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The large-scale structure of homogeneous turbulence

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A field of homogeneous turbulence generated at an initial instant by a distribution of random impulsive forces is considered. The statistical properties of the forces are assumed to be such that the integral moments of the cumulants of the force system all exist. The motion generated has the property that at the initial instant

$$E(\kappa) = C\kappa^2 + o(\kappa^2),$$

where $E(\kappa)$ is the energy spectrum function, κ is the wave-number magnitude, and C is a positive number which is not in general zero. The corresponding forms of the velocity covariance spectral tensor and correlation tensor are determined. It is found that the terms in the velocity covariance $R_{ij}(\mathbf{r})$ are $O(r^{-3})$ for large values of the separation magnitude r.

An argument based on the conservation of momentum is used to show that C is a dynamical invariant and that the forms of the velocity covariance at large separation and the spectral tensor at small wave number are likewise invariant. For isotropic turbulence, the Loitsianski integral diverges but the integral

$$\int_0^\infty r^2 R(r)\,dr = \tfrac{1}{2}\pi C$$

exists and is invariant.

1. Introduction

Batchelor & Proudman (1956) have studied the asymptotic form for large separations of the second- and third-order velocity correlation tensors in homogeneous turbulence and determined the corresponding forms of the spectral tensors near the origin of wave-number space. Their work was based on the following hypothesis (my italics):

'Homogeneous turbulence that is generated by placing a grid in a uniform stream carrying small velocity fluctuations has a large-scale structure like that which would develop, by dynamical action, in a field of turbulence which at some instant t_0 is homogeneous and has convergent integral moments of cumulants of the *velocity* distribution.'

The purpose of the present note is to explore the consequences of a different hypothesis, which is as follows:

Homogeneous turbulence has a large-scale structure like that which would develop, by dynamical action, in a field of turbulence which at some instant t_0 is homogeneous and has convergent integral moments of cumulants of the *vorticity* distribution or, equivalently, is generated at t_0 by random impulsive forces with convergent integral moments of cumulants.

The modified hypothesis is less restrictive because the convergence of velocity moments implies the convergence of vorticity moments, but not vice versa. Before we give the reasons for believing that the less restrictive hypothesis may apply to general cases of homogeneous turbulence, it is as well to see that major changes in the large-scale structure may result. We follow as closely as possible the notation of Batchelor & Proudman (1956) and Batchelor (1953). Consider the initial instant. If all integral moments of the cumulants of the vorticity distribution exist, then the spectral tensor of the vorticity correlation has the expansion:

$$\begin{aligned}\frac{1}{(8\pi)^3}\int \overline{\omega_i \omega_j'} \exp(-i\boldsymbol{\kappa}.\mathbf{r})\,d\mathbf{r} &= \Omega_{ij}(\boldsymbol{\kappa}) \\ &= B_{ij} + \kappa_k B_{ijk} + \kappa_k \kappa_l B_{ijkl} + O(\kappa^3),\end{aligned} \tag{1.1}$$

where the tensor coefficients are independent of $\boldsymbol{\kappa}$. Since Ω_{ij} is solenoidal in both the suffixes i and j, it follows that $B_{ij} = 0$. Cramer's theorem (see Batchelor 1953, §2.4) implies that $B_{ijk} = 0$. The solenoidal condition then implies that

$$\kappa_i \kappa_k \kappa_l B_{ijkl} = \kappa_j \kappa_k \kappa_l B_{ijkl} = 0. \tag{1.2}$$

It can be shown (see appendix) that the general form of B_{ijkl} is

$$B_{ijkl} = \epsilon_{ik\alpha}\epsilon_{jl\beta} M_{\alpha\beta}, \tag{1.3}$$

where $M_{\alpha\beta}$ is an arbitrary symmetrical second-order tensor independent of $\boldsymbol{\kappa}$. (The symmetry is a consequence of the fact that B_{ijkl} is symmetrical in the i and j indices because $\Omega_{ij}(\boldsymbol{\kappa}) = \Omega_{ji}(-\boldsymbol{\kappa})$). Then it follows that

$$\begin{aligned}\Omega_{ij}(\boldsymbol{\kappa}) &= \epsilon_{ik\alpha}\epsilon_{jl\beta}\kappa_k\kappa_l M_{\alpha\beta} + O(\kappa^3) \\ &= \left[\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right)\left(\delta_{\alpha\beta} - \frac{\kappa_\alpha\kappa_\beta}{\kappa^2}\right) - \left(\delta_{i\alpha} - \frac{\kappa_i\kappa_\alpha}{\kappa^2}\right)\left(\delta_{\beta j} - \frac{\kappa_j\kappa_\beta}{\kappa^2}\right)\right]\kappa^2 M_{\alpha\beta} + O(\kappa^3),\end{aligned} \tag{1.4}$$

and in particular

$$\Omega_{ii}(\boldsymbol{\kappa}) = \left(\delta_{\alpha\beta} - \frac{\kappa_\alpha\kappa_\beta}{\kappa^2}\right)\kappa^2 M_{\alpha\beta} + O(\kappa^3). \tag{1.5}$$

The relation between Ω_{ij} and the spectral tensor of the velocity correlation Φ_{ij} is (Batchelor 1953, §3.2)

$$\Omega_{ij} = (\delta_{ij}\kappa^2 - \kappa_i\kappa_j)\Phi_{kk} - \kappa^2\Phi_{ji}, \tag{1.6}$$

from which it follows that

$$\Phi_{ij} = \left(\delta_{i\alpha} - \frac{\kappa_i\kappa_\alpha}{\kappa^2}\right)\left(\delta_{j\beta} - \frac{\kappa_j\kappa_\beta}{\kappa^2}\right) M_{\alpha\beta} + O(\kappa), \tag{1.7}$$

$$\Phi_{ii} = \left(\delta_{\alpha\beta} - \frac{\kappa_\alpha\kappa_\beta}{\kappa^2}\right) M_{\alpha\beta} + O(\kappa), \tag{1.8}$$

at the initial instant t_0.

