

UNFRIENDLY COLORINGS OF GRAPHS WITH FINITE AVERAGE DEGREE

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ABSTRACT. In an unfriendly coloring of a graph the color of every node mismatches that of the majority of its neighbors. We show that every probability measure preserving Borel graph with finite average degree admits a Borel unfriendly coloring almost everywhere. We also show that every bounded degree Borel graph of subexponential growth admits a Borel unfriendly coloring.

1. INTRODUCTION

Suppose that G is a locally finite graph on the vertex set X . We say that $c: X \rightarrow 2$ is an *unfriendly coloring* of G if for all $x \in X$ at least half of x 's neighbors receive a different color than x does. More formally, letting G_x denote the set of G -neighbors of x , such a function c is an unfriendly coloring if $|\{y \in G_x : c(x) \neq c(y)\}| \geq |\{y \in G_x : c(x) = c(y)\}|$. By a compactness argument unfriendly colorings exist for all locally finite graphs (see, e.g., [1]). There exist graphs with uncountable vertex sets that have no unfriendly colorings [7]; it is not known if this is possible for graphs with countably many vertices.

A large and growing literature considers measure-theoretical analogues of classical combinatorial questions (see, e.g., a survey by Kechris and Marks [5]). Following [3], we consider a measure-theoretical analogue of the question of unfriendly colorings. Suppose that G is a locally finite Borel graph on the standard Borel space X , and that μ is a Borel probability measure on X . We say that G is μ -*preserving* if there are countably many μ -preserving Borel involutions whose graphs cover the edges of G . Equivalently, G is μ -preserving if its connectedness relation E_G is a μ -preserving equivalence relation.

An important example of such graphs comes from probability measure preserving actions of finitely generated groups. Indeed let a group, generated by the finite symmetric set S , act by measure preserving transformations on a standard borel probability space (X, μ) . Then

Clinton T. Conley was supported by NSF grant DMS-1500906. Omer Tamuz was supported by a grant from the Simons Foundation (#419427).

the associated graph $G = (X, E)$ whose edges are

$$E = \{(x, y) : y = sx \text{ for some } s \in S\}$$

is a μ -preserving graph.

In [3] it is shown that every free probability measure preserving action of a finitely generated group is weakly equivalent to another such action whose associated graph admits an unfriendly coloring. Note that such graphs are regular: (almost) every node has degree $|G_x| = |S|$. Recall that the (μ -)cost of a μ -preserving locally finite Borel graph G is simply half its average degree: $\text{cost}(G) = \frac{1}{2} \int |G_x| d\mu(x)$. Our first result shows that every measure preserving graph with finite cost admits an (almost everywhere) unfriendly coloring.

Theorem 1. *Suppose that (X, μ) is a standard probability space and that G is a μ -preserving locally finite Borel graph on X with finite cost. Then there is a μ -conull G -invariant Borel set A such that $G \upharpoonright A$ admits a Borel unfriendly coloring.*

We next explore how the invariance assumption can be weakened. Recall that a Borel probability measure is G -quasi-invariant if the G -saturation of every μ -null set remains μ -null. Such measures admit a Radon-Nikodym cocycle $\rho: G \rightarrow \mathbb{R}^+$ so that whenever $A \subseteq X$ is Borel and $f: A \rightarrow X$ a Borel partial injection whose graph is contained in G , then $\mu(f[A]) = \int_A \rho(x, f(x)) d\mu$.

Theorem 2. *Suppose that (X, μ) is a standard probability space, that G is a Borel graph on X with bounded degree d , and that μ is G -quasi-invariant, with corresponding Radon-Nikodym cocycle ρ . Suppose also that for all $(x, y) \in G$, $(x, y) \in G$, $1 - \frac{1}{d} \leq \rho(x, y) \leq 1 + \frac{1}{d}$. Then there is a μ -conull G -invariant Borel set A such that $G \upharpoonright A$ admits a Borel unfriendly coloring.*

The proofs of Theorems 1 and 2 build on a potential function technique used in [8] (see also [2]) to study majority dynamics on infinite graphs; in the context of finite graphs, these techniques go back to Goles and Olivos [4]. Indeed, we show that in our settings (anti)-majority dynamics converge to an unfriendly coloring. The combinatorial nature of this technique allows us to extend our results to the Borel setting.

