

Negligible Cooperation: Contrasting the Maximal- and Average-Error Cases

Parham Noorzad, Michael Langberg, Michelle Effros

Abstract

In communication networks, cooperative strategies are coding schemes where network nodes work together to improve network performance metrics such as the total rate delivered across the network. This work studies *encoder* cooperation in the setting of a discrete multiple access channel (MAC) with two encoders and a single decoder. A network node, here called the cooperation facilitator (CF), that is connected to both encoders via rate-limited links, enables the cooperation strategy. Previous work by the authors presents two classes of MACs: (i) one class where the *average-error* sum-capacity has an infinite derivative in the limit where CF output link capacities approach zero, and (ii) a second class of MACs where the *maximal-error* sum-capacity is not continuous at the point where the output link capacities of the CF equal zero. This work contrasts the power of the CF in the maximal- and average-error cases, showing that a *constant number of bits* communicated over the CF output link can yield a positive gain in the maximal-error sum-capacity, while a far greater number of bits, even numbers that grow sublinearly in the blocklength, can never yield a non-negligible gain in the average-error sum-capacity.

Index Terms

Continuity, cooperation facilitator, edge removal problem, maximal-error capacity region, multiple access channel.

I. INTRODUCTION

Interference is an important limiting factor in the capacities of many communication networks. One way to reduce interference is to enable network nodes to work together to coordinate their transmissions. Strategies that employ coordinated transmissions are called cooperation strategies.

Perhaps the simplest cooperation strategy is “time-sharing” (e.g., [3, Theorem 15.3.2]), where nodes avoid interference by taking turns transmitting. A popular alternative model is the “conferencing” cooperation model [4]; in conferencing, unlike in time-sharing, encoders share information about the messages they wish to transmit and use that shared information to coordinate their channel inputs. In this work, we employ a similar approach, but in our cooperation model, encoders communicate indirectly. Specifically, the encoders communicate through

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P. Noorzad was with the California Institute of Technology, Pasadena, CA 91125 USA. He is now with Qualcomm Technologies, Inc., San Diego, CA 92121 USA (email: parham@qti.qualcomm.com).

M. Langberg is with the State University of New York at Buffalo, Buffalo, NY 14260 USA (email: mikel@buffalo.edu).

M. Effros is with the California Institute of Technology, Pasadena, CA 91125 USA (email: effros@caltech.edu).

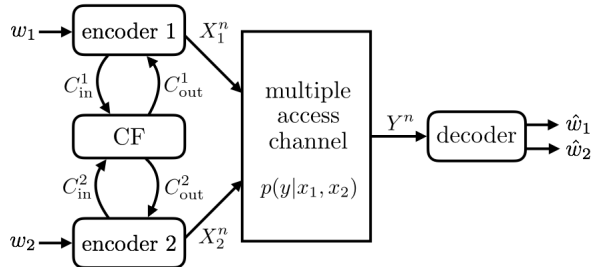


Figure 1: A network consisting of two encoders that initially cooperate via a CF and then, based on the information they receive, transmit their codewords over the MAC to the decoder.

another node, which we call the cooperation facilitator (CF) [5], [6]. Figure 1 depicts the CF model in the two-user multiple access channel (MAC) scenario.

The CF enables cooperation between the encoders through its rate-limited input and output links. Prior to choosing a codeword to transmit over the channel, each encoder sends a function of its message to the CF. The CF uses the information it receives from *both* encoders to compute a rate-limited function for each encoder. It then transmits the computed values over its output links. Finally, each encoder selects a codeword using its message and the information it receives from the CF.

To simplify our discussion in this section, suppose the CF input link capacities both equal C_{in} and the CF output link capacities both equal C_{out} . If $C_{in} \leq C_{out}$, then the optimal strategy for the CF is to simply forward the information it receives from one encoder to the other. Using the capacity region of the MAC with conferencing encoders [4], it follows that the average-error sum-capacity gain of CF cooperation in this case is bounded from above by $2C_{in}$ and does not depend on the precise value of $C_{out} \geq C_{in}$. If $C_{in} > C_{out}$, however, the situation is more complicated since the CF can no longer forward all of its incoming information. While the $2C_{in}$ upper bound is still valid, the dependence of the sum-capacity gain on C_{out} is less clear. If the CF simply forwards part of the information it receives, then again by [4], the average-error sum-capacity gain is at most $2C_{out}$. The $2C_{out}$ bound has an intuitive interpretation: it reflects the amount of information the CF shares with the encoders, perhaps suggesting that the benefit of sharing information with rate $2C_{out}$ with the encoders is at most $2C_{out}$. It turns out, though, that a much larger gain is possible through more sophisticated coding techniques. Specifically, in prior work [6, Theorem 3], we show that for a class of MACs, for fixed $C_{in} > 0$, the average-error sum-capacity has a derivative in C_{out} that is infinite at $C_{out} = 0$; that is, for small C_{out} , the gain resulting from cooperation exceeds any function that goes through the origin and has bounded derivative.

The large sum-capacity gain described above is not limited to the average-error scenario. In fact, in related work [5, Proposition 5], we show that for any MAC for which, in the absence of cooperation, the average-error sum-capacity is strictly greater than the maximal-error sum-capacity, adding a CF and measuring the *maximal-error* sum-capacity for fixed $C_{in} > 0$ gives a curve that is discontinuous at $C_{out} = 0$. In this case, we say that “negligible cooperation” results in a non-negligible capacity benefit.

Given these earlier results, a number of important questions remain open. For example, we wish to understand how many bits from the CF are needed to achieve the discontinuities already shown to be possible in the maximal-error case. We also seek to understand, in the average-error case, whether the sum-capacity gain can be discontinuous in C_{out} .

For the first question, we note that while the demonstration of discontinuity at $C_{\text{out}} = 0$ for the maximal-error case proves that negligible cooperation can yield a non-negligible benefit, it does not distinguish how many bits are required to effect that change nor whether that number of bits must grow with the blocklength n . We therefore begin by pushing that question to its extreme: we seek to understand the minimal output rate from the CF that can change network capacity. Our central result for the maximal-error case demonstrates that even a constant number of bits from the CF can yield a non-negligible impact on network capacity in the maximal-error case.

For the second question, we seek to gain a similar understanding of how many bits from the CF are required to obtain a non-negligible change to network capacity in the average-error case. Since in this case there are no prior results demonstrating the possibility of a discontinuity in C_{out} , we begin by investigating whether the sum-capacity in the average-error case can ever be discontinuous. Our central result for the average-error case is that the average-error sum-capacity is continuous even at $C_{\text{out}} = 0$. (See Corollary V.1.) Our proof relies on tools developed by Dueck [7] to prove the strong converse for the MAC. Saeedi Bidokhti and Kramer [8] and Kosut and Kliever [9, Proposition 15] also use Dueck’s method to address similar problems. Our application of Dueck’s method first appears in [10, Appendix C].

In addition to the contributions above, our work also explicitly strengthens earlier results. Specifically, we refer the reader to Theorem IV.1 in Section IV and Corollary V.2 in Section V, which provide stronger versions of results derived in [6] and [5], respectively.

