

M5-Brane Amplitudes

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Abstract

The dynamics of a probe M5-brane, embedded as a hypersurface in eleven-dimensional Minkowski spacetime, is described by a six-dimensional world-volume theory. This theory has a variety of interesting symmetries some of which are obscure in the Lagrangian formulation of the theory. However, as summarized in this review, an alternative approach is to construct all of its on-shell tree-level scattering amplitudes. This enables understanding all of the symmetries in a satisfying way. This work is dedicated to the memory of Peter Freund.

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1 Introduction

I am pleased to have an opportunity to contribute to this volume in memory of a great physicist, my good friend Peter Freund. We never collaborated, but I often enjoyed discussing physics with Peter, and I admired his research. Reading his 2007 autobiographical reminiscences, aptly entitled *A Passion for Discovery*, I learned about Peter’s courage as a student activist in Romania, where in 1956 he faced a row of tanks that were aimed at him and his friends. It was fortunate for him, and the future of physics, that the tanks did not fire and that he was able to make his way to the United States. As it happens, my father was born in a nearby town in Romania that belonged to Hungary when he was there. The border was moved after the first world war.

Peter’s most famous and highly-cited paper, the 1980 “Freund–Rubin” paper [1], is only three pages long, yet it was remarkably insightful. It showed that 11-dimensional supergravity has a maximally supersymmetric solution, involving four-form flux, in which the 11d geometry is $AdS_4 \times S^7$, a product of a 4d anti de Sitter space and a 7d sphere. (There is also an analogous solution of the form $AdS_7 \times S^4$.) Even though I found this result interesting when it appeared, there were two reasons why I failed to fully appreciate it at that time.

The first reason for my failure to fully appreciate the Freund–Rubin solution was that it appeared when Michael Green and I were developing superstring theory, which requires ten dimensions. I was skeptical that 11d supergravity, which has severe UV divergences, could have a quantum completion. On the other hand, I was sure that superstring theory is UV finite. Therefore I mockingly described eleven dimensions as “a 10% error”. Of course, I now know better. The quantum completion of 11d supergravity even has a name – M theory. There might be a better formulation of M theory than is currently known, but the existence of M theory as a consistent quantum theory is beyond question.

The second reason for my failure to fully appreciate the Freund–Rubin solution was that I was interested in finding new ways to compactify extra dimensions that can break supersymmetries and leave four dimensions exactly or approximately flat while the extra dimensions are compact and highly curved, so that they are unobservable at sufficiently low energies. The Freund–Rubin solution requires that the anti de Sitter and sphere curvatures are comparable, which is certainly not realistic. Despite this misgiving, three years later, when I was studying type IIB supergravity – the low-energy effective description of type IIB superstring theory – Nick Warner and I realized that this theory has an $AdS_5 \times S^5$ solution that is analogous to the Freund–Rubin solution. This fact is mentioned for the first time in the concluding paragraph of [2].

The profound importance of such anti de Sitter solutions of superstring theory and M theory became clear to the theory community with Maldacena’s famous 1997 paper [3] proposing that such solutions have holographic dual descriptions as conformal field theories (CFTs), which can be regarded as residing at the boundary (at spatial infinity) of the anti de Sitter space.

2 D-branes and M-branes

The 3d CFT that is dual to the Freund–Rubin solution of M theory is associated to a stack of M2-branes, which are half-BPS extended objects (with two spatial dimensions) that exist in M theory. Similarly, the 6d CFT that is dual to the $AdS_7 \times S^4$ solution of M theory is given by a stack of M5-branes and the 4d CFT that is dual to the $AdS_5 \times S^5$ solution of type IIB superstring theory is given by a stack of D3-branes. The latter CFT, which is the one that has been studied in the greatest detail, is $\mathcal{N} = 4$ super Yang–Mills theory [4]. The rank N of the $U(N)$ gauge group corresponds to the number of units of 5-form flux threading the S^5 (and the AdS_5). In each case, N , the number of branes in the stack, is also the number of units of flux in the dual configuration. Indeed, the branes are the sources for such flux. The study of these three and other examples of AdS/CFT duality has been a major theme of theoretical physics for more than two decades.

D-branes and M-branes are also of interest from other perspectives. In particular, the world-volume theory of a single flat brane can be studied as a probe of M theory or superstring theory. In this setting, the word “probe” refers to an approximation in which the back reaction of the brane on the M theory or string theory background is neglected. This type of approximation is usually taken for granted in particle physics. The relevant world-volume theories of such probe branes are not conformal.² Rather, they are extensions of Born–Infeld theory, which is a nonlinear extension of Maxwell theory whose Lagrangian has the form $L \sim \sqrt{-\det(g_{\mu\nu} + kF_{\mu\nu})}$. Even though such theories probably do not have a quantum completion without including additional massless degrees of freedom, they are intriguing classical field theories with lots of symmetry and other special features that are useful for various purposes.