Theorem 3. *Suppose that G is a bounded-degree Borel graph of subexponential growth. Then G admits a Borel unfriendly coloring.*

A natural question remains open: is there a locally finite Borel graph that does not admit a Borel unfriendly coloring? To the best of our knowledge this is not known, even with regards to the restricted class

of bounded degree graphs. In contrast, Theorem 1 shows that for this class unfriendly colorings exist in the measure preserving case. Still, we do not know if the finite cost assumption in Theorem 1 is necessary, or whether every locally finite measure preserving graph admits an almost everywhere unfriendly coloring.

2. PROOFS

Proof of Theorem 1. By Kechris-Solecki-Todorcevic [6, Proposition 4.5], fix a sequence $(X_n)_{n \in \mathbb{N}}$ of G -independent Borel sets so that each $x \in X$ is in infinitely many X_n . We will recursively build for each $n \in \mathbb{N}$ a Borel function $c_n: X \rightarrow 2$ which converge μ -almost everywhere to an unfriendly coloring of G .

The choice of c_0 is arbitrary, but we may as well declare it to be the constant 0 function.

Suppose now that c_n has been defined. We build c_{n+1} by “flipping” the color of vertices in X_n with too many neighbors of the same color, and leaving everything else unchanged. More precisely, $c_{n+1}(x) = 1 - c_n(x)$ if $x \in X_n$ and $|\{y \in G_x : c_n(x) \neq c_n(y)\}| < |\{y \in G_x : c_n(x) = c_n(y)\}|$; otherwise, $c_{n+1}(x) = c_n(x)$.

To show that this sequence c_n converges μ -a.e. to an unfriendly coloring, we introduce some auxiliary graphs. Let G_n be the subgraph of G containing exactly those edges between vertices of the same c_n -color, so $x G_n y$ iff $x G y$ and $c_n(x) = c_n(y)$. Certainly for all $n \in \mathbb{N}$, $\text{cost}(G_n) \leq \text{cost}(G)$.

For $n \in \mathbb{N}$, let $B_n = \{x \in X : c_n(x) \neq c_{n+1}(x)\}$.

Claim. $\text{cost}(G_n) - \text{cost}(G_{n+1}) \geq \mu(B_n)$.

Proof of the claim. Recall that, by the definition of c_{n+1} , $x \in B_n$ iff $x \in X_n$ and $|\{y \in G_x : c_n(x) \neq c_n(y)\}| < |\{y \in G_x : c_n(x) = c_n(y)\}|$. In particular, $B_n \subseteq X_n$ and hence is G -independent. Thus $G_{n+1} = G_n \triangle \{(x, y) : x G y \text{ and } \{x, y\} \cap B_n \neq \emptyset\}$. But for each $x \in B_n$, the above characterization of membership in B_n ensures that its G_{n+1} -degree is strictly smaller than its G_n -degree. The claim follows. \square

In particular, since the sum telescopes we see $\sum_{n \in \mathbb{N}} \mu(B_n) \leq \text{cost}(G) < \infty$. Hence the set $C = \{x \in X : x \in B_n \text{ for infinitely many } n\}$ is μ -null by the Borel-Cantelli lemma. Let $A = X \setminus [C]_G$, so A is μ -conull.

Claim. c is an unfriendly coloring of $G \upharpoonright A$.

Proof of the claim. Fix $x \in A$ and fix $k \in \mathbb{N}$ sufficiently large so that c_n has stabilized for x and all its (finitely many) neighbors beyond k . Fix $n > k$ so that $x \in X_n$. Since $c_n(x) = c_{n+1}(x)$, the definition of c_{n+1}

implies that $|\{y \in G_x : c_n(x) \neq c_n(y)\}| \geq |\{y \in G_x : c_n(x) = c_n(y)\}|$. But $c_n = c$ on $G_x \cup \{x\}$, and hence c is unfriendly as desired. \square

This completes the proof of the theorem. \square

We next analyze the extent to which the measure-theoretic hypotheses may be weakened in this argument. Note that the sequence c_n of colorings is defined without using the measure at all (in fact it is determined by the graph G and the sequence (X_n) of independent sets); the measure only appears in the argument that sequence converges to a limit coloring. And even in this convergence argument, invariance only shows up in the critical estimate $\text{cost}(G_n) - \text{cost}(G_{n+1}) \geq \mu(B_n)$.