II. RELATED WORK

A continuity problem similar to the one considered here appears in studying rate-limited feedback over the MAC. In that setting, Sarwate and Gastpar [11] use the dependence-balance bounds of Hekstra and Willems [12] to show that as the feedback rate converges to zero, the average-error capacity region converges to the average-error capacity region of the same MAC in the absence of feedback.

The problem we study here can also be formulated as an “edge removal problem” as introduced by Ho, Effros, and Jalali [13], [14]. The edge removal problem seeks to quantify the capacity effect of removing a single edge from a network. While bounds on this capacity impact exist in a number of limited scenarios (see, for example, [13] and [14]), the problem remains open in the general case. In the context of network coding, Langberg and Effros show that this problem is connected to a number of other open problems, including the difference between the 0-error and ϵ -error capacity regions [15] and the difference between the lossless source coding regions for independent and dependent sources [16].

In [9], Kosut and Kliever present different variations of the edge removal problem in a unified setting. In their terminology, the present work investigates whether the network consisting of a MAC and a CF satisfies the “weak

edge removal property” with respect to the average-error reliability criterion. A discussion in [10, Chapter 1] summarizes the known results for each variation of the edge removal problem.

The question of whether the capacity region of a network consisting of noiseless links is continuous with respect to the link capacities is studied by Gu, Effros, and Bakshi [17] and Chan and Grant [18]. The present work differs from [17], [18] in the network under consideration; while our network does have noiseless links (the CF input and output links), it also contains a multiterminal component (the MAC) which may exhibit interference or noise; no such component appears in [17], [18].

For the maximal-error case, our study focuses on the effect of a constant number of bits of communication in the memoryless setting. For noisy networks *with memory*, it is not difficult to see that even one bit of communication may indeed affect the capacity region. For example, consider a binary symmetric channel whose error probability θ is chosen at random and then fixed for all time. If for $i \in \{1, 2\}$, θ equals θ_i with positive probability p_i , and $0 \leq \theta_1 < \theta_2 \leq 1/2$, then a single bit of feedback (not rate 1, but exactly one bit no matter how large the blocklength) from the receiver to the transmitter suffices to increase the capacity. For memoryless channels, the question is far more subtle and is the subject of our study.

In the next section, we present the cooperation model we consider in this work.

III. THE COOPERATION FACILITATOR MODEL

In this work, we study cooperation between two encoders that communicate their messages to a decoder over a stationary, memoryless, and discrete MAC. Such a MAC can be represented by the triple

$$(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y}),$$

where \mathcal{X}_1 , \mathcal{X}_2 , and \mathcal{Y} are finite sets and $p(y|x_1, x_2)$ is a conditional probability mass function. For any positive integer $n \geq 2$, the n th extension of this MAC is given by

$$p(y^n|x_1^n, x_2^n) := \prod_{t=1}^n p(y_t|x_{1t}, x_{2t}).$$

For each positive integer n , called the blocklength, and nonnegative real numbers R_1 and R_2 , called the rates, we next define a $(2^{nR_1}, 2^{nR_2}, n)$ -code for communication over a MAC with a $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ -CF. Here $\mathbf{C}_{\text{in}} = (C_{\text{in}}^1, C_{\text{in}}^2)$ and $\mathbf{C}_{\text{out}} = (C_{\text{out}}^1, C_{\text{out}}^2)$ represent the capacities of the CF input and output links, respectively. (See Figure 1.)

A. Positive Rate Cooperation

For every $x \geq 1$, let $[x]$ denote the set $\{1, \dots, [x]\}$. For $i \in \{1, 2\}$, the transmission of encoder i to the CF is represented by a mapping

$$\varphi_i: [2^{nR_i}] \rightarrow [2^{nC_{\text{in}}^i}].$$

The CF uses the information it receives from the encoders to compute a function

$$\psi_i: [2^{nC_{\text{in}}^1}] \times [2^{nC_{\text{in}}^2}] \rightarrow [2^{nC_{\text{out}}^i}]$$

for encoder i , where $i \in \{1, 2\}$. Encoder i uses its message and what it receives from the CF to select a codeword according to

$$f_i: [2^{nR_i}] \times [2^{nC_{\text{out}}^i}] \rightarrow \mathcal{X}_i^n.$$

The decoder finds estimates of the transmitted messages using the channel output. It is represented by a mapping

$$g: \mathcal{Y}^n \rightarrow [2^{nR_1}] \times [2^{nR_2}].$$

The collection of mappings

$$(\varphi_1, \varphi_2, \psi_1, \psi_2, f_1, f_2, g)$$

defines a $(2^{nR_1}, 2^{nR_2}, n)$ -code for the MAC with a $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ -CF.¹

B. Constant Size Cooperation

To address the setting of a constant number of cooperation bits, we modify the output link of the CF to have support $[2^k]$ for some fixed integer k ; unlike the prior support $[2^{nC_{\text{out}}^i}]$, the support of this link is independent of the blocklength n . Then, for $i \in \{1, 2\}$, the transmission of encoder i to the CF is represented by a mapping

$$\varphi_i: [2^{nR_i}] \rightarrow [2^{nC_{\text{in}}^i}].$$

The CF uses the information it receives from the encoders to compute a function

$$\psi_i: [2^{nC_{\text{in}}^1}] \times [2^{nC_{\text{in}}^2}] \rightarrow [2^k]$$

for encoder i , where $i \in \{1, 2\}$. Encoder i , as before, uses its message and what it receives from the CF to select a codeword according to

$$f_i: [2^{nR_i}] \times [2^k] \rightarrow \mathcal{X}_i^n.$$

We now say that

$$(\varphi_1, \varphi_2, \psi_1, \psi_2, f_1, f_2, g)$$

defines a $(2^{nR_1}, 2^{nR_2}, n)$ -code for the MAC with a $(\mathbf{C}_{\text{in}}, \frac{k}{n})$ -CF.

C. Capacity Region

For a fixed code, the probability of decoding a particular transmitted message pair (w_1, w_2) incorrectly equals

$$\lambda_n(w_1, w_2) := \sum_{y^n: g(y^n) \neq (w_1, w_2)} p(y^n | f_1(w_1, z_1), f_2(w_2, z_2)),$$

where z_1 and z_2 are the CF outputs and are calculated, for $i \in \{1, 2\}$, according to

$$z_i = \psi_i(\varphi_1(w_1), \varphi_2(w_2)).$$

¹Technically, the definition we present here is for a single round of cooperation. As discussed in [5], it is possible to define cooperation via a CF over multiple rounds. However, this general scenario does not alter our main proofs. This is due to the fact that in Lemma VI.2, the lower bound only needs one round of cooperation, while the upper bound holds regardless of the number of rounds.

