d$, $g \neq 1$, such that $\nu(\mathrm{Fix}_X(g)) > 0$. Choose an index $k \in \mathbb{N}$ sufficiently large such that the balls of radius $2|g|_{\mathbf{S}}$ around identities in (Γ_k, S_k) and $(\mathbf{F}_d, \mathbf{S})$ coincide and $h_{\mathrm{RW}}(\Gamma_k, \mu) < \epsilon$. Take the stationary joining $(Z_k, \nu \curlywedge \eta_k)$ of (X, ν) and (Π_k, η_k). By the general inequality, we have

$$h(Z_k, \nu \curlywedge \eta_k) \leq h(X, \nu) + h(\Pi_k, \eta_k) \leq h(X, \nu) + \epsilon.$$

It remains to show that $h(Z_k, \nu \curlywedge \eta_k) > h(X, \nu)$. Suppose on the contrary equality holds, then by Lemma 2.2, the equality would imply $(Z_k, \nu \curlywedge \eta_k) =$

(X,ν). However the action of Γ_k on (Π_k, η_k) is essentially free, which implies $\nu \curlywedge \eta_k\left(\mathrm{Fix}_{Z_k}(g)\right) = 0$, contradicting $\nu(\mathrm{Fix}_X(g)) > 0$.

□

We now show an analogous result on spectral radii stated as Theorem 1.4 in the Introduction. Consider a symmetric non-degenerate probability measure μ on Γ. In [33, 34] Kesten proved the following theorem: let μ be a symmetric non-degenerate probability measure on Γ and N be a normal subgroup of Γ, then the following are equivalent:

(i): $\rho(\Gamma, \mu) = \rho(\Gamma/N, \mu)$,

(ii): N is amenable.

Given a proper quotient Γ of $\mathbf{F}_d$ and $\epsilon > 0$, to prove Theorem 1.4 we take $\tilde{\Gamma}$ to be a diagonal product $\Gamma \otimes H$, for some appropriate choice of H similar to the groups used in Theorem 1.1.

Proof of Theorem 1.4. Let $\epsilon > 0$ be a constant given. Let Γ be a proper quotient of $\mathbf{F}_d$ and fix a choice of $g \in \ker(\mathbf{F}_d \to \Gamma)$, $g \neq id$. Fix a choice of d-marking T of the Fabrykowski-Gupta group G such that the ball of radius $2|g|_{\mathbf{S}}$ around identity is the same as the ball of same radius in $(\mathbf{F}, \mathbf{S})$. Take first the diagonal product $\Gamma \otimes G$. By the choice of g and marking on G we have that $N_0 = \ker(\Gamma \otimes G \to \Gamma)$ is non-trivial.

Denote by (W_n) a μ-random walk on $\mathbf{F}_d$. For a marked group (H, S), we write π_H for the quotient map $\mathbf{F}_d \to H$ when the marking is clear from the context.

Take a small constant $\epsilon_1 > 0$, choose n large enough such that

$$\mathbb{P}\left(\pi_\Gamma\left(W_n\right) = id_\Gamma\right) \geq ((1-\epsilon_1)\rho(\Gamma, \mu))^n.$$

For $\gamma \in N_0$, set

$$Q(\gamma) = \frac{\mathbb{P}(\pi_{\Gamma\otimes G}(W_n) = \gamma)}{\mathbb{P}(\pi_\Gamma(W_n) = id_\Gamma)}.$$

Then Q is a symmetric probability measure on N_0. Equip N_0 with the induced metric $|\cdot|_T$ from (G,T). Let R be a sufficiently large radius such that $Q(\{\gamma \in N_0 : |\gamma|_T > R\}) \leq \epsilon_1$. Truncate the measure Q at R and let

$$Q_R(g) = \frac{1}{Q\left(\{\gamma : |\gamma|_T \leq R\}\right)} Q(g) \mathbf{1}_{\{|g|_T \leq R\}}.$$

Since N_0 is a normal subgroup of G, thus amenable, there exists an integer m such that

$$Q_R^{2m}\left(id_{N_0}\right) \geq (1-\epsilon_1)^{2m}.$$

With n, m, R chosen, take a sufficiently large index ℓ (to be specified shortly) and consider the group $\Gamma_\ell = \langle a_\ell, b_\ell \rangle$ where the generators a_ℓ, b_ℓ are defined in (3.1). Mark the group Γ_ℓ with the generating tuple S_ℓ inherited from T as in Claim 3.4. Choose ℓ to be sufficiently large such that the ball of radius $2mR$ in (Γ_ℓ, S_ℓ) coincide with the ball of the same radius in (G,T). Consider the diagonal product $\Gamma \otimes \Gamma_\ell$.

Now we follow the original argument in Kesten's theorem $(ii) \Rightarrow (i)$ above to show $\rho(\Gamma \otimes \Gamma_\ell, \mu) > \rho(\Gamma, \mu) - \epsilon$. Write $W^{kn}_{(k-1)n} = W^{-1}_{(k-1)n} W_{kn}$.

$$\begin{aligned}
&\mathbb{P}\left(\pi_{\Gamma\otimes\Gamma_\ell}\left(W_{2nm}\right) = id_{\Gamma\otimes\Gamma_\ell}\right)\\
&\geq \mathbb{P}\left(\cap_{k=1}^{2m}\left\{\pi_\Gamma\left(W^{kn}_{(k-1)n}\right) = id_\Gamma, \left|\pi_{\Gamma_\ell}\left(W^{kn}_{(k-1)n}\right)\right|_{S_\ell} \leq R\right\} \cap \left\{\pi_{\Gamma_\ell}\left(W_{2mn}\right) = id_{\Gamma_\ell}\right\}\right)\\
&\qquad \geq ((1-\epsilon_1)\rho(\Gamma,\mu))^{2mn}(1-\epsilon_1)^{2m} Q_R^{2m}\left(id_{N_0}\right) \geq (1-\epsilon_1)^{2mn+4m}\rho^{2mn}.
\end{aligned}$$

Choose $\epsilon_1 < \epsilon/3$, we have that $\rho(\Gamma \otimes \Gamma_\ell, \mu) > (1-\epsilon)\rho(\Gamma, \mu)$.

Finally, in Lemma 3.3 it is proved that Γ_ℓ has no nontrivial amenable normal subgroups. Since by the choice of markings $\ker(\tilde{\Gamma} \to \Gamma)$ is nontrivial, it follows that $\ker(\tilde{\Gamma} \to \Gamma)$ is non-amenable. By Kesten's theorem $(i) \Rightarrow (ii)$, we conclude that $\rho(\tilde{\Gamma}, \mu) < \rho(\Gamma, \mu)$.

□

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