The energy spectrum function follows from (1.8) and is found to be

$$E(\kappa) = \tfrac{4}{3}\pi M_{\alpha\alpha}\kappa^2 + O(\kappa^3), \tag{1.9}$$

which differs from the hitherto generally accepted form in which $E(\kappa) \propto \kappa^4$ for small κ.† If the form (1.9) persists for all time, it is evident that the large-scale structure will differ in many ways. In particular, the energy in the final period of decay will decay like $t^{-\frac{3}{2}}$ instead of $t^{-\frac{5}{2}}$.

If we impose the initial condition that Φ_{ij} be analytic, then we must have $M_{\alpha\beta} = 0$, and the usual $E(\kappa) \propto \kappa^4$ is obtained. However, we have found no kinematic reason why $M_{\alpha\beta}$ should be zero at the initial instant and in fact it will now be shown that plausible initial conditions (which may, however, for special reasons not be those applicable to the approximately homogeneous turbulence downstream of a grid in a uniform stream with small velocity fluctuations; see §6) give non-zero values to $M_{\alpha\beta}$.

For future reference, we list the forms taken by the spectral tensors for isotropic turbulence, where isotropy implies that

$$M_{\alpha\beta} = M\delta_{\alpha\beta}. \tag{1.10}$$

The vorticity spectral tensor is

$$\Omega_{ij} = M(\kappa^2\delta_{ij} - \kappa_i\kappa_j) + O(\kappa^4), \tag{1.11}$$

the velocity spectral tensor is

$$\Phi_{ij} = M\{\delta_{ij} - (\kappa_i\kappa_j/\kappa^2)\} + O(\kappa^2), \tag{1.12}$$

and the energy spectrum function is

$$E(\kappa) = 4\pi M\kappa^2 + O(\kappa^4). \tag{1.13}$$

It is clear that M must be positive.

2. A motion generated by random impulsive forces

Let us suppose that the turbulent motion is generated by an impulsive force system $\mathbf{f}(\mathbf{x})$ per unit mass distributed throughout the fluid, where $\mathbf{f}(\mathbf{x})$ is a stationary random function of $\mathbf{x}$ with zero mean. Moreover, we shall assume that the correlation $\overline{f_i(\mathbf{x})f_j(\mathbf{x}')}$ decreases exponentially as the separation increases so that all integral moments of the cumulants of the force correlation exist. Then the spectral tensor of the force correlation exists and is analytic. We denote it by $\mathscr{M}_{ij}(\boldsymbol{\kappa})$, and we have the expansion

$$\mathscr{M}_{ij}(\boldsymbol{\kappa}) = M_{ij} + O(\kappa). \tag{2.1}$$

(The use of the symbol M_{ij} is deliberate as we intend to show that (1.4) and (1.7) are the vorticity and velocity spectral tensors generated by this impulsive force field.)

The velocity field generated initially by this impulsive force is

$$\mathbf{u} = \nabla\phi + \mathbf{f}, \tag{2.2}$$

where $$\nabla^2\phi = -\operatorname{div}\mathbf{f}. \tag{2.3}$$

The initial vorticity is $$\boldsymbol{\omega} = \operatorname{curl}\mathbf{f}. \tag{2.4}$$

† Birkhoff (1954) claimed that the κ^4 dependence was not necessary, and that $E(\kappa) \propto \kappa^2$ was a theoretical possibility compatible with the Navier–Stokes equations.

Hence, the integral moments of the cumulants of $\boldsymbol{\omega}$ converge if those of $\mathbf{f}$ do. We have from (2.4) on taking the Fourier transform of the relation

$$\overline{\omega_i(\mathbf{x})\omega_j(\mathbf{x}')} = V_{ij}(\mathbf{r}) = -\epsilon_{ik\alpha}\epsilon_{jl\beta}\partial^2(\overline{f_\alpha(\mathbf{x})f_\beta(\mathbf{x}')})/\partial r_k \partial r_l$$

that

$$\Omega_{ij}(\boldsymbol{\kappa}) = \epsilon_{ik\alpha}\epsilon_{jl\beta}\kappa_k\kappa_l\mathscr{M}_{\alpha\beta}(\boldsymbol{\kappa}). \tag{2.5}$$

Since any vorticity spectral tensor can be written in the form (2.5), where $\mathscr{M}_{\alpha\beta}(\boldsymbol{\kappa})$ is analytic if $\Omega_{ij}(\boldsymbol{\kappa})$ is, convergent integral moments of the vorticity cumulants imply the existence of an equivalent impulsive force distribution with convergent integral moments of the cumulants. It follows from (1.5) that

$$\Phi_{ij}(\boldsymbol{\kappa}) = \{\delta_{i\alpha} - (\kappa_i\kappa_\alpha)/\kappa^2\}\{\delta_{j\beta} - (\kappa_j\kappa_\beta)/\kappa^2\}\mathscr{M}_{\alpha\beta}(\boldsymbol{\kappa}). \tag{2.6}$$

On substituting (2.1), we have immediately (1.4) and (1.6), showing the relation as claimed between the spectral tensor of the impulsive forces and the spectral tensor of the velocity and vorticity fields.

It is now clear from (2.5) that the hypothesis that the turbulence is generated by random impulsive forces, is equivalent to the statement that the integral moments of the vorticity correlation all exist.