There are two basic approaches to describing these theories. One is to formulate the Lagrangian or, equivalently, the corresponding equations of motion. In the mid-1990s my students and I expended considerable effort formulating D-brane and M5-brane Lagrangians,

²However, if the brane is embedded in an anti de Sitter space and localized at a point in the other dimensions, then its world-volume theory has spontaneously broken conformal symmetry.

as did others. The Lagrangians for probe D-branes embedded in 10d Minkowski spacetime are precisely of the Born–Infeld type. In addition to the $U(1)$ gauge field of Born–Infeld theory, they incorporate additional bosonic and fermionic degrees of freedoms that, together with the $U(1)$ gauge field, comprise maximally supersymmetric vector multiplets. These are precisely the same kinds of supermultiplets that appear in maximally supersymmetric Yang–Mills theories.

The 6d M5-brane theory is also DBI-like even though it involves a two-form tensor field, with a self-dual field strength, instead of a Maxwell field. This type of field is quite awkward to incorporate in a Lagrangian description, though various ways to do so have been developed [5][6][7]. Despite the considerable effort expended in finding the formula for the Lagrangian, I think it is fair to say that it is not very illuminating. On dimensional reduction to 5d the self-dual tensor can be replaced by a vector, by means of a duality transformation, and then the M5-brane theory becomes the D4-brane theory. This procedure is sometimes called “double dimensional reduction”, because the 11d space is simultaneously reduced to 10d. The D-brane Lagrangians are less awkward than the M5-brane one, because they do not involve self-dual tensors.

A method of exploring properties of field theories that has become popular in recent years is to present formulas for all of their on-shell scattering amplitudes. Of course, this is the approach to the study of string theory that was utilized from its beginning in 1968, building on the S-matrix program that had been developed in the preceding decade, which is what I had been raised on in graduate school. Many clever methods have been developed in the past decade for constructing amplitudes more efficiently than by Feynman diagrams. Some of them have been inspired by string theory [8]. In some cases these amplitudes are given by remarkably elegant formulas. My research during the last few years has focused on such field theory amplitudes. Since the case of four dimensions appeared to be already in pretty good shape, I decided to focus on 6d supersymmetric field theories. I would prefer to be studying ten or eleven dimensions, but the spinor-helicity method that my collaborators and I have used, is not easily applied to those cases. The hope is that six dimensions is a useful step in that direction. It may be possible to apply spinor-helicity methods to ten or eleven dimensions, and there is already some literature on the subject, but they involve complications that I am not yet ready to confront.

The plan for the remainder of this paper is to summarize what my collaborators and I have learned about M5-brane scattering amplitudes [9][10] [11][12]. Specifically, we have derived explicit formulas for the on-shell n -particle tree superamplitudes of the M5-brane theory, which will be presented here. We have also obtained the amplitudes for several

other supersymmetric 6d theories, but I will not discuss them in this paper. The M5-brane formulas describe the scattering of n massless particles in 6d, each of which belongs to a single tensor supermultiplet with $(2, 0)$ supersymmetry. The formulas for these amplitudes are quite concise and they have a lot of symmetry, most of which is made manifest. For one thing, the amplitudes vanish unless n is even. This \mathbb{Z}_2 symmetry is generic for all Dirac–Born–Infeld (DBI)-like theories.

The n -particle tree superamplitude $\mathcal{A}_n(\lambda, \eta)$ will be presented in the form of an integral of an expression that contains a product of bosonic and fermionic delta functions. Carrying out the integrations is an algebraic problem, whose solution gives a rational function on the supermanifold times six momentum-conservation delta functions and eight fermionic delta functions that describe conservation of half of the supercharges. The supermanifold is parametrized by bosonic spinor-helicity coordinates λ_{ia}^A and Grassmann coordinates η_{ia}^I , where $i = 1, 2, \dots, n$ labels the particles. The other indices will be described later. One goal of the work that I am reviewing was to explore the extent to which we can determine these rational functions from symmetry considerations. Perhaps the lessons we have learned can be utilized in the study of other interesting theories. Since symmetries are central to our work, the plan is to discuss them in some detail before describing the amplitudes themselves in the remainder of this paper.

3 Symmetries of M5-brane superamplitudes

As we have said, we are interested in studying the effective 6d field theory associated to a flat probe M5-brane embedded in 11d Minkowski spacetime. Before describing the tree-level scattering amplitudes of this field theory, it is useful to understand their symmetries.

3.1 Total permutation symmetry of the n scattered superparticles

An entire supermultiplet, which consists of eight bosonic and eight fermionic degrees of freedom, is represented by a scalar function of four Grassmann coordinates, η_a^I . The superamplitude, describing all possible n -particle scattering amplitudes of the M5-brane theory, requires one such multiplet for each of the particles, η_{ia}^I , $i = 1, 2, \dots, n$. Total permutation symmetry, S_n , of $\mathcal{A}_n(\lambda_i, \eta_i)$ ensures that Bose and Fermi statistics are incorporated correctly. Note that there is no nonabelian gauge symmetry. Therefore, unlike Yang–Mills theories, there are no additional group-theory factors. In this respect Born–Infeld theories and world-volume theories of single branes are more like gravitational theories.