Definition 4. Suppose that G is a locally finite Borel graph on standard Borel X , that $(X_n)_{n \in \mathbb{N}}$ is a sequence of G -independent Borel sets so that each $x \in X$ is in infinitely many X_n . We define the *flip sequence* $(c_n)_{n \in \mathbb{N}}$ of Borel functions from X to 2 as follows:

- c_0 is the constant 0 function,
- $c_{n+1}(x) = 1 - c_n(x)$ if $x \in X_n$ and $|\{y \in G_x : c_n(x) \neq c_n(y)\}| < |\{y \in G_x : c_n(x) = c_n(y)\}|$; otherwise, $c_{n+1}(x) = c_n(x)$.

Definition 5. Given a locally finite Borel graph G on X and a sequence $(X_n)_{n \in \mathbb{N}}$ of independent sets as above, we say that a Borel measure μ on X is *compatible* with G and (X_n) if the corresponding flip sequence c_n converges on a μ -conull set.

So the proof of Theorem 1 shows that whenever μ is a G -invariant Borel probability measure with respect to which the average degree of G is finite, then μ is compatible with every sequence of independent sets. We seek to weaken the invariance assumption when G has bounded degree.

Proposition 6. *Suppose that G is a Borel graph on X with bounded degree d , and that μ is a G -quasi-invariant Borel probability measure with corresponding Radon-Nikodym cocycle ρ . Suppose further that for all $(x, y) \in G$, $1 - \frac{1}{d} \leq \rho(x, y) \leq 1 + \frac{1}{d}$. Then μ is compatible with every sequence of independent sets.*

Theorem 2 is an immediate consequence of this proposition.

Proof of Proposition 6. Put $\varepsilon = \frac{1}{d}$. Define a measure M on G by putting for all Borel $H \subseteq G$,

$$M(H) = \int |H_x| d\mu$$

This new measure M will replace the occurrences of cost in the proof of Theorem 1.

Consider the flip sequence c_n , and define corresponding graphs $G_n \subseteq G$ by $x G_n y$ iff $x G y$ and $c_n(x) = c_n(y)$. As before, let B_n denote those $x \in X_n$ for which $c_{n+1}(x) \neq c_n(x)$. Note that the “double counting” that occurred in the proof of Theorem 1 may no longer be true double counting, but the bound on ρ ensures that each edge is counted at most $(2 + \varepsilon)$ times and at least $(2 - \varepsilon)$ times.

Claim. $M(G_{n+1}) \leq M(G_n) - \mu(B_n)$.

Proof of the claim. It suffices to show that $M(G_n) - M(G_{n+1}) \geq \mu(B_n)$. Partition B_n into finitely many Borel sets $A_{r,s}$ where $x \in A_{r,s}$ iff x has r -many G_n neighbors and s -many G_{n+1} neighbors (so $r > s$ and $r + s \leq d$). We compute

$$\begin{aligned}
M(G_n) - M(G_{n+1}) &= \int_X |(G_n)_x| - |(G_{n+1})_x| \, d\mu \\
&\geq \int_{B_n} (2 - \varepsilon)|(G_n)_x| - (2 + \varepsilon)|(G_{n+1})_x| \, d\mu \\
&= \sum_{r,s} \int_{A_{r,s}} (2 - \varepsilon)r - (2 + \varepsilon)s \, d\mu \\
&= \sum_{r,s} \int_{A_{r,s}} 2(r - s) - \varepsilon(r + s) \, d\mu \\
&\geq \sum_{r,s} \int_{A_{r,s}} 2 - d\varepsilon \, d\mu \\
&= \mu(B_n)
\end{aligned}$$

as required. \square

The remainder of the argument is as in the proof of Theorem 1. \square

Given Proposition 6 the proof of Theorem 3 is straightforward.

Proof of Theorem 3. Fix a degree bound d for G and put $\varepsilon = \frac{1}{d}$. It suffices to construct for each $x \in X$ a G -quasi-invariant Borel probability measure μ_x whose Radon-Nikodym cocycle is ε -bounded on G such that $\mu_x(\{x\}) > 0$. If we do so, Proposition 6 ensures that the flip sequence c_n converges μ_x -everywhere for each x , and thus it converges everywhere. The limit is then an unfriendly coloring by the same argument as in the final claim in the proof of Theorem 1.

To construct μ_x , first define a purely atomic measure ν_x supported on the G -component of x by declaring $\nu_x(\{y\}) = (1 - \varepsilon)^{\delta(x,y)}$, where δ denotes the graph metric. Subexponential growth of G ensures that $K = \sum_{y \in [x]_G} \nu_x(\{y\}) < \infty$. Finally, put $\mu_x = \frac{1}{K}\nu_x$. \square

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