Any motion of an incompressible fluid can be produced from rest by a suitable distribution of impulsive forces. Thus turbulent motions generated in accordance with the hypothesis of Batchelor & Proudman (1956) can be regarded as produced by an impulsive force system with convergent integral moments of cumulants, and subject to the additional restriction that the force system generates a velocity field with an analytic spectral tensor. For this additional restriction to be satisfied, it is not sufficient that $\mathscr{M}_{ij}(0) = M_{ij}$ should be zero; the necessary condition is that $\mathscr{M}_{ij}(\boldsymbol{\kappa})$ should be solenoidal in both indices in order that $\Phi_{ij}(\boldsymbol{\kappa})$ should be analytic. The equivalent condition on the forces is

$$\operatorname{div} \mathbf{f} = 0. \tag{2.7}$$

Thus the Batchelor & Proudman hypothesis is entirely equivalent to the hypothesis of the present paper with the additional restriction that the impulsive forces are solenoidal.

It is important to consider the significance of a non-zero value of M_{ij}. We have that

$$M_{ij} = \frac{1}{8\pi^3}\int \overline{f_i(\mathbf{x})f_j(\mathbf{x}+\mathbf{r})}\,d\mathbf{r}$$
$$= \frac{1}{8\pi^3}\lim_{V\to\infty}\frac{1}{V}\int_V\int_V f_i(\mathbf{x})f_j(\mathbf{x}')\,d\mathbf{x}\,d\mathbf{x}'$$

(using the ergodic hypothesis that mean values are equivalent to spatial averages)

$$= \frac{1}{8\pi^3}\lim_{V\to\infty}\left\{\frac{1}{V^{\frac{1}{2}}}\int_V f_i(\mathbf{x})\,d\mathbf{x}\right\}\left\{\frac{1}{V^{\frac{1}{2}}}\int f_j(\mathbf{x}')\,d\mathbf{x}'\right\}. \tag{2.8}$$

Thus a non-zero value of M_{ij} results when the force applied to a volume V in a realization of the flow field is proportional to $V^{\frac{1}{2}}$, which is what one would expect if the forces add up randomly without the existence of a negative correlation.

It will clearly depend on how the turbulence is generated whether the momentum applied to a volume V will increase like $V^{\frac{1}{2}}$ as V increases, or whether it will be bounded in which case the limit (2.8) will be zero and $M_{ij} = 0$. For turbulent motion which develops out of a dynamical instability of fluid motion, it is perhaps intuitively appealing to suppose that the fluctuating momentum in a large volume V remains bounded as V increases indefinitely, or in other words that the mechanism producing the turbulence can only produce a finite momentum in an infinitely large volume. In this case, we would have in general

$$\mathscr{M}_{ij} = M_{ijkl}\kappa_k\kappa_l + O(\kappa^3), \tag{2.9}$$

where M_{ijkl} is independent of $\boldsymbol{\kappa}$, from which we deduce

$$\Phi_{ij}(\boldsymbol{\kappa}) = \left(\delta_{i\alpha} - \frac{\kappa_i\kappa_\alpha}{\kappa^2}\right)\left(\delta_{j\beta} - \frac{\kappa_j\kappa_\beta}{\kappa^2}\right)\kappa_k\kappa_l M_{\alpha\beta kl} + O(\kappa^3). \tag{2.10}$$

This is exactly the form of the spectral tensor obtained by Batchelor & Proudman (1956, equation (5.32)) for times $t > t_0$, given that Φ_{ij} was analytic at $t = t_0$. Hence the results for the case $M_{ij} = 0$ are those of Batchelor & Proudman. In particular, $E(\kappa) \propto \kappa^4$. Moreover, Batchelor & Proudman have worked out the corresponding form of the velocity correlation tensor and the properties of the final period of decay. It should be noted, however, that although the present hypothesis with $M_{ij} = 0$ gives the same results as the Batchelor & Proudman hypothesis, the two postulates are still not entirely equivalent since $\operatorname{div} \mathbf{f} = 0$ implies $M_{ij} = 0$ but not vice versa.

On the other hand, it is reasonable to suppose that for some cases of homogeneous turbulence $M_{ij} \neq 0$. Certainly it is easy to visualize a laboratory experiment whereby a turbulent flow field with this property is produced, although practical achievement is another matter. We shall now examine the flow generated according to our hypothesis for the general case $M_{ij} \neq 0$.

3. The permanence of the big eddies

The velocity spectral tensor will develop with time under the dynamical action of the turbulence as described by the Navier–Stokes equations. We now present intuitive arguments to show that the leading term in the expansion of $\Phi_{ij}(\boldsymbol{\kappa}, t)$ about $\kappa = 0$ remains of the form (1.7), and moreover that the tensor coefficients are constants independent of time. At a later time t, there will be an impulsive force system $-\mathbf{f}(\mathbf{x}, t)$ which when applied to the turbulent motion generated by the initial forces $\mathbf{f}(\mathbf{x}, t_0)$ will make the velocity everywhere zero. Now the momentum applied initially to a large volume V is $O(V^{\frac{1}{2}})$. Momentum escapes from this volume due to the action of pressure forces across the surface of V and the convection of momentum by the fluid. However, both these processes are random, and the net transfer of momentum will be proportional to the square root of the surface area, i.e. the momentum loss is $O(V^{\frac{1}{3}})$. Thus the impulse of the force system required to stop the motion will differ from that of the initial impulse applied to the fluid by an amount of order $V^{\frac{1}{3}}$ in a volume of order V. That is

$$\frac{1}{V^{\frac{1}{2}}}\int_V \mathbf{f}(\mathbf{x}, t)\, d\mathbf{x} = \frac{1}{V^{\frac{1}{2}}}\int_V \mathbf{f}(\mathbf{x}, t_0)\, d\mathbf{x} + O(V^{-\frac{1}{6}}). \tag{3.1}$$

It follows that $$\int \overline{f_i(\mathbf{x},t)f_j(\mathbf{x}',t)}\,d\mathbf{r} = \int \overline{f_i(\mathbf{x},t_0)f_j(\mathbf{x}',t_0)}\,d\mathbf{r} = M_{ij}, \tag{3.2}$$

the value at the initial instant. As shown in §2, the left-hand side of (3.2) determines the leading term in the expansion of $\Phi_{ij}(\boldsymbol{\kappa},t)$ about $\boldsymbol{\kappa}=0$, for quite generally

$$\Omega_{ij}(\boldsymbol{\kappa},t) = \epsilon_{ik\alpha}\epsilon_{jl\beta}\kappa_k\kappa_l\mathscr{M}_{\alpha\beta}(\boldsymbol{\kappa},t), \tag{3.3}$$

where $\mathscr{M}_{\alpha\beta}(\boldsymbol{\kappa},t)$ is the spectral tensor of the impulsive force system required to generate from rest the velocity field that exists at time t. The result (3.2) shows that $\mathscr{M}_{\alpha\beta}(0,t)$ exists and is constant.