3.2 Poincaré and little-group invariance

The M5-brane theory is defined on a 6d Minkowski hypersurface embedded in an 11d Minkowski spacetime. Therefore $\text{Spin}(5, 1) \times \text{Spin}(5)$ is the unbroken subgroup of the 11d Lorentz group $\text{Spin}(10, 1)$. The six translation symmetries along the brane are preserved, whereas the five transverse to the brane are spontaneously broken. Indeed, the spectrum includes five massless scalars, which are the corresponding Goldstone bosons.

The 6d Lorentz group, $\text{Spin}(5, 1)$, is a noncompact version of $\text{SU}(4)$. Like $\text{SU}(4)$, it has two inequivalent four-dimensional spinor representations. In the case of $\text{Spin}(5, 1)$ they have opposite chirality. Their components are represented by indices $A, B = 1, 2, 3, 4$, written as superscripts for the “left-handed” representation and as subscripts for the “right-handed” one. Invariant tensors are ε_{ABCD} , ε^{ABCD} , and δ_A^B . The last one is used to contract superscripts with subscripts. The momentum six-vector can be expressed as an antisymmetric matrix, $P_{AB} = \sigma_{AB}^\mu P_\mu$ or $P^{AB} = \frac{1}{2}\varepsilon^{ABCD}P_{CD}$. Translation invariance leads to momentum conservation $P^{AB} = \sum_{i=1}^n p_i^{AB} = 0$. When we discuss amplitudes it will sometimes be useful to extract the momentum-conservation delta function and to define

$$\mathcal{A}_n = \delta^6(P^{AB})A_n. \quad (1)$$

In addition, the M5-brane theory has a $\text{USp}(4) = \text{Spin}(5)$ R-symmetry group. As implied above, this symmetry arises from invariance of the brane under rotations of the five M-theory directions transverse to the M5-brane. By definition, this is an R symmetry, because the supersymmetry charges belong to nontrivial representations of this group. R symmetry and supersymmetry will be described further in the subsequent subsections.

Massless particles in 6d have on-shell degrees of freedom classified by representations of the little group $\text{Spin}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$. There is a distinct such group for each particle. Each M5-brane excitation is described by a $(2, 0)$ tensor supermultiplet, which is a singlet of its $\text{SU}(2)_R$. Therefore, $\text{SU}(2)_R$ symmetry is trivial in the M5-brane theory, and we will not refer to $\text{SU}(2)_R$ any more in this work.³ Thus, on-shell particles are characterized by representations of the R-symmetry group $\text{USp}(4)$ and the little group $\text{SU}(2)_L$. Note that there is just one $\text{USp}(4)$ R-symmetry group, which classifies all n of the particles, but there are n separate $\text{SU}(2)_L$ little groups – one for each particle.

Indices $a, b, \dots = +, -$ are used to label doublets of $\text{SU}(2)_L$, but such labels are meaningless unless they are accompanied by a label $i, j, \dots = 1, 2, \dots, n$, so that we know which

³This property is not shared by the other 6d theories that we have studied. For this reason their amplitudes are somewhat more complicated. Also, super Yang–Mills and supergravity theories have nonvanishing amplitudes for odd n , which is another complicating feature.

particle's little group they refer to. (We prefer not to use the more cumbersome notation a_i, b_i, \dots) Invariant tensors (for a given little group) are ε^{ab} and ε_{ab} . The spectrum of the M5-brane theory consists of a single $(2, 0)$ tensor supermultiplet, which contains the representations $(\mathbf{5}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}) + (\mathbf{1}, \mathbf{3})$ of $\text{USp}(4) \times \text{SU}(2)_L$. For example, the third term is the chiral tensor, denoted $\beta^{ab} = \beta^{ba}$. In a covariant Lagrangian formalism a chiral tensor is described by a two-form with a self-dual field strength and the appropriate gauge symmetry. This formalism introduces complications that are circumvented when on-shell amplitudes are described using the spinor-helicity formalism that we will now describe. A $(2, 0)$ tensor multiplet is conveniently expressed as a scalar function of four Grassmann parameters η_a^I , where $I = 1, 2$. It is easy to verify that it contains the $\text{SU}(2)_L$ representations listed above with the correct multiplicities. The way that $\text{USp}(4)$ is realized will be explained later.

The only way that non-trivial Lorentz-group representations appear in our description of M5-brane amplitudes is through spinor-helicity coordinates λ_{ia}^A [13]. The basic idea is that the momentum of the i -th particle is given by

$$p_i^{AB} = \varepsilon^{ab} \lambda_{ia}^A \lambda_{ib}^B, \quad (2)$$

which is invariant under the i -th little group (and all the others). Because of this, three of the eight components of λ_i are redundant. The remaining five encode the five independent components of the momentum vector of a massless particle in 6d. (The mass squared is proportional to the Pfaffian of p_i^{AB} , which vanishes because the matrix only has rank two.) The 15 generators of the 6d Lorentz group are given by the traceless matrix

$$J_B^A = \lambda^A \cdot \frac{\partial}{\partial \lambda^B} - \frac{1}{4} \delta_B^A \lambda^C \cdot \frac{\partial}{\partial \lambda^C}. \quad (3)$$