The *average* probability of error is defined as

$$P_{e,\text{avg}}^{(n)} := \frac{1}{2^{n(R_1+R_2)}} \sum_{w_1, w_2} \lambda_n(w_1, w_2),$$

and the *maximal* probability of error is given by

$$P_{e,\text{max}}^{(n)} := \max_{w_1, w_2} \lambda_n(w_1, w_2).$$

A rate pair (R_1, R_2) is achievable with respect to the average-error reliability criterion if there exists an infinite sequence of $(2^{nR_1}, 2^{nR_2}, n)$ -codes such that $P_{e,\text{avg}}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$. The average-error capacity region of a MAC with a $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ -CF, denoted by $\mathcal{C}_{\text{avg}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$, is the closure of the set of all rate pairs that are achievable with respect to the average-error reliability criterion. The average-error sum-capacity is defined as

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) := \max_{(R_1, R_2) \in \mathcal{C}_{\text{avg}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})} (R_1 + R_2).$$

By replacing $P_{e,\text{avg}}^{(n)}$ with $P_{e,\text{max}}^{(n)}$, we can similarly define achievable rates with respect to the maximal-error reliability criterion, the maximal-error capacity region, and the maximal-error sum-capacity. For a MAC with a $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ -CF, we denote the maximal-error capacity region and sum-capacity by $\mathcal{C}_{\text{max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ and $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$, respectively.

IV. PRIOR RESULTS ON THE SUM-CAPACITY GAIN OF COOPERATION

We next review a number of results from [5], [6] which describe the sum-capacity gain of cooperation under the CF model. We begin with the average-error case.

Consider a discrete MAC $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$. Let $p_{\text{ind}}(x_1, x_2) = p_{\text{ind}}(x_1)p_{\text{ind}}(x_2)$ be a distribution that satisfies

$$I_{\text{ind}}(X_1, X_2; Y) := I(X_1, X_2; Y) \Big|_{p_{\text{ind}}(x_1, x_2)} = \max_{p(x_1)p(x_2)} I(X_1, X_2; Y); \quad (1)$$

subscript “ind” here denotes independence between the output of encoders 1 and 2 in the absence of cooperation. In addition, suppose that there exists a distribution $p_{\text{dep}}(x_1, x_2)$ such that the support of $p_{\text{dep}}(x_1, x_2)$ is contained in the support of $p_{\text{ind}}(x_1)p_{\text{ind}}(x_2)$,

$$I_{\text{dep}}(X_1, X_2; Y) := I(X_1, X_2; Y) \Big|_{p_{\text{dep}}(x_1, x_2)},$$

and

$$I_{\text{dep}}(X_1, X_2; Y) + D(p_{\text{dep}}(y) \| p_{\text{ind}}(y)) > I_{\text{ind}}(X_1, X_2; Y); \quad (2)$$

here $p_{\text{ind}}(y)$ and $p_{\text{dep}}(y)$ are the marginals on \mathcal{Y} resulting from $p_{\text{ind}}(x_1, x_2)$ and $p_{\text{dep}}(x_1, x_2)$, respectively. Let \mathcal{C}^* denote the class of all discrete MACs for which input distributions p_{ind} and p_{dep} , as described above, exist. Theorem IV.1 below is a stronger version of [6, Theorem 3] in the two-user case; the latter result is stated as a corollary below. A similar result holds for the Gaussian MAC [6, Prop. 9].

Theorem IV.1. *Let $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$ be a MAC in \mathcal{C}^* , and suppose $(\mathbf{C}_{\text{in}}, \mathbf{v}) \in \mathbb{R}_{>0}^2 \times \mathbb{R}_{>0}^2$. Then there exists a constant $K > 0$, which depends only on the MAC and $(\mathbf{C}_{\text{in}}, \mathbf{v})$, such that when h is sufficiently small,*

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}, h\mathbf{v}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) \geq K\sqrt{h} + o(\sqrt{h}). \quad (3)$$

The proof of Theorem IV.1 appears in Subsection IX-A.

In the above theorem, dividing both sides of (3) by h and letting $h \rightarrow 0^+$ results in the next corollary.²

Corollary IV.1. *For any MAC in \mathcal{C}^* and any $(\mathbf{C}_{\text{in}}, \mathbf{v}) \in \mathbb{R}_{>0}^2 \times \mathbb{R}_{>0}^2$,*

$$\lim_{h \rightarrow 0^+} \frac{C_{\text{sum}}(\mathbf{C}_{\text{in}}, h\mathbf{v}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0})}{h} = \infty.$$

We next describe the maximal-error sum-capacity gain. While it is possible in the average-error scenario to achieve a sum-capacity that has an infinite slope, a stronger result is known in the maximal-error case. There exists a class of MACs for which the maximal-error sum-capacity exhibits a discontinuity in the capacities of the CF output links. This is stated formally in the next proposition, which is a special case of [5, Proposition 5]. The proposition relies on the existence of a discrete MAC with average-error sum-capacity larger than its maximal-error sum-capacity; that existence was first proven by Dueck [19]. We investigate further properties of Dueck's MAC in [5, Subsection VI-E].

Proposition IV.1. *Consider a discrete MAC for which*

$$C_{\text{sum}}(\mathbf{0}, \mathbf{0}) > C_{\text{sum}, \max}(\mathbf{0}, \mathbf{0}). \quad (4)$$

Fix $\mathbf{C}_{\text{in}} \in \mathbb{R}_{>0}^2$. Then $C_{\text{sum}, \max}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ is not continuous at $\mathbf{C}_{\text{out}} = \mathbf{0}$.

We next present the main results of this work.

V. OUR RESULTS: CONTINUITY OF AVERAGE- AND MAXIMAL-ERROR SUM-CAPACITIES

In the prior section, for a fixed \mathbf{C}_{in} , we discussed previous results regarding the value of $C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ and $C_{\text{sum}, \max}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ as a function of \mathbf{C}_{out} at $\mathbf{C}_{\text{out}} = \mathbf{0}$. In this section, we do not limit ourselves to the point $\mathbf{C}_{\text{out}} = \mathbf{0}$; rather, we study $C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ over its entire domain.

We begin by considering the case where the CF has full access to the messages. Formally, for a given discrete MAC $(\mathcal{X}_1 \times \mathcal{X}_2, p(y|x_1, x_2), \mathcal{Y})$, let the components of $\mathbf{C}_{\text{in}}^* = (C_{\text{in}}^{*1}, C_{\text{in}}^{*2})$ be sufficiently large so that any CF with input link capacities C_{in}^{*1} and C_{in}^{*2} has full knowledge of the encoders' messages. For example, we can choose \mathbf{C}_{in}^* such that

$$\min\{C_{\text{in}}^{*1}, C_{\text{in}}^{*2}\} > \max_{p(x_1, x_2)} I(X_1, X_2; Y).$$

Our first result addresses the continuity of $C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$ as a function of \mathbf{C}_{out} over $\mathbb{R}_{\geq 0}^2$.

Theorem V.1. *For any discrete MAC, the mapping*

$$\mathbf{C}_{\text{out}} \mapsto C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}}),$$

defined on $\mathbb{R}_{\geq 0}^2$ is continuous.

²Note that Corollary IV.1 does not lead to any conclusions regarding continuity; a function $f(x)$ with infinite derivative at $x = 0$ can be continuous (e.g., $f(x) = \sqrt{x}$) or discontinuous (e.g., $f(x) = \lceil x \rceil$).