We have not shown that $\mathscr{M}_{\alpha\beta}(\boldsymbol{\kappa},t)$ is analytic at general time t, because our argument does not say anything about the integral moments of $\overline{f_i(\mathbf{x},t)f_j(\mathbf{x}',t)}$, and it is possible that the redistribution of momentum by the turbulent motion implies that integral moments of high order may not exist. But if the integral on the left-hand side of (3.2) exists, then

$$\mathscr{M}_{\alpha\beta}(\boldsymbol{\kappa},t) = M_{\alpha\beta}+o(1), \tag{3.4}$$

and we have therefore our main result which we repeat for emphasis

$$\Phi_{ij}(\boldsymbol{\kappa},t) = \left(\delta_{i\alpha}-\frac{\kappa_i\kappa_\alpha}{\kappa^2}\right)\left(\delta_{j\beta}-\frac{\kappa_j\kappa_\beta}{\kappa^2}\right)M_{\alpha\beta}+o(1), \tag{3.5}$$

$$\Omega_{ij}(\boldsymbol{\kappa},t) = \epsilon_{ik\alpha}\epsilon_{jl\beta}\kappa_k\kappa_l M_{\alpha\beta}+o(\kappa^2), \tag{3.6}$$

for $t \geqslant t_0$, where $M_{\alpha\beta}$ is constant. In other words, the form of the spectral tensor near $\kappa = 0$, which is commonly referred to as the big eddy structure, is a dynamical invariant of the motion determined by the initial conditions.

An immediate consequence of (3.5) is that

$$E(\kappa) = \tfrac{8}{3}\pi M_{\alpha\alpha}\kappa^2+o(\kappa^2), \tag{3.7}$$

and the curvature of the energy spectral function at $\kappa = 0$ is a dynamical invariant.

It is interesting that our argument for dynamical invariance does not depend on the Navier–Stokes equations but only on the conservation of linear momentum and the incompressibility of the fluid. To add further conviction, the outline of a more formal proof is given in §7. The question of what happens if the fluid is slightly compressible is an interesting one, but will be left for further study.

4. The asymptotic form of the velocity correlation tensor

The form of the spectral tensors near $\kappa = 0$ is related to the asymptotic behaviour of the correlation tensor as $r\to\infty$. The relations can be readily deduced using the concept of generalized functions (see Lighthill 1958). We have

$$\int e^{i\boldsymbol{\kappa}.\mathbf{r}}d\boldsymbol{\kappa} = 8\pi^3\delta(\mathbf{r}), \tag{4.1}$$

where $\delta(\mathbf{r}) = \delta(r_1)\delta(r_2)\delta(r_3)$ is the three-dimensional delta function. It follows that

$$\int \frac{e^{i\boldsymbol{\kappa}.\mathbf{r}}d\boldsymbol{\kappa}}{\kappa^2} = \frac{2\pi^2}{r}, \quad \int \frac{e^{i\boldsymbol{\kappa}.\mathbf{r}}d\boldsymbol{\kappa}}{\kappa^4} = -\pi^2 r, \tag{4.2}$$

since
$$\nabla^2(1/r) = -4\pi\delta(\mathbf{r}). \tag{4.3}$$

Further,
$$\int \kappa_j \frac{e^{i\boldsymbol{\kappa}.\mathbf{r}}}{\kappa^2}\, d\boldsymbol{\kappa} = 2\pi^2 i \frac{\partial}{\partial r_j}\left(\frac{1}{r}\right), \tag{4.4}$$

and so on.

Then from (3.5) the asymptotic form of the velocity correlation tensor $R_{ij}(\mathbf{r})$ for large $\mathbf{r}$ is

$$R_{ij}(\mathbf{r}) \sim -M_{\alpha\beta}\pi^2\left(\delta_{i\alpha}\nabla^2 - \frac{\partial^2}{\partial r_i \partial r_\alpha}\right)\left(\delta_{j\beta}\nabla^2 - \frac{\partial^2}{\partial r_j \partial r_\beta}\right) r. \tag{4.5}$$

The important conclusion is that for turbulence generated in accordance with our hypothesis
$$R_{ij}(\mathbf{r}) = O(r^{-3}), \tag{4.6}$$

so the singularity in the spectral tensor corresponds to the fact that the integral of $R_{ij}(\mathbf{r})$ is not absolutely convergent and the velocity correlation goes down like the inverse cube of the distance. This behaviour is in accord with intuitive ideas, for any local region of the motion will behave like a dipole as far as the fluid a long way from it is concerned (it cannot act like a source because of the incompressibility condition), unless there is a strong negative correlation between neighbouring regions so that the dipoles cancel, which would require special initial conditions. Now the velocity field due to a dipole goes down like r^{-3}, and hence we expect a correlation of order r^{-3} between the velocity at a point P and the motion induced at a distant point by the motion in the neighbourhood of P.