The index contraction implied by the dots in this formula corresponds to the tensor δ_{jb}^{ia} . The rest of the Poincaré group is generated by the total momentum $P^{AB} = \lambda^A \cdot \lambda^B$. In this case the dot corresponds to $\delta^{ij} \varepsilon^{ab}$. We could also define generators of conformal transformations

$$K_{AB} = \frac{\partial}{\partial \lambda^A} \cdot \frac{\partial}{\partial \lambda^B}, \quad (4)$$

though the M5-brane theory does not have conformal symmetry. Similarly, the dilatation operator

$$D = \lambda^A \cdot \frac{\partial}{\partial \lambda^A} \quad (5)$$

also does not generate a symmetry. However, it is useful to know that $(D - 2n)A_n = 0$, which implies that $A_n \sim \lambda^{2n}$.

The $\text{SU}(2)_L$ little group of the i -th particle is generated by the traceless operator

$$j_{ia}^b = \eta_{ia}^I \frac{\partial}{\partial \eta_{ib}^I} - \frac{1}{2} \delta_a^b \eta_{ic}^I \frac{\partial}{\partial \eta_{ic}^I}. \quad (6)$$

The sum of the trace terms,

$$d = \eta^I \cdot \frac{\partial}{\partial \eta^I}, \quad (7)$$

is analogous to D . It even has the same eigenvalue, $(d - 2n)A_n = 0$. Thus, $A_n \sim \eta^{2n}$.

3.3 R symmetry

As explained in the previous subsection, the M5-brane theory has USp(4) R symmetry. However, since it is convenient to use the Grassmann parameters introduced in the previous subsection, only an SU(2) \times U(1) subgroup is manifest. The SU(2) factor is generated by

$$R_J^I = \eta^I \cdot \frac{\partial}{\partial \eta^J} - \frac{1}{2} \delta_J^I \eta^K \cdot \frac{\partial}{\partial \eta^K}. \quad (8)$$

The U(1) subgroup of the R-symmetry group, identified at the end of the previous subsection, is generated by

$$R = d - 2n = \frac{1}{2} \left(\eta^I \cdot \frac{\partial}{\partial \eta^I} - \frac{\partial}{\partial \eta^I} \cdot \eta^I \right). \quad (9)$$

Thus, the requirement that $d = 2n$ is a consequence of this U(1) subgroup of the R symmetry.

The remaining six R symmetries will not be manifest, though we will explain how to prove that they are symmetries of the amplitudes. These generators are given by a pair of symmetric matrices:

$$R^{IJ} = \eta^I \cdot \eta^J \quad \text{and} \quad R_{IJ} = \frac{\partial}{\partial \eta^I} \cdot \frac{\partial}{\partial \eta^J}. \quad (10)$$

It is easy to verify that these 10 generators give the USp(4) Lie algebra. One way to think about this is that generators $R^{\tilde{I}\tilde{J}} = R^{\tilde{J}\tilde{I}}$, where \tilde{I} and \tilde{J} take values from 1 to 4, can be obtained by replacing *subscripts* I, J by *superscripts* $I+2, J+2$. Thus $R_1^1 \rightarrow R^{13}$, $R_{12} \rightarrow R^{34}$, etc. This corresponds to raising an index using the antisymmetric symplectic metric $\omega^{\tilde{I}\tilde{J}}$ with $\omega^{13} = \omega^{24} = 1$. In this notation, the full R symmetry algebra is

$$[R^{\tilde{I}\tilde{J}}, R^{\tilde{K}\tilde{L}}] = \omega^{\tilde{I}\tilde{K}} R^{\tilde{J}\tilde{L}} + \omega^{\tilde{J}\tilde{K}} R^{\tilde{I}\tilde{L}} + \omega^{\tilde{I}\tilde{L}} R^{\tilde{J}\tilde{K}} + \omega^{\tilde{J}\tilde{L}} R^{\tilde{I}\tilde{K}}. \quad (11)$$

So the algebra is easy enough to understand. What is more challenging is to ensure that the amplitudes actually possess the symmetries generated by R^{IJ} and R_{IJ} .

As was already mentioned, the supermultiplet $\Phi(\eta)$ contains the representation $(\mathbf{5}, \mathbf{1}) + (\mathbf{4}, \mathbf{2}) + (\mathbf{1}, \mathbf{3})$ of USp(4) \times SU(2)_L. In the expansion of $\Phi(\eta)$ in powers of η the five scalars appear in three different terms, even though they form an irreducible R-symmetry multiplet. Similarly, the spinors appear in two terms. The five scalars can be described by an

antisymmetric matrix $\phi^{\bar{I}\bar{J}}$ with a vanishing symplectic trace ($\omega_{\bar{I}\bar{J}}\phi^{\bar{I}\bar{J}} = 0$).⁴ Similarly, the four little-group spinors are denoted $\psi^{a\bar{I}}$ and the little-group triplet remains β^{ab} as before. Writing these indices as superscripts is a matter of convention, since they can be lowered using the symplectic metrics ε_{ab} and $\omega_{\bar{I}\bar{J}}$.