While Theorem V.1 focuses on the scenario where $\mathbf{C}_{\text{in}} = \mathbf{C}_{\text{in}}^*$, the result is sufficiently strong to address the continuity problem for a fixed, arbitrary \mathbf{C}_{in} at $\mathbf{C}_{\text{out}} = \mathbf{0}$. To see this, note that for all $\mathbf{C}_{\text{in}} \in \mathbb{R}_{\geq 0}^2$,

$$\begin{aligned} C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) &\leq C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) \\ &\leq C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}}). \end{aligned} \quad (5)$$

Corollary V.1, below, now follows from Theorem V.1 by letting \mathbf{C}_{out} approach zero in (5) and noting that for all $\mathbf{C}_{\text{in}} \in \mathbb{R}_{\geq 0}^2$,

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{0}) = C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) = C_{\text{sum}}(\mathbf{0}, \mathbf{0}).$$

Corollary V.1. *For any discrete MAC and any fixed $\mathbf{C}_{\text{in}} \in \mathbb{R}_{\geq 0}^2$, the mapping*

$$\mathbf{C}_{\text{out}} \mapsto C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}),$$

is continuous at $\mathbf{C}_{\text{out}} = \mathbf{0}$.

Recall that Proposition IV.1 gives a sufficient condition under which $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ is not continuous at $\mathbf{C}_{\text{out}} = \mathbf{0}$ for a fixed $\mathbf{C}_{\text{in}} \in \mathbb{R}_{> 0}^2$. Corollary V.1 implies that the sufficient condition is also necessary. This is stated in the next corollary. We prove this corollary in Subsection IX-B.

Corollary V.2. *Fix a discrete MAC and $\mathbf{C}_{\text{in}} \in \mathbb{R}_{> 0}^2$. Then $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ is not continuous at $\mathbf{C}_{\text{out}} = \mathbf{0}$ if and only if*

$$C_{\text{sum}}(\mathbf{0}, \mathbf{0}) > C_{\text{sum,max}}(\mathbf{0}, \mathbf{0}). \quad (6)$$

We next describe the second main result of this paper. Our first main result, Theorem V.1, shows that $C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$ is continuous in \mathbf{C}_{out} over $\mathbb{R}_{\geq 0}^2$. The next result shows that proving the continuity of $C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ over $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$ is equivalent to demonstrating its continuity on certain axes. Specifically, it suffices to check the continuity of C_{sum} when one of C_{out}^1 and C_{out}^2 approaches zero, while the other arguments of C_{sum} are fixed positive numbers.

Theorem V.2. *For any discrete MAC, the mapping*

$$(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) \mapsto C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}),$$

defined on $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$ is continuous if and only if for all $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) \in \mathbb{R}_{> 0}^2 \times \mathbb{R}_{> 0}^2$, we have

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}, (\tilde{C}_{\text{out}}^1, C_{\text{out}}^2)) \rightarrow C_{\text{sum}}(\mathbf{C}_{\text{in}}, (0, C_{\text{out}}^2))$$

as $\tilde{C}_{\text{out}}^1 \rightarrow 0^+$ and

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, \tilde{C}_{\text{out}}^2)) \rightarrow C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, 0))$$

as $\tilde{C}_{\text{out}}^2 \rightarrow 0^+$.

We remark that using a time-sharing argument, it is possible to show that C_{sum} is concave on $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$ and thus continuous on its interior. Therefore, it suffices to study the continuity of C_{sum} on the boundary of $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$. Note

that Theorem V.2 leaves the continuity problem of the average-error sum-capacity open in one case. If C_{in}^1 and C_{in}^2 are positive but not sufficiently large and $C_{\text{out}}^1 > 0$, then we have not established the continuity of the sum-capacity, solely as function of C_{out}^2 , at $C_{\text{out}}^2 = 0^+$. (Clearly, the symmetric case where $C_{\text{out}}^2 > 0$ and $C_{\text{out}}^1 \rightarrow 0^+$ remains open as well.) This last scenario remains a subject for future work.

Finally, we present the third main contribution of our work, the discontinuity of sum-capacity in the maximum-error setting when the outgoing edges of the CF can send only a constant number of bits.

Theorem V.3. *For $k = 6$, the discrete MAC presented in [19] satisfies*

$$C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, k/n) > C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, \mathbf{0}). \quad (7)$$

We prove our key results in the following sections. In Sections VI, VII, and VIII, we outline the proofs of Theorems V.1, V.2, and V.3, respectively. We provide detailed proofs of our claims in Section IX.

VI. CONTINUITY OF SUM-CAPACITY: THE $\mathbf{C}_{\text{in}} = \mathbf{C}_{\text{in}}^*$ CASE

We start our study of the continuity of $C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$ by presenting lower and upper bounds in terms of an auxiliary function $\sigma(\delta)$ defined for $\delta \geq 0$ (Lemma VI.2). This function is similar to a tool used by Dueck in [7] but differs with [7] in its reliance on a time-sharing random variable denoted by U . The random variable U plays two roles. First it ensures that σ is concave, which immediately proves the continuity of σ over $\mathbb{R}_{>0}$. Second, together with a lemma from [7] (Lemma VI.5 below), it helps us find a single-letter upper bound for σ (Corollary VIII.1). We then use the single-letter upper bound to prove continuity at $\delta = 0$.

The following definitions are useful for the description of our lower and upper bounds for $C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$. For every finite alphabet \mathcal{U} and all $\delta \geq 0$, define the set of probability mass functions $\mathcal{P}_{\mathcal{U}}^{(n)}(\delta)$ on $\mathcal{U} \times \mathcal{X}_1^n \times \mathcal{X}_2^n$ as

$$\mathcal{P}_{\mathcal{U}}^{(n)}(\delta) := \left\{ p(u, x_1^n, x_2^n) \mid I(X_1^n; X_2^n | U) \leq n\delta \right\}.$$

Intuitively, $\mathcal{P}_{\mathcal{U}}^{(n)}(\delta)$ captures a family of “mildly dependent” input distributions for our MAC; this mild dependence is parametrized by a bound δ on the per-symbol mutual information. In the discussion that follows, we relate δ to the amount of information that the CF shares with the encoders. For every positive integer n , let $\sigma_n: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ denote the function³

$$\sigma_n(\delta) := \sup_{\mathcal{U}} \max_{p \in \mathcal{P}_{\mathcal{U}}^{(n)}(\delta)} \frac{1}{n} I(X_1^n, X_2^n; Y^n | U), \quad (8)$$

where the supremum is over all finite sets \mathcal{U} . Thus $\sigma_n(\delta)$ captures something like the maximal sum-rate achievable under the mild dependence described above. As we see in Lemma VI.4, conditioning on the random variable U in (8) ensures that σ_n is concave.

For every $\delta \geq 0$, $(\sigma_n(\delta))_{n=1}^{\infty}$ satisfies a superadditivity property which appears in Lemma VI.1, below. Intuitively, this property says that the sum-rate of the best code of blocklength $m+n$ is bounded from below by the sum-rate of the concatenation of the best codes of blocklengths m and n . We prove this Lemma in Subsection IX-C.