By the condition of incompressibility

$$\int_V R_{ij}(\mathbf{r})\, d\mathbf{r} = \int_S n_k r_i R_{kj}(\mathbf{r})\, dS, \tag{4.7}$$

where $\mathbf{n}$ is the outward normal to the surface. It follows from (4.5) that (4.7) is a dynamical invariant independent of the size of V, but the value of the integral depends on its shape. For the particular case of a sphere, it is found that

$$\int_{\text{sphere}} R_{ij}(\mathbf{r})\, d\mathbf{r} = 8\pi^3(\tfrac{7}{15}M_{ij} + \tfrac{1}{15}\delta_{ij}M_{kk}). \tag{4.8}$$

Batchelor & Proudman (1956, §2) point out that the volume integral of the correlation tensor may be expected to be a dynamical invariant of the turbulence, but they take it as zero. The present work can be regarded as a generalization of their study to the case when this dynamical invariant is not zero.

The dynamical invariant is more closely related to the second integral moments of the vorticity correlation tensor, for it follows from (1.4) that

$$\frac{\partial^2 \Omega_{ij}(0,t)}{\partial \kappa_p \partial \kappa_p} = 2\delta_{ij}M_{kk} - M_{ij} = -\frac{1}{8\pi^3}\int r^2 V_{ij}(\mathbf{r},t)\, d\mathbf{r}. \tag{4.9}$$

Hence the second integral moments of the vorticity correlation tensor are dynamical invariants. Note that it does not follow from the result that $R_{ij} = O(r^{-3})$ that $V_{ij} = O(r^{-5})$, for the terms that are $O(r^{-3})$ in R_{ij} are curl free in each index. In fact, the behaviour of Ω_{ij} near $\boldsymbol{\kappa} = 0$ shows that $V_{ij} = o(r^{-5})$.

5. Isotropic turbulence

Although the large-scale structure is unlikely to be isotropic in any real situation, it is useful to give the special results for this case because of their relative simplicity.

We have from § 1,

$$\frac{\Omega_{ij}}{\kappa^2}(\boldsymbol{\kappa},t) = \Phi_{ij}(\boldsymbol{\kappa},t) = M\left(\delta_{ij}-\frac{\kappa_i\kappa_j}{\kappa^2}\right)+o(1), \tag{5.1}$$

$$E(\kappa,t) = 4\pi M\kappa^2+o(\kappa^2), \tag{5.2}$$

where M is a dynamical invariant.

The corresponding asymptotic form of the velocity correlation tensor is

$$R_{ij}(\mathbf{r}) \sim 2\pi^2 M\left(\frac{3r_ir_j}{r^5}-\frac{\delta_{ij}}{r^3}\right). \tag{5.3}$$

It is usual to express $R_{ij}(\mathbf{r})$ in terms of the longitudinal velocity correlation $f(r) = R_{11}/u^2$, where u is the root-mean-square component of velocity, i.e.

$$R_{ij} = u^2(f+\tfrac{1}{2}rf')\,\delta_{ij}-\tfrac{1}{2}(f'/r)r_ir_j. \tag{5.4}$$

Thus

$$u^2f(r) \sim 4\pi^2M/r^3. \tag{5.5†}$$

Prior to the work of Proudman & Reid (1954) and Batchelor & Proudman (1956), it was thought that the expression

$$\int_0^\infty u^2r^4f(r)\,dr \tag{5.6}$$

was an invariant of the dynamical motion (the Loitsianski invariant) provided that it existed. Batchelor & Proudman showed that under their hypothesis the integral exists, but is not an invariant. It now appears that with the more general hypothesis, the expression (5.6) does not in fact converge.

There is, however, another invariant to replace it. Define $R(r)$ by

$$R_{ii}(\mathbf{r}) = 2R(r) = u^2(3f+rf'). \tag{5.7}$$

Note that $R_{ii}(\mathbf{r})$ is $o(r^{-3})$. Then from equation (4.8), we have

$$\int_0^\infty r^2R(r)\,dr = 2\pi^2M. \tag{5.8}$$

Note that from (5.7),

$$\int_0^\infty r^2R(r)\,dr = \tfrac{1}{2}u^2\lim_{r\to\infty}(r^3f). \tag{5.9}$$

Thus it has always been the case that the expression (5.8) has been a dynamical invariant, but it has previously been taken to be zero.‡

† Synge & Lin (1943) found this same asymptotic form for their model of isotropic turbulence as a random assembly of Hill spherical vortices.

‡ Birkhoff (1954) examined the consequences of assuming that $E(\kappa)\propto\kappa^2$ and showed that it implied the divergence of the Loitsianski integral. He also deduced the invariance of (5·9).

The result (5.8) can be deduced from the Navier–Stokes equations and the assumption that the triple velocity correlation $\overline{u_i u_j u_k'}$ is $O(r^{-3})$. The Karman–Howarth equation (see Batchelor 1953, §5.5) is equivalent to

$$\frac{\partial R}{\partial t} = \frac{1}{2}\left(r\frac{\partial}{\partial r}+3\right)\left(\frac{\partial}{\partial r}+\frac{4}{r}\right)u^3k(r)+2\nu\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)R, \tag{5.10}$$

where $u^3k(r)$ is the longitudinal triple velocity correlation. Hence

$$\frac{\partial}{\partial t}\int_0^\infty r^2 R\,dr = \frac{u^3}{2}\lim_{r\to\infty}\frac{1}{r}\frac{\partial}{\partial r}(r^4k(r)), \tag{5.11}$$

and the right-hand side vanishes if $k(r) = O(r^{-3})$.

A correlation of this order of magnitude is expected from the idea that the long-range effect of the fluid in one neighbourhood should be like that of a dipole.