3.4 Supersymmetry

M theory in an 11d Minkowski background has 32 conserved supercharges, and only half of them preserve a flat 6D hypersurface. Therefore the M5-brane theory has 16 conserved supercharges, which transform as $(\mathbf{4}, \mathbf{4})$ with respect to $\text{Spin}(5, 1) \times \text{USp}(4)$. Anticommutators of the broken supercharges give momenta that generate five transverse translations of the M5-brane, which are also broken symmetries. Like the scalars, the massless fermions of the M5-brane theory can be interpreted as Goldstone particles associated to the spontaneously broken symmetries.

Eight anticommuting supercharges are given by

$$Q^{AI} = \sum_{i=1}^n q_i^{AI} = \lambda^A \cdot \eta^I \quad \text{where} \quad q_i^{AI} = \varepsilon^{ab} \lambda_{ia}^A \eta_{ib}^I. \quad (12)$$

Their conservation can be implemented by Grassmann delta functions:

$$A_n(\lambda, \eta) = \delta^8(Q^{AI}) F_n(\lambda, \eta). \quad (13)$$

The other eight supercharges, which are also mutually anticommuting, are represented by

$$\bar{Q}_I^A = \sum_{i=1}^n \bar{q}_{iI}^A = \lambda^A \cdot \frac{\partial}{\partial \eta^I}, \quad \text{where} \quad \bar{q}_{iI}^A = \lambda_{ia}^A \frac{\partial}{\partial \eta_{ia}^I}. \quad (14)$$

The nonzero supersymmetry anticommutators are

$$\{Q^{AI}, \bar{Q}_J^B\} = \delta_J^I P^{AB}. \quad (15)$$

The entire supersymmetry algebra can be combined into the single equation

$$\{Q^{A\bar{I}}, Q^{B\bar{J}}\} = \omega^{\bar{I}\bar{J}} P^{AB}. \quad (16)$$

Because of the appearance of the factor $\delta^8(Q^{AI})$ in the amplitudes, half of the supersymmetry is manifest, and the other half needs to be proved. Also, only a subgroup of the $\text{USp}(4)$ R symmetry is manifest. The approach taken in [9] is to explicitly verify the rest of the R symmetry of the n -particle amplitudes, since this together with $\delta^8(Q^{AI})$ implies the rest of the supersymmetry.

⁴Equivalently, one could utilize a five-vector $\phi^\alpha = \frac{1}{2} \sigma_{\bar{I}\bar{J}}^\alpha \phi^{\bar{I}\bar{J}}$.

3.5 R symmetry of the four-particle amplitude

The four-particle amplitude of the M5-brane theory is given (up to a constant) by the deceptively simple formula

$$A_4 = \delta^8(Q^{AI}). \quad (17)$$

As before, $Q^{AI} = \sum_{i=1}^4 \varepsilon^{ab} \lambda_{ia}^A \eta_{ib}^I$. The formula (17) makes half of the supersymmetry manifest. Its main deficiency is that the USp(4) R symmetry is very obscure. The point is that the index $I = 1, 2$ labels a doublet of an SU(2) subgroup of the R symmetry. As mentioned earlier, the symmetry associated to the six generators R^{IJ} and R_{IJ} is far from obvious. The solution to this problem for $n = 4$, presented in [9], is repeated here. The generalization to all n is also given in [9].

For this purpose it is convenient to rename η_{i-}^I as η_i^I and η_{i+}^I as $\tilde{\eta}_i^I$. Then we Fourier transform the latter coordinates to conjugate Grassmann coordinates denoted ζ_{iI} . Thus, we consider

$$\tilde{A}_4 = \int d^8 \tilde{\eta}_i^I e^{\sum_{iI} \tilde{\eta}_i^I \zeta_{iI}} \delta^8 \left(\sum_{i=1}^4 \varepsilon^{ab} \lambda_{ia}^A \eta_{ib}^I \right). \quad (18)$$

Substituting an integral representation of the delta functions,

$$\delta^8(Q^{AI}) = \int d^8 \theta_{AI} e^{\theta_{AI} Q^{AI}}, \quad (19)$$

and carrying out the $\tilde{\eta}$ integrations gives

$$\tilde{A}_4 = \int d^8 \theta_{AI} \delta^8(\zeta_{iI} + \sum_A \theta_{AI} \lambda_{i-}^A) e^{\sum_{AI} \theta_{AI} \lambda_{i+}^A \eta_i^I}. \quad (20)$$

If the 4×4 matrix λ_{i-}^A is nonsingular, which is generically the case, then

$$\delta^8(\zeta_{iI} + \sum_A \theta_{AI} \lambda_{i-}^A) = (\det \lambda_-)^2 \delta^8((\zeta \lambda_-^{-1})_{IA} + \theta_{AI}), \quad (21)$$

where $(\zeta \lambda_-^{-1})_{IA} = \sum_i \zeta_{iI} (\lambda_-^{-1})_{iA}$. Thus,

$$\tilde{A}_4 = (\det \lambda_-)^2 \exp(-\text{tr}(\zeta \lambda_-^{-1} \lambda_+ \eta)). \quad (22)$$

More explicitly, the exponent is

$$-\text{tr}(\zeta \lambda_-^{-1} \lambda_+ \eta) = \text{tr}(\lambda_-^{-1} \lambda_+ \eta \zeta) = \sum_{ij} (\lambda_-^{-1} \lambda_+)_{ij} (\eta \zeta)_{ji}, \quad (23)$$

where $(\eta \zeta)_{ji} = \eta_j^I \zeta_{iI}$.