³For $n = 1$, this function also appears in the study of the MAC with negligible feedback [11].

Lemma VI.1. For all $m, n \geq 1$, all $\delta \geq 0$, and $\sigma_n(\delta)$ defined as in (8), we have

$$(m+n)\sigma_{m+n}(\delta) \geq m\sigma_m(\delta) + n\sigma_n(\delta).$$

Given Lemma VI.1, [20, Appendix 4A, Lemma 2] now implies that the sequence of mappings $(\sigma_n)_{n=1}^{\infty}$ converges pointwise to some mapping $\sigma: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, and

$$\sigma(\delta) := \lim_{n \rightarrow \infty} \sigma_n(\delta) = \sup_n \sigma_n(\delta). \quad (9)$$

We next present our lower and upper bounds for $C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$ in terms of σ . The lower bound follows directly from [6, Corollary 8]. We prove the upper bound in Subsection IX-D.

Lemma VI.2. For any discrete MAC and any $\mathbf{C}_{\text{out}} \in \mathbb{R}_{\geq 0}^2$, we have

$$\sigma(C_{\text{out}}^1 + C_{\text{out}}^2) - \min\{C_{\text{out}}^1, C_{\text{out}}^2\} \leq C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}}) \leq \sigma(C_{\text{out}}^1 + C_{\text{out}}^2).$$

From the remark following Theorem V.2, we only need to prove that $C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$ is continuous on the boundary of $\mathbb{R}_{\geq 0}^2$. On the boundary of $\mathbb{R}_{\geq 0}^2$, however, $\min\{C_{\text{out}}^1, C_{\text{out}}^2\} = 0$. Thus it suffices to show that σ is continuous on $\mathbb{R}_{\geq 0}$, which is stated in the next lemma.

Lemma VI.3. For any finite alphabet MAC, the function σ , defined by (9), is continuous on $\mathbb{R}_{\geq 0}$.

To prove Lemma VI.3, we first consider the continuity of σ on $\mathbb{R}_{> 0}$ and then focus on the point $\delta = 0$. Note that σ is the pointwise limit of the sequence of functions $(\sigma_n)_{n=1}^{\infty}$. Lemma VI.4 uses a time-sharing argument as in [21] to show that each σ_n is concave. (See Subsection IX-E for the proof.) Therefore, σ is concave as well, and since $\mathbb{R}_{> 0}$ is open, σ is continuous on $\mathbb{R}_{> 0}$.

Lemma VI.4. For all $n \geq 1$, σ_n is concave on $\mathbb{R}_{\geq 0}$.

To prove the continuity of $\sigma(\delta)$ at $\delta = 0$, we find an upper bound for σ in terms of σ_1 . For some finite set \mathcal{U} and $\delta > 0$, consider a distribution $p(u, x_1^n, x_2^n) \in \mathcal{P}_{\mathcal{U}}^{(n)}(\delta)$. By the definition of $\mathcal{P}_{\mathcal{U}}^{(n)}(\delta)$,

$$I(X_1^n; X_2^n | U) \leq n\delta. \quad (10)$$

Finding a bound for σ in terms of σ_1 requires a single-letter version of (10). In [7], Dueck presents the necessary result. We present Dueck's result in the next lemma and provide the proof in Subsection IX-F.

Lemma VI.5 (Dueck's Lemma [7]). Fix positive reals ϵ and δ , positive integer n , and finite alphabet \mathcal{U} . If $p \in \mathcal{P}_{\mathcal{U}}^{(n)}(\delta)$, then there exists a set $T \subseteq [n]$ satisfying $|T| \leq n\delta/\epsilon$ such that

$$\forall t \notin T: I(X_{1t}; X_{2t} | U, X_1^T, X_2^T) \leq \epsilon,$$

where for $i \in \{1, 2\}$, $X_i^T := (X_{it})_{t \in T}$.

Corollary VI.1 uses Lemma VI.5 to find an upper bound for σ in terms of σ_1 . The proof of this corollary, in Subsection IX-G, combines ideas from [7] with results derived here.

Corollary VI.1. *For all $\epsilon, \delta > 0$, we have*

$$\sigma(\delta) \leq \frac{\delta}{\epsilon} \log |\mathcal{X}_1| |\mathcal{X}_2| + \sigma_1(\epsilon).$$

By Corollary VI.1, we have

$$\sigma(0) \leq \lim_{\delta \rightarrow 0^+} \sigma(\delta) \leq \sigma_1(\epsilon).$$

If we calculate the limit $\epsilon \rightarrow 0^+$, we get

$$\sigma(0) \leq \lim_{\delta \rightarrow 0^+} \sigma(\delta) \leq \lim_{\epsilon \rightarrow 0^+} \sigma_1(\epsilon).$$

Since $\sigma(0) = \sigma_1(0)$,⁴ it suffices to show that $\sigma_1(\delta)$ is continuous at $\delta = 0$. Recall that σ_1 is defined as

$$\sigma_1(\delta) := \sup_{\mathcal{U}} \max_{p \in \mathcal{P}_{\mathcal{U}}^{(1)}(\delta)} I(X_1, X_2; Y|U). \quad (11)$$

Since in (11), the supremum is over *all* finite sets \mathcal{U} , it is difficult to find an upper bound for $\sigma_1(\delta)$ near $\delta = 0$ directly. Instead we first show, in Subsection IX-H, that it is possible to assume that \mathcal{U} has at most two elements.

Lemma VI.6 (Cardinality of \mathcal{U}). *In the definition of $\sigma_1(\delta)$, it suffices to calculate the supremum over all sets \mathcal{U} with $|\mathcal{U}| \leq 2$.*

In Subsection IX-I, we prove the continuity of σ_1 at $\delta = 0$ from Lemma VI.6 using standard tools, such as Pinsker's inequality [3, Lemma 17.3.3] and the L_1 lower bound of KL divergence [3, Lemma 11.6.1]. The continuity of σ_1 on $\mathbb{R}_{>0}$ follows from the concavity of σ_1 on $\mathbb{R}_{\geq 0}$.

Lemma VI.7 (Continuity of σ_1). *The function σ_1 is continuous on $\mathbb{R}_{\geq 0}$.*

VII. CONTINUITY OF SUM-CAPACITY: ARBITRARY \mathbf{C}_{in}

In this section, we study the continuity of $C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ over $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$ with the aim of proving Theorem V.2.

Fix $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$. For arbitrary $(\tilde{\mathbf{C}}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}})$, the triangle inequality implies

$$\begin{aligned} & |C_{\text{sum}}(\tilde{\mathbf{C}}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})| \\ & \leq |C_{\text{sum}}(\tilde{\mathbf{C}}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}})| + |C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})| \end{aligned} \quad (12)$$

We study this bound in the limit $(\tilde{\mathbf{C}}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) \rightarrow (\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$. We begin by considering the first term in (12).