6. The final period of decay

The structure of the large eddies is of little practical importance because of the small amounts of energy involved, but there is one prediction that should be capable of some experimental check. In the final period of decay when the non-linear inertial terms in the equations of motion are supposedly negligible, the energy spectral tensor is

$$\Phi_{ij}(\boldsymbol{\kappa},t) = \Phi_{ij}(\boldsymbol{\kappa},t_1)\,e^{-2\nu\kappa^2(t-t_1)}, \tag{6.1}$$

where t_1 denotes some instant of time inside the final period (see Batchelor 1953, §5.4). It follows that as $t-t_1\to\infty$,

$$\overline{u_i u_j} \sim \tfrac{1}{120}\,[2\pi\nu(t-t_1)]^{-\frac{5}{2}}\,(7M_{ij}+\delta_{ij}M_{kk}), \tag{6.2}$$

showing that each component of energy decays like $t^{-\frac{5}{2}}$.

The experimental evidence is not too certain, but the measurements plotted in Batchelor & Townsend (1948) for the decay of $\overline{u_1^2}$ in a wind tunnel downstream of a grid (the x-axis being taken parallel to the stream) would appear to fit a $t^{-\frac{5}{2}}$ law. (Measurements of $\lambda^2 = \overline{u_1^2}/\overline{(\partial u_1/\partial x_1)^2}$ are not helpful as although $\lambda^2 \propto t$ in the final period, the coefficient depends not only on the index of decay but also on the degree of anisotropy.) Now the turbulence produced by a grid is created by the fluctuating forces on the rods or bars of the grid. A volume V of the turbulence comprising a cross-section A and a length L would correspond to the volume of fluid swept past area A of the grid in time L/U, where U is the velocity of the stream. The momentum imparted to the fluid by a single rod in this time is $\int \mathbf{F}\,dt$ where $-\mathbf{F}$ is the fluctuating aerodynamic force on the rod. A sufficient condition for the momentum in the volume V to be unbounded is that the force on different rods should be uncorrelated and that $\int \mathbf{F}\,dt$ should increase like $t^{\frac{1}{2}}$, or equivalently that $\int \overline{\mathbf{F}(t)\,\mathbf{F}(t+\tau)}\,d\tau$ should be non-zero. If the force fluctuated periodically, as would be the case if the wake was formed like a vortex street (and the turbulent wake of a rod appears to retain a periodic structure at large Reynolds numbers) the correlation integral would be zero and the momentum applied to the volume V would be less than $V^{\frac{1}{2}}$. The dynamical invariant would then be zero. For this reason, it is not surprising that grid turbulence in a wind tunnel appears to obey the $t^{-\frac{5}{2}}$ law, and a more general type of energy production is needed to test the theory, such as might obtain if the grids were given a random shaking. (Incidentally, it is worth stressing that the existence of a final period in which inertia is negligible is an assumption without convincing theoretical support beyond self consistency.)

It has been suggested that, because of the relation valid in the final period,

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{[4\pi\nu(t-t_1)]^{\frac{3}{2}}}\int \mathbf{u}(\mathbf{x}',t_1)\exp\left[-\frac{(\mathbf{x}-\mathbf{x}')^2}{4\nu(t-t_1)}\right]d\mathbf{x}', \tag{6.3}$$

a form of the Central Limit Theorem will apply as $t-t_1\to\infty$ so that the asymptotic probability distribution of $\mathbf{u}(\mathbf{x},t)$ is Gaussian. However, if the r^{-3} correlation at large separations exists the correlation between the values of $\mathbf{u}(\mathbf{x}',t_1)$ at different points $\mathbf{x}'$ is not sufficiently weak for the theorem to be valid; and indeed it can be shown for instance that the skewness factor for the x_1-component of the velocity, say, in the final period is proportional to

$$\iint \overline{f_1(\mathbf{x},t_0)f_1(\mathbf{x}+\mathbf{r},t_0)f_1(\mathbf{x}+\mathbf{s},t_0)}\;d\mathbf{r}\,d\mathbf{s}\Big/\left[\int \overline{f_1(\mathbf{x},t_0)f_1(\mathbf{x}+\mathbf{r},t_0)}\;d\mathbf{r}\right]^{\frac{3}{2}},$$

which is not necessarily zero. It seems reasonable to believe that a fairly simple relation will hold between the probability distribution of $\mathbf{u}$ in the final period and the probability distribution of the impulsive force system which starts the motion, but the relationship has not yet been found.

7. An analytical proof of the invariance of the big eddies

We now approach the problem in the spirit of Batchelor & Proudman (1956, §4) and consider the time derivatives of the velocity and vorticity correlation functions evaluated at the initial instant. We wish to show that $\partial^n R_{ij}/\partial t^n$ at $t = t_0$ does not contain terms that are $O(r^{-3})$ for arbitrary n, or equivalently that there are no terms $O(r^{-5})$ in $\partial^n V_{ij}/\partial t^n$, given that all integral moments of the vorticity cumulants exist at $t = t_0$ but that velocity cumulants may be $O(r^{-3})$. The detailed approach used by Batchelor & Proudman depends heavily on their hypothesis that velocity cumulants are transcendentally small, and rather than modify their analysis it is more convenient to work directly in terms of the Fourier components.