Momentum conservation implies that $(\lambda_+ \lambda_-^T)^{AB} = (\lambda_- \lambda_+^T)^{AB}$, and therefore $(\lambda_-^{-1} \lambda_+)^{ij}$ is a symmetric matrix. Since only the symmetric part of $(\eta \zeta)_{ji}$ contributes, it can be replaced by half of

$$E_{ij} = \sum_{I=1}^2 (\eta_i^I \zeta_{Ij} + \eta_j^I \zeta_{Ii}). \quad (24)$$

E can now be rewritten in a form with manifest $\text{USp}(4)$ R symmetry

$$E_{ij} = \sum_{\tilde{I}, \tilde{J}=1}^4 \omega_{\tilde{I}\tilde{J}} \eta_i^{\tilde{I}} \eta_j^{\tilde{J}}, \quad (25)$$

where we have renamed $\zeta_{Ii} = \eta_i^{I+2}$. As before, the only nonzero elements of the symplectic metric $\omega_{\tilde{I}\tilde{J}}$ are $\omega_{13} = \omega_{24} = -\omega_{31} = -\omega_{42} = 1$. Then $\eta_i^{\tilde{I}}$, with $\tilde{I} = 1, 2, 3, 4$, belongs to the fundamental representation of the $\text{USp}(4)$ R-symmetry group. To summarize, we have shown that \tilde{A}_4 can be written in the manifestly R-symmetric form

$$\tilde{A}_4 = \Delta_4 e^{-\frac{1}{2} \sum_{ij} (K_4)_{ij} E_{ij}}, \quad (26)$$

where

$$\Delta_4 = (\det \lambda_-)^2 \quad \text{and} \quad (K_4)_{ij} = (\lambda_-^{-1} \lambda_+)^{ij}. \quad (27)$$

A_4 can be recovered as the inverse Grassmann Fourier transform. In conclusion, $\text{USp}(4)$ R symmetry is a property of \tilde{A}_4 , and not A_4 , which is good enough. However, the formula for A_4 is simpler and makes the supersymmetry more transparent.

3.6 Symplectic Grassmannian

The n -particle M5-brane superamplitude can be written in a way that depends on a symplectic Grassmannian, $\mathbb{L}\mathbb{G}(n, 2n)$, which has $\text{USp}(2n)$ symmetry [12]. The way this works is that the subscripts ia on the λ 's and η 's are combined to label the fundamental $2n$ -dimensional representation of $\text{USp}(2n)$. $\text{USp}(2)^n$, the product of the n little groups, each of which is $\text{SU}(2)_L = \text{USp}(2)$, is a subgroup of this $\text{USp}(2n)$.⁵ This combines and extends the little-group symmetries and the permutation symmetry of the amplitude.

The symmetry appears in the form of a symplectic Grassmannian, denoted $\mathbb{L}\mathbb{G}(n, 2n)$, which is a homogeneous space of $n(n+1)/2$ complex dimensions. One description of this space is $\text{USp}(2n)/\text{U}(n)$. In particular, the four-particle amplitude is given (up to a constant) by $\mathcal{A}_4 = \delta^6(P^{AB}) \delta^8(Q^{AI})$. Since $P^{AB} = \lambda^A \cdot \lambda^B$ and $Q^{AI} = \lambda^A \cdot \eta^I$, the $\text{USp}(8)$ symmetry

⁵Other analogous examples include the D3-brane theory in which $\text{U}(1)^n$ is enhanced to $\text{U}(n)$ and the M2-brane theory in which \mathbf{Z}_2^n is enhanced to $\text{O}(n)$.

of this amplitude is manifest. Also, these factors generalize to all n , though only even n is relevant.

4 n-particle M5-brane amplitudes

In the spirit of the CHY construction of n -particle scattering amplitudes [14][15], the formulas can be schematically summarized in the following form,

$$\mathcal{A}_n = \int d\mu_n^{6D} \mathcal{I}_L \mathcal{I}_R, \quad (28)$$

where the measure $d\mu_n^{6D}$, which is theory independent, encodes the 6d massless kinematics including momentum conservation. It provides a map from 6d kinematics to punctures of the Riemann sphere. The factors \mathcal{I}_L and \mathcal{I}_R determine the specific theory under consideration.