Lemma VII.1 (Continuity of Sum-Capacity in \mathbf{C}_{in}). *There exists a function*

$$\Delta: \mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}$$

that satisfies

$$\lim_{\tilde{\mathbf{C}}_{\text{in}} \rightarrow \mathbf{C}_{\text{in}}} \Delta(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}}) = 0,$$

⁴This follows from the converse proof of the MAC capacity region in the absence of cooperation [3, Theorem 15.3.1].

and for any finite alphabet MAC and $(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}}, \mathbf{C}_{\text{out}}) \in \mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$, we have

$$|C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) - C_{\text{sum}}(\tilde{\mathbf{C}}_{\text{in}}, \mathbf{C}_{\text{out}})| \leq \Delta(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}}).$$

We prove Lemma VII.1 in Subsection IX-J.

Applying Lemma VII.1 to (12), we get

$$|C_{\text{sum}}(\tilde{\mathbf{C}}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})| \leq \Delta(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}}) + |C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})|.$$

Thus to calculate the limit $(\tilde{\mathbf{C}}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) \rightarrow (\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$, Lemma VII.2 investigates

$$\lim_{\tilde{\mathbf{C}}_{\text{out}} \rightarrow \mathbf{C}_{\text{out}}} |C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})|.$$

We prove this lemma in Subsection IX-K.

Lemma VII.2 (Continuity of Sum-Capacity in \mathbf{C}_{out}). *For any finite alphabet MAC and $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) \in \mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$, proving that*

$$\lim_{\tilde{\mathbf{C}}_{\text{out}} \rightarrow \mathbf{C}_{\text{out}}} C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) = C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$$

is equivalent to showing that for all $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) \in \mathbb{R}_{> 0}^2 \times \mathbb{R}_{> 0}^2$, we have

$$\begin{aligned} \lim_{\tilde{C}_{\text{out}}^1 \rightarrow 0^+} C_{\text{sum}}(\mathbf{C}_{\text{in}}, (\tilde{C}_{\text{out}}^1, C_{\text{out}}^2)) &= C_{\text{sum}}(\mathbf{C}_{\text{in}}, (0, C_{\text{out}}^2)) \\ \lim_{\tilde{C}_{\text{out}}^2 \rightarrow 0^+} C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, \tilde{C}_{\text{out}}^2)) &= C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, 0)). \end{aligned}$$

VIII. DISCONTINUITY OF SUM-CAPACITY WITH A CONSTANT NUMBER OF COOPERATION BITS

In this section we prove Theorem V.3. We start by presenting Dueck's deterministic memoryless MAC from [19]. Consider the MAC $(\mathcal{X}_1 \times \mathcal{X}_2, p_{\text{Dueck}}(y|x_1, x_2), \mathcal{Y})$ where $\mathcal{X}_1 = \{a, b, A, B\}$, $\mathcal{X}_2 = \{0, 1\}$, $\mathcal{Y} = \{a, b, c, A, B, C\} \times \{0, 1\}$. The probability transition matrix $p_{\text{Dueck}}(y|x_1, x_2) = 1$ where for the deterministic mapping $W : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{Y}$, $y = W(x_1, x_2)$. The mapping W is defined as

$$W(x_1, x_2) := \begin{cases} (c, 0) & \text{if } (x_1, x_2) \in \{(a, 0), (b, 0)\} \\ (C, 1) & \text{if } (x_1, x_2) \in \{(A, 1), (B, 1)\} \\ (x_1, x_2) & \text{otherwise.} \end{cases}$$

For positive integer n , we define the mapping $W^n : \mathcal{X}_1^n \times \mathcal{X}_2^n \rightarrow \mathcal{Y}^n$ as $(y_1^n, y_2^n) = W^n(x_1^n, x_2^n)$ if and only if for all $1 \leq i \leq n$,

$$(y_{1i}, y_{2i}) = W(x_{1i}, x_{2i}).$$

Set $\mathbf{C}_{\text{in}}^* = (\log \mathcal{X}_1, \log \mathcal{X}_2) = (2, 1)$ to allow the CF to have access to both source messages w_1 and w_2 . We use the following theorem from [19].

Theorem VIII.1 (Outer Bound on the Maximal-Error Sum-Capacity [19]). *For the MAC $(\mathcal{X}_1 \times \mathcal{X}_2, p_{\text{Dueck}}(y|x_1, x_2), \mathcal{Y})$ defined above, we have*

$$C_{\text{sum, max}}(\mathbf{0}, \mathbf{0}) \leq \max_{0 \leq p \leq 1/2} \left[H(1/3) + 2/3 - p + H(p) \right].$$

Optimizing over p , and noting that $C_{\text{sum,max}}(\mathbf{0}, \mathbf{0}) = C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, \mathbf{0})$, Theorem VIII.1 directly implies the following corollary.

Corollary VIII.1. *For the MAC $(\mathcal{X}_1 \times \mathcal{X}_2, p_{\text{Dueck}}(y|x_1, x_2), \mathcal{Y})$ and $p^* = 1/3$, we have*

$$C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, \mathbf{0}) \leq H(p^*) + H(1/3) + 2/3 - p^* \leq 2.1632.$$

To conclude the proof of Theorem V.3, we now show that for some integer k , $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, k/n) > C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, \mathbf{0})$. Specifically, for any $\epsilon > 0$, $\delta > 0$, $k = \lceil \log 2 \lceil 3/\delta \rceil \rceil + 1$, and n sufficiently large, we define a $(2^{nR_1}, 2^{nR_2}, n)$ -code for our MAC with a $(\mathbf{C}_{\text{in}}^*, k/n)$ -CF with zero maximum error in which $R_1 = (1.5 - \delta)(1 - \epsilon)$ and $R_2 = 1 - \epsilon$. Corollary VIII.1 then implies that for $\delta = 1/4$, $k = 6$, and $\epsilon > 0$ sufficiently small we have $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, k/n) > C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, \mathbf{0})$. We specify our code by presenting the functions $(\varphi_1, \varphi_2, \psi_1, \psi_2, f_1, f_2, g)$.

The functions φ_1 and φ_2 are the identity functions. Function ψ_1 is a constant function which always returns 0^k . That is, for our construction, the CF only needs to send the cooperation information to encoder 2. Thus the function f_1 does not depend on ψ_1 and maps message w_1 to x_1^n deterministically according to a codebook $\{x_1^n(w_1)\}_{w_1 \in [2^{nR_1}]}$ to be specified later. Before we define the codebook $\{x_1^n(w_1)\}_{w_1 \in [2^{nR_1}]}$, and functions ψ_2 , f_2 , and g , we set some notation and definitions.

Recall that $\mathcal{Y} = \{a, b, c, A, B, C\} \times \{0, 1\}$. We therefore define $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$, where $\mathcal{Y}_1 = \{A, B, C, a, b, c\}$ and $\mathcal{Y}_2 = \{0, 1\}$. Accordingly, let $y_1^n = (y_{1;1}, \dots, y_{1;n}) \in \mathcal{Y}_1^n$ and $y_2^n = (y_{2;1}, \dots, y_{2;n}) \in \mathcal{Y}_2^n$. Given transmitted codewords x_1^n and x_2^n , recall that $(y_1^n, y_2^n) = W^n(x_1^n, x_2^n)$, where W^n is the blocklength- n extension of W defined above. We use notation $y_1^n = W_1^n(x_1^n, x_2^n)$ and $y_2^n = W_2^n(x_1^n, x_2^n)$ to describe W^n . Formally, we use $W(x_1, x_2) = (W_1(x_1, x_2), W_2(x_1, x_2))$. Note that $y_2^n = W_2^n(x_1^n, x_2^n) = x_2^n$ for all $x_1^n \in \mathcal{X}_1^n$.

Our communication scheme is divided into two phases, the first of blocklength $n_1 = (1 - \epsilon)n$, and the second of blocklength $n_2 = \epsilon n$, with $n = n_1 + n_2$. Roughly speaking, after the first phase the decoder will be able to *list-decode* the message (w_1, w_2) , and after the second it will be able to determine the correct message from its list. We thus refine our notation, and define for $i = 1, 2$: $x_i^{n_1} = (x_{i;1}, \dots, x_{i;n_1})$, $x_i^{n_2} = (x_{i;n_1+1}, \dots, x_{i;n})$, $y_i^{n_1} = (y_{i;1}, \dots, y_{i;n_1})$, and $y_i^{n_2} = (y_{i;n_1+1}, \dots, y_{i;n})$. Accordingly, we represent the two phases in the encoding functions f_1 and f_2 as $f_{1;1}, f_{1;2}$ and $f_{2;1}, f_{2;2}$. Namely, for messages w_1 and w_2 we have $x_1^{n_1} = f_{1;1}(w_1)$, $x_1^{n_2} = f_{1;2}(w_1)$, $x_2^{n_1} = f_{2;1}(w_2, \psi_2(w_1, w_2))$, and $x_2^{n_2} = f_{2;2}(w_2, \psi_2(w_1, w_2))$. We start by discussing the first phase of communication.

Consider a vector $y^{n_1} = (y_1^{n_1}, y_2^{n_1})$. Let $\mathcal{E}(y^{n_1}) = |\{i \mid y_{1;i}^{n_1} \in \{c, C\}\}|$ be the number of symbols in $y_1^{n_1}$ that equal c or C . Note that both c and C are the result of an *erasure*, since the channel maps both a and b , or both A and B , to the same output.⁵ Let $(W^{-1})^{n_1}(y_1^{n_1}, y_2^{n_1})$ be the set of inputs $(x_1^{n_1}, x_2^{n_1}) \in \mathcal{X}_1^{n_1} \times \mathcal{X}_2^{n_1}$ for which $W^{n_1}(x_1^{n_1}, x_2^{n_1}) = (y_1^{n_1}, y_2^{n_1})$; then from the definition of our channel it holds that $|(W^{-1})^{n_1}(y_1^{n_1}, y_2^{n_1})| = 2^{\mathcal{E}(y_1^{n_1})}$ since each erasure symbol must have resulted from one of two possible input symbols.

⁵Since output $Y_1 = c$ occurs only with output $Y_2 = 0$ and output $Y_1 = C$ occurs only with output $Y_2 = 1$, we could, alternatively, denote both by the erasure symbol “ E ” without losing any information.

We say that the pair $(y_1^{n_1}, y_2^{n_1})$ is *good* if $\mathcal{E}(y_1^{n_1}) \leq n_1/2$ or equivalently, $|(W^{-1})^{n_1}(y_1^{n_1}, y_2^{n_1})| \leq 2^{n_1/2}$; thus the first phase of a channel output is good if at most half of the input symbols are erased. In our encoding scheme, we would like to guarantee that $(y_1^{n_1}, y_2^{n_1})$ is always good. This is accomplished using the cooperation bits of ψ_2 , or more specifically, using the first bit $\psi_{2;1}$ of ψ_2 . Note that since we are interested in designing a code with $R_2 = 1 - \epsilon$, we may represent w_2 as a binary vector of length $n(1 - \epsilon)$. With this representation in mind, we specify $\psi_{2;1}$. The remaining bits of ψ_2 will be defined later.

$$\psi_{2;1}(w_1, w_2) = \begin{cases} 0 & \text{if } W^{n_1}(f_{1;1}(w_1), w_2) \text{ is good,} \\ 1 & \text{otherwise.} \end{cases}$$

Note that our MAC and $f_{1;1}$ are deterministic, and thus the CF can compute $W^{n_1}(f_{1;1}(w_1), w_2)$.

We are now ready to define the first phase of encoding, namely $f_{1;1}$ and $f_{2;1}$. We start with $f_{2;1}$ which will equal either w_2 or its element-wise complement \bar{w}_2 .

$$f_{2;1}(w_2, \psi_{2;1}(w_1, w_2)) = \begin{cases} w_2 & \text{if } \psi_{2;1}(w_1, w_2) = 0, \\ \bar{w}_2 & \text{otherwise.} \end{cases}$$

Since $\psi_{2;1}(w_1, w_2)$ is 0 when $W^{n_1}(f_{1;1}(w_1), w_2)$ is good and 1 otherwise, $f_{2;1}$ flips the message w_2 if and only if the unflipped channel input yields a bad channel output. The following claim now guarantees that $y^{n_1} = (y_1^{n_1}, y_2^{n_1})$ is good.

Claim VIII.1. *For any w_1 and w_2 , and any deterministic mapping $f_{1;1}$, if $W^{n_1}(f_{1;1}(w_1), w_2)$ is bad then $W^{n_1}(f_{1;1}(w_1), \bar{w}_2)$ is good.*

Proof: see Section IX-L.

We now define $f_{1;1}$, i.e., the codebook $\{x_1^{n_1}(w_1)\}_{w_1 \in [2^{nR_1}]}$. Consider first choosing the codebook uniformly at random from all subsets of size 2^{nR_1} of $\mathcal{X}_1^{n_1}$. Namely, let the function $f_{1;1}$ be distributed uniformly over all injective functions $[2^{nR_1}] \rightarrow \mathcal{X}_1^{n_1}$. We show that with high probability over $f_{1;1}$, for any good received word $(y_1^{n_1}, y_2^{n_1})$, there are at most $2\lceil 3/\delta \rceil$ codeword pairs $(x_1^{n_1}, x_2^{n_1})$ that satisfy $W^{n_1}(x_1^{n_1}, x_2^{n_1}) = (y_1^{n_1}, y_2^{n_1})$, where δ is defined in our choice of R_1 and is independent of the blocklength n_1 .

Claim VIII.2. *For any sufficiently large n_1 , with probability at least $1 - 2^{-0.4n_1}$ over $f_{1;1}$, for any good pair $(y_1^{n_1}, y_2^{n_1})$, there are at most $2\lceil 3/\delta \rceil$ message pairs (w_1, w_2) such that $W^{n_1}(f_{1;1}(w_1), f_{2;1}(w_2, \psi_{2;1}(w_1, w_2))) = (y_1^{n_1}, y_2^{n_1})$.*

Proof: see Section IX-M.

Consider any function $f_{1;1}$ that satisfies the conditions of Claim VIII.2. Function $f_{1;1}$ defines the codebook $\{x_1^{n_1}(w_1)\}_{w_1 \in [2^{nR_1}]}$ by setting $x_1^{n_1}(w_1) = f_{1;1}(w_1)$. By Claim VIII.1 and our definition of $\psi_{2;1}(w_1, w_2)$, for any message pair (w_1, w_2) , $W^{n_1}(f_{1;1}(w_1), f_{2;1}(w_2, \psi_{2;1}(w_1, w_2))) = (y_1^{n_1}, y_2^{n_1})$ is always good. Consider a preliminary decoding function g_{list} that, given $y^{n_1} = (y_1^{n_1}, y_2^{n_1})$, returns all possible (w_1, w_2) such that $W^{n_1}(f_{1;1}(w_1), f_{2;1}(w_2, \psi_{2;1}(w_1, w_2))) = (y_1^{n_1}, y_2^{n_1})$. By Claim VIII.2, the decoder g_{list} is a *list-decoder* with list size $2\lceil 3/\delta \rceil$. Here, a list-decoder is a decoding function that returns a list of potential messages (of a limited

size) which, under our deterministic channel model and code design, is guaranteed to include the original source messages.

We thus conclude that for $R_1 = (1.5 - \delta)(1 - \epsilon)$ and $R_2 = 1 - \epsilon$, after the first phase of communication, the decoder can recover a list L of size $\ell = 2\lceil 3/\delta \rceil$ of potential message pairs that includes the original message pair (w_1, w_2) of the encoder. Moreover, we note, due to the deterministic nature of our channel, that the CF can calculate the list L and the location (in lexicographic order) of the original message (w_1, w_2) in the list. As the list size is ℓ , $\lceil \log \ell \rceil = \lceil \log 2\lceil 3/\delta \rceil \rceil$ bits suffice to specify the location of the original message (w_1, w_2) in the list. We therefore define the remaining bits of $\psi_2(w_1, w_2)$, denoted by $\psi_{2;2}(w_1, w_2)$, to be the binary representation of the location of the original message (w_1, w_2) in the list L decoded by the decoder. We now show how encoders 1 and 2 can send $\psi_{2;2}(w_1, w_2)$ to the decoder in the second phase of our communication scheme, allowing it to recover the original message (w_1, w_2) from the list L obtained after the first phase.

To send $\psi_{2;2}(w_1, w_2)$ to the decoder during the second phase of communication (of blocklength $n_2 = \epsilon n$) we use the fact that the rate pair $(0, 1)$ is in the zero error capacity region of our MAC. This follows by noticing that $W_2(x_1, x_2) = x_2$. Thus, we set $f_{1;2}(w_1)$ to be constant and $f_{2;2}(w_2, \psi_{2;2}(w_1, w_2))$ to equal $\psi_{2;2}(w_1, w_2)$ padded on the left by zeros (here, we use the fact that for sufficiently large n it holds that $n_2 = \epsilon n \geq \lceil \log 2\lceil 3/\delta \rceil \rceil$). The decoder can now recover $\psi_{2;2}(w_1, w_2)$ and in turn the original message pair (w_1, w_2) from its list. This concludes the proof of Theorem V.3.

IX. PROOFS

In this section, we begin with the proof of Corollary V.2. We then provide detailed proofs of the lemmas appearing in Sections VI, VII, and VIII.

A. Proof of Theorem IV.1

We seek to prove that for any MAC where cooperation increases sum-capacity (formally specified by the definition of \mathcal{C}^* in Section IV), the benefit of cooperation grows at least as fast as the square root of C_{out} .

Since $\mathbf{C}_{\text{in}} \in \mathbb{R}_{>0}^2$, there exists $\lambda = \lambda(\mathbf{C}_{\text{in}}) \in (0, 1)$ such that for $i \in \{1, 2\}$,

$$C_{\text{in}}^i > \lambda C_{\text{in}}^{*i}.$$

Using a time-sharing argument, it follows that

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}, h\mathbf{v}) \geq C_{\text{sum}}(\lambda\mathbf{C}_{\text{in}}^*, \lambda h\mathbf{v}) \geq \lambda C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, h\mathbf{v}) + (1 - \lambda)C_{\text{sum}}(\mathbf{0}, \mathbf{0}). \quad (13)$$

Applying in (13) the fact that

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{0}) = C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) = C_{\text{sum}}(\mathbf{0}, \mathbf{0})$$

results in

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}, h\mathbf{v}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) \geq \lambda(C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, h\mathbf{v}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{0})). \quad (14)$$

On the other hand, setting $\mathbf{C}_{\text{out}} = h\mathbf{v} = h(v_1, v_2)$ in the lower bound of Lemma VI.2 gives

$$\begin{aligned} C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, h\mathbf{v}) &\geq \sigma(h(v_1 + v_2)) - \min\{hv_1, hv_2\} \\ &\geq \sigma_1(h(v_1 + v_2)) - \min\{hv_1, hv_2\}, \end{aligned} \quad (15)$$

where (15) follows by (9). Note that

$$\sigma_1(0) = C_{\text{sum}}(\mathbf{0}, \mathbf{0}) = C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{0}).$$

Therefore, by combining (14) and (15), we get

$$C_{\text{sum}}(\mathbf{C}_{\text{in}}, h\mathbf{v}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) \geq \lambda(\sigma_1(hv_1 + hv_2) - \sigma_1(0)) - \lambda h \min\{v_1, v_2\}. \quad (16)$$

Given (16), the following lemma completes the proof.

Lemma IX.1. *For any discrete memoryless MAC in \mathcal{C}^* , there exists a constant $K > 0$, such that*

$$\sigma_1(\delta) \geq \sigma_1(0) + K\sqrt{\delta} + o(\sqrt{\delta}).$$

Remark. The key result in proving Lemma IX.1 is the asymptotic equivalence of the Kullback-Liebler (KL) divergence and the chi-squared divergence [22, Theorem 4.1, p. 448]. Precisely, let $p(x)$ and $q(x)$ be distributions on a finite alphabet \mathcal{X} . Then the KL and the chi-squared divergences between p and q are given by

$$\begin{aligned} D(p(x)\|q(x)) &:= \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} \\ \chi^2(p(x), q(x)) &:= \sum_{x \in \mathcal{X}} \frac{(p(x) - q(x))^2}{q(x)}, \end{aligned}$$

respectively. Suppose q has full support; that is, for all $x \in \mathcal{X}$, $q(x) > 0$. Then if p tends to q pointwise on \mathcal{X} , then

$$\frac{D(p(x)\|q(x))}{\chi^2(p(x), q(x))} \rightarrow \frac{1}{2 \ln 2}.$$

We remark that in a different setting, the authors of [23] also use this property of the KL divergence to obtain a ‘‘square-root law’’ similar to Lemma IX.1 above.

Proof of Lemma IX.1: Since our MAC is in \mathcal{C}^* , by definition, there exists a distribution $p_0(x_1)p_0(x_2)$ with support $\mathcal{S}_0 \subseteq \mathcal{X}_1 \times \mathcal{X}_2$ that satisfies

$$I_0(X_1, X_2; Y) = \max_{p(x_1)p(x_2)} I(X_1, X_2; Y)$