We denote by $\mathbf{a}(\boldsymbol{\kappa})$ and $\mathbf{m}(\boldsymbol{\kappa})$ the (generalized) Fourier transforms of the velocity field and the impulsive force system that generates the motion initially, i.e.

$$\mathbf{a}(\boldsymbol{\kappa}) = \frac{1}{8\pi^3}\int \mathbf{u}(\mathbf{x})e^{-i\boldsymbol{\kappa}.\mathbf{x}}d\mathbf{x}, \quad \mathbf{m}(\boldsymbol{\kappa}) = \frac{1}{8\pi^3}\int \mathbf{f}(\mathbf{x})e^{-i\boldsymbol{\kappa}.\mathbf{x}}d\mathbf{x}. \tag{7.1}$$

The Fourier components satisfy the relations (which are simply consequences of the homogeneity)

$$\overline{a_i(\boldsymbol{\kappa})a_j(\boldsymbol{\kappa}')} = \Phi_{ij}(\boldsymbol{\kappa})\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}'); \quad \overline{m_i(\boldsymbol{\kappa})m_j(\boldsymbol{\kappa}')} = M_{ij}(\boldsymbol{\kappa})\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}'). \tag{7.2}$$

The relation between the Fourier components of the velocity at $t = t_0$ and the force system is

$$a_i(\boldsymbol{\kappa}) = \{\delta_{ij}-(\kappa_i\kappa_j/\kappa^2)\}m_j(\boldsymbol{\kappa}). \tag{7.3}$$

If we define the Fourier transform of the triple velocity correlation at three points by

$$\Gamma_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}') = \frac{1}{(2\pi)^6}\int \overline{u_i(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r}')\,u_k(\mathbf{x}+\mathbf{r})}\exp\left[-i(\boldsymbol{\kappa}'.\mathbf{r}'+\boldsymbol{\kappa}.\mathbf{r})\right]d\mathbf{r}\,d\mathbf{r}', \quad (7.4)$$

the triple correlation at two points is

$$\overline{u_i(\mathbf{x})\,u_j(\mathbf{x})\,u_k(\mathbf{x}+\mathbf{r})} = S_{ijk}(\mathbf{r}) = \int \gamma_{ijk}(\boldsymbol{\kappa})\,e^{i\boldsymbol{\kappa}.\mathbf{x}}d\boldsymbol{\kappa}, \quad (7.5)$$

where

$$\gamma_{ijk}(\boldsymbol{\kappa}) = \int \Gamma_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}'. \quad (7.6)$$

The relation between the Fourier components and the triple velocity spectral tensor is

$$\overline{a_i(\boldsymbol{\kappa}'')\,a_j(\boldsymbol{\kappa}')\,a_k(\boldsymbol{\kappa})} = \Gamma_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}')\,\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}'+\boldsymbol{\kappa}''). \quad (7.7)$$

Then if $G_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}')$ denotes the corresponding spectral tensor of the force distribution, it follows from (7.3) that

$$\Gamma_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}') = \left(\delta_{i\alpha}-\frac{(\kappa_i'+\kappa_i)(\kappa_\alpha'+\kappa_\alpha)}{|\boldsymbol{\kappa}+\boldsymbol{\kappa}'|^2}\right)\left(\delta_{j\beta}-\frac{\kappa_j'\kappa_\beta'}{\kappa'^2}\right)\left(\delta_{k\gamma}-\frac{\kappa_k\kappa_\gamma}{\kappa^2}\right)G_{\alpha\beta\gamma}(\boldsymbol{\kappa},\boldsymbol{\kappa}'). \quad (7.8)$$

Now by hypothesis, $G_{\alpha\beta\gamma}(\boldsymbol{\kappa},\boldsymbol{\kappa}')$ is an analytic function which is exponentially small as $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ tend to infinity. It now follows from (7.6) on integrating with respect to $\boldsymbol{\kappa}'$ and expanding about $\boldsymbol{\kappa} = 0$ that

$$\begin{aligned}\gamma_{ijk}(\boldsymbol{\kappa}) &= \left(\delta_{k\gamma}-\frac{\kappa_k\kappa_\gamma}{\kappa^2}\right)\int\left(\delta_{i\alpha}-\frac{\kappa_i'\kappa_\alpha'}{\kappa'^2}\right)\left(\delta_{j\beta}-\frac{\kappa_j'\kappa_\beta'}{\kappa'^2}\right)G_{\alpha\beta\gamma}(O,\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}'+O(\kappa)\\ &= \left(\delta_{k\gamma}-\frac{\kappa_k\kappa_\gamma}{\kappa^2}\right)U_{ij\gamma}+O(\kappa),\quad \text{say.}\end{aligned} \quad (7.9)$$

This demonstrates that $S_{ijk}(\mathbf{r})$ is $O(r^{-3})$ at the initial instant as $r\to\infty$, except in isotropic turbulence where the dependence is $O(r^{-4})$ at most because there is no isotropic constant tensor of the third order which is symmetrical under reflexion.

A similar argument can be given for all the velocity cumulants to demonstrate the initial r^{-3} behaviour, which is related to the presence of the $(\delta_{k\gamma}-\kappa_k\kappa_\gamma/\kappa^2)$ term in the spectral tensors.

The Fourier transform of the Navier–Stokes equations gives

$$\frac{\partial a_i(\boldsymbol{\kappa})}{\partial t} = -i\kappa_i\,\Phi(\boldsymbol{\kappa})-i\kappa_j\int a_j(\boldsymbol{\kappa}'')\,a_i(\boldsymbol{\kappa}-\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}''-\nu\kappa^2a_i(\boldsymbol{\kappa}), \quad (7.10)$$

where $\Phi(\boldsymbol{\kappa})$ is the Fourier transform of the kinematic pressure. Now $\kappa_i a_i = 0$ by virtue of the equation of continuity and hence

$$\Phi(\boldsymbol{\kappa}) = -\frac{\kappa_i\kappa_j}{\kappa^2}\int a_j(\boldsymbol{\kappa}'')\,a_i(\boldsymbol{\kappa}-\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}''. \quad (7.11)$$

Then from (7.10),

$$\frac{\partial a_i(\boldsymbol{\kappa})}{\partial t} = -i\kappa_j\left(\delta_{ik}-\frac{\kappa_i\kappa_k}{\kappa^2}\right)\int a_j(\boldsymbol{\kappa}'')\,a_k(\boldsymbol{\kappa}-\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}''-\nu\kappa^2a_i(\boldsymbol{\kappa}). \quad (7.12)$$