A coordinate σ_i is assigned to each of the n massless external particles. These coordinates are defined up to an overall common $\text{SL}(2, \mathbb{C})_\sigma$ Möbius-group transformation,

$$\sigma_i \rightarrow \frac{a\sigma_i + b}{c\sigma_i + d}, \quad ad - bc = 1. \quad (29)$$

This allows the coordinates of three of the punctures to be given arbitrary distinct values. The contribution to the integration measure, $\prod_{i=1}^n d\sigma_i / \text{vol}(\text{SL}(2, \mathbb{C})_\sigma)$, is defined in a standard way.

A function of the σ coordinates $F(\{\sigma_i\})$ is said to have weight w if it transforms under a Möbius transformation by the rule

$$F\left(\left\{\frac{a\sigma_i + b}{c\sigma_i + d}\right\}\right) = \left[\prod_{i=1}^n (c\sigma_i + d)\right]^w F(\{\sigma_i\}). \quad (30)$$

The measure $d\mu_n^{6D}$, defined below, transforms with weight $w = -4$. Therefore, $\mathcal{I}_L \mathcal{I}_R$ must have weight 4. In practice, each of the two factors, \mathcal{I}_L and \mathcal{I}_R , has weight 2.

The presentation that follows is based on [12]. For even n , which is all that is required for the M5-brane theory, the n -particle measure is given by

$$\prod_{i=1}^n \delta(p_i^2) \int d\mu_n^{6D} = \int \frac{\prod_{i=1}^n d\sigma_i \prod_{k=0}^m d^8 \rho_k}{\text{vol}(\text{SL}(2, \mathbb{C})_\sigma \times \text{SL}(2, \mathbb{C})_\rho)} \frac{1}{V_n^2} \prod_{i=1}^n \delta^6 \left(p_i^{AB} - \frac{\langle \rho^A(\sigma_i) \rho^B(\sigma_i) \rangle}{\prod_{j \neq i} \sigma_{ij}} \right), \quad (31)$$

where

$$n = 2m + 2, \quad \sigma_{ij} = \sigma_i - \sigma_j, \quad \text{and} \quad V_n = \prod_{i < j} \sigma_{ij}. \quad (32)$$

The mass-shell delta functions allow the momenta p_i^{AB} to be expressed in terms of spinor-helicity coordinates λ_{ia}^A . The maps in the delta functions are given by degree- m polynomials,

$$\rho_a^A(\sigma) = \sum_{k=0}^m \rho_{a,k}^A \sigma^k. \quad (33)$$

They are determined up to an overall $\text{SL}(2, \mathbb{C})_\rho$ transformation, which is a complexification of $\text{SU}(2)_L$, and its volume also is divided out. The $\text{SL}(2, \mathbb{C})_\sigma$ transformations of the coordinates $\rho_{a,k}^A$ are determined by requiring that the expressions inside the delta functions in (31) are invariant. Then one can show that $d\mu_n^{6\text{D}}$ has weight -4 .

As shown in [10], by introducing n additional 2×2 matrices $(W_i)_a^b$, the dependence on the ρ coordinates can be recast in the linear form

$$\int d\mu_n^{6\text{D}} = \int \frac{\prod_{i=1}^n d\sigma_i d^4 W_i \prod_{k=0}^m d^8 \rho_k}{\text{vol}(\text{SL}(2, \mathbb{C})_\sigma \times \text{SL}(2, \mathbb{C})_\rho)} \prod_{i=1}^n \delta^8(\lambda_{ia}^A - (W_i)_a^b \rho_b^A(\sigma_i)) \delta\left(|W_i| - \frac{1}{\prod_{j \neq i} \sigma_{ij}}\right), \quad (34)$$

where $|W_i| = \det W_i$. The indices a and b of $(W_i)_a^b$ refer to different groups. Specifically, b is contracted with the $\text{SL}(2, \mathbb{C})$ index of the moduli $\rho_{k,b}^A$, and therefore it is a global index, whereas a is associated with the little group of the i -th particle.

Integrating out the moduli $\rho_{a,k}^A$ [16] leaves

$$\int d\mu_n^{6\text{D}} = \int \frac{\prod_{i=1}^n d\sigma_i d^4 W_i}{\text{vol}(\text{SL}(2, \mathbb{C})_\sigma \times \text{SL}(2, \mathbb{C})_W)} \prod_{k=0}^m \delta^{2 \times 4} \left(\sum_{i=1}^n (W_i)_a^b \sigma_i^k \lambda_i^{Aa} \right) \prod_{i=1}^n \delta\left(|W_i| - \frac{1}{\prod_{j \neq i} \sigma_{ij}}\right), \quad (35)$$

where $\text{SL}(2, \mathbb{C})_\rho$ has become $\text{SL}(2, \mathbb{C})_W$, which is the symmetry acting on the global little-group index b . We can now show the emergence of the symplectic Grassmannian by defining the $n \times 2n$ matrix

$$C_{k,b;i,a} = (W_i)_a^b \sigma_i^k. \quad (36)$$

We have grouped the exponent k with the global $\text{SL}(2, \mathbb{C})$ index b to label n rows and the index i with the i -th little-group index a to label $2n$ columns. The matrix C formed in this way satisfies the identity

$$C \cdot \Omega \cdot C^T = 0, \quad (37)$$

where Ω is the $\text{USp}(2n)$ metric

$$\Omega = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}, \quad (38)$$

and \mathbb{I}_n is the $n \times n$ identity matrix. (37) is proved by using the delta-function constraints and the theorem⁶

$$\sum_{i=1}^n \frac{\sigma_i^K}{\prod_{j \neq i} \sigma_{ij}} = 0 \quad \text{for } K = 0, 1, \dots, n-2. \quad (39)$$

If M is a symplectic matrix belonging to $\text{USp}(2n)$, satisfying $M^T \cdot \Omega \cdot M = \Omega$, the identity (37) is invariant under the symplectic transformation $C \rightarrow C \cdot M^T$. The scattering-equation constraints can then be encoded as

$$\prod_{k=0}^m \delta^{2 \times 4} \left(\sum_{i=1}^n C_{k,b;i,a} \lambda_i^{Aa} \right) = \delta^{n \times 4} (C \cdot \Omega \cdot \Lambda^A), \quad (40)$$

with $C \cdot \Omega \cdot C^T = 0$. We have introduced a $2n$ -dimensional vector Λ^A , which is also a Lorentz spinor, built out of the spinor-helicity coordinates $\lambda_{i,a}^A$,

$$\Lambda^A := \{\lambda_{1,1}^A, \lambda_{2,1}^A, \dots, \lambda_{n,1}^A, \lambda_{1,2}^A, \lambda_{2,2}^A, \dots, \lambda_{n,2}^A\}. \quad (41)$$

Invariance under symplectic transformations requires that $\Lambda^A \rightarrow M \cdot \Lambda^A$.

Let us verify that C parametrizes $\mathbb{LG}(n, 2n)$. First of all, it has $\text{USp}(2n)$ symmetry, as required. We can also check the complex dimension of the space that it parametrizes, which is supposed to be $n(n+1)/2$. C is an $n \times 2n$ complex matrix, which has $2n^2$ complex dimensions. However, it can be multiplied on the left by an arbitrary $\text{GL}(n, \mathbb{C})$ matrix, without changing the constraint equations $C \cdot \Omega \cdot C^T = 0$. Altogether, we are left with $2n^2 - n^2 - n(n-1)/2 = n(n+1)/2$ complex dimensions, as required. These two facts uniquely characterize $\mathbb{LG}(n, 2n)$. $\text{GL}(n, \mathbb{C})$ transformations also preserve the scattering equations $C \cdot \Omega \cdot \Lambda^A = 0$

The formula that describes the tree amplitudes of the M5-brane theory is given by [12]

$$\mathcal{A}_n^{\text{M5}} = \int d\mu_n^{6\text{D}} \mathcal{I}_L^{(2,0)} \mathcal{I}_R^{\text{DBI}}, \quad (42)$$

where the factors $\mathcal{I}_L^{(2,0)}$ and $\mathcal{I}_R^{\text{DBI}}$ in the integrand are

$$\mathcal{I}_L^{(2,0)} = \delta^{n \times 2} (C \cdot \Omega \cdot \eta^I) V_n \text{Pf}' S_n, \quad \mathcal{I}_R^{\text{DBI}} = (\text{Pf}' S_n)^2 = \det' S_n. \quad (43)$$

S_n is an $n \times n$ antisymmetric matrix that has rank $n-2$ and is given by

$$[S_n]_{ij} = \frac{p_i \cdot p_j}{\sigma_{ij}}. \quad (44)$$

⁶This theorem is easy to establish by showing that the residues of the poles vanish when K is a non-negative integer and that the expression vanishes at infinity for $K < n-1$.

The reduced Pfaffian of S_n is defined as

$$\text{Pf}' S_n = \frac{(-1)^{k+l}}{\sigma_{kl}} \text{Pf}(S_n)_{kl}, \quad (45)$$

where $(S_n)_{kl}^{kl}$ is an $(n-2) \times (n-2)$ matrix with the k -th and l -th rows and columns of S_n removed, and the result is independent of the choice of k, l . Since $\text{Pf}' S_n$ has conformal weight $w = 1$, we see that $\mathcal{I}_R^{\text{DBI}}$ has conformal weight $w = 2$, as required. This factor, which is only nonzero for even n , appears in all DBI-type theories. These theories only have nonvanishing amplitudes when n is even. The factor $\mathcal{I}_L^{(2,0)}$ also has weight 2, since $\delta^{n \times 2}(C \cdot \Omega \cdot \eta^I)$ has weight n and V_n has weight $1 - n$. As the notation suggests, it implements $(2, 0)$ supersymmetry.

In conclusion, the M5-brane theory has an interesting mix of symmetries, and its tree amplitudes can be written in a form that enables one to understand all of them. The form of the scattering equations described in this paper is sometimes referred to as the “rational maps” approach. An alternative, based on “polarized scattering equations,” has been proposed by Geyer and Mason [17]. One of the main results of [12] is to show that the two descriptions are related by a $\text{GL}(n, \mathbb{C})$ transformation of C . A possible goal for the future is to extend this type of analysis to string theory and M-theory amplitudes.

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