Hence,

$$\overline{a_j(\boldsymbol{\kappa}')\frac{\partial a_i(\boldsymbol{\kappa})}{\partial t}} = -i\kappa_l\left(\delta_{ik}-\frac{\kappa_i\kappa_k}{\kappa^2}\right)\int \Gamma_{klj}(\boldsymbol{\kappa}',\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}''\,\delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}')-\nu\kappa^2\overline{a_i(\boldsymbol{\kappa})\,a_j(\boldsymbol{\kappa}')}. \quad (7.13)$$

Thus remembering (7.2) we have

$$\frac{\partial}{\partial t}\Phi_{ij}(\boldsymbol{\kappa}) = -i\kappa_l\left(\delta_{ik}-\frac{\kappa_i\kappa_k}{\kappa^2}\right)\int\Gamma_{klj}(-\boldsymbol{\kappa},\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}''+i\kappa_l\left(\delta_{jk}-\frac{\kappa_j\kappa_k}{\kappa^2}\right) \times\int\Gamma_{kli}(\boldsymbol{\kappa},\boldsymbol{\kappa}'')\,d\boldsymbol{\kappa}''-2\nu\kappa^2\Phi_{ij}(\boldsymbol{\kappa}). \tag{7.14}$$

This equation holds at all times. At the initial instant, $\Gamma_{kli}(\boldsymbol{\kappa},\boldsymbol{\kappa}'')$ is given by (7.8) in terms of the triple correlation spectral tensor of the force system. It is clear that

$$\frac{\partial}{\partial t}\Phi_{ij}(\boldsymbol{\kappa},t_0) = -i\kappa_l\left(\delta_{ik}-\frac{\kappa_i\kappa_k}{\kappa^2}\right)\left(\delta_{jm}-\frac{\kappa_j\kappa_m}{\kappa^2}\right)(U_{klm}-U_{mlk})+O(\kappa^2), \tag{7.15}$$

where the time dependence has been put in explicitly to show that the spectral tensor is evaluated at the initial instant, and it is to be remembered that the $O(\kappa^2)$ term is not necessarily analytic.

It follows immediately from the relations between $\Phi_{ij}(\boldsymbol{\kappa})$ near $\boldsymbol{\kappa}=0$ and $R_{ij}(\mathbf{r})$ at infinity that

$$\frac{\partial}{\partial t}R_{ij}(\mathbf{r},t_0) = -\pi^2(U_{klm}-U_{mlk})\frac{\partial}{\partial r_l}\left(\delta_{ik}\nabla^2-\frac{\partial^2}{\partial r_i\partial r_k}\right)\left(\delta_{jm}\nabla^2-\frac{\partial^2}{\partial r_j\partial r_m}\right)r+o(r^{-4}). \tag{7.16}$$

Thus the coefficient of the terms $O(r^{-3})$ in $R_{ij}(\mathbf{r})$ has zero first-order time derivative at t_0. We have given the argument in some detail for the first derivative with respect to time. It is clear that in the spirit of Batchelor & Proudman, we can obtain expressions for the time derivatives of all orders, by differentiating (7.12) with respect to time and eliminating the lower-order derivatives, in terms of the Fourier coefficients of the force system at the initial instant. Moreover, because of the $\kappa_l(\delta_{ik}-\kappa_i\kappa_k/\kappa^2)$ factor in $\partial a_i(\boldsymbol{\kappa})/\partial t$ which must persist in derivatives of all orders, it is clear that $\partial^n\Phi_{ij}(\boldsymbol{\kappa},t_0)/\partial t^n$ must be of the form (7.15) near $\boldsymbol{\kappa}=0$, where the third-order tensor is given by a multiple integral like (7.9) with many factors. Hence, $\partial^n R_{ij}(\mathbf{r},t_0)/\partial t^n$ is at most $O(r^{-4})$, and as the same is clearly true for the cumulants of any order (although the details in general are messy but reasonably straightforward), the invariance is proved granted that the solution of the Navier–Stokes equations can be expanded as a Taylor series in $t-t_0$, an assumption that is by no means obviously true. (It is not true for a velocity field generated by the sudden motion of rigid walls as the vorticity is not an analytic function of $t-t_0$).

Appendix

To show that (1.3) is the most general form of (1.2), consider the second-order tensor

$$A_{ij} = \kappa_k \kappa_l B_{ijkl}. \tag{A 1}$$

Since $\kappa_i A_{ij} = 0$, we can write quite generally that

$$A_{ij} = \epsilon_{ipq} \kappa_p C_{qj}(\boldsymbol{\kappa}). \tag{A 2}$$

However, A_{ij} is a linear quadratic form in the components of $\boldsymbol{\kappa}$. Hence

$$C_{qj}(\boldsymbol{\kappa}) = \Gamma_{qjr} \kappa_r, \tag{A 3}$$

and

$$A_{ij} = \epsilon_{ipq} \Gamma_{qjr} \kappa_p \kappa_r. \tag{A 4}$$

Similarly,

$$A_{ij} = \epsilon_{jrq} \Gamma'_{qip} \kappa_p \kappa_r, \tag{A 5}$$

because $\kappa_j A_{ij} = 0$. Since (A 4) and (A 5) must be true for all κ,

$$B_{ijpr} = \epsilon_{ipq} \Gamma_{qjr} = \epsilon_{jrq} \Gamma'_{qip}. \tag{A 6}$$

The last term is zero when $r = j$ and is antisymmetric with respect to interchange of r and j. Hence Γ_{qjr} has these properties. Therefore

$$\Gamma_{qjr} = \epsilon_{jr\beta} M_{q\beta}, \tag{A 7}$$

where $M_{q\beta}$ is arbitrary. The expression (A 7) is the most general form since Γ_{qjr} possesses nine independent coefficients and $M_{q\beta}$ has the same number. The result now follows.

This work was stimulated by the belief of Prof. H. W. Liepmann that the κ^4 dependence of the energy spectrum function was not a general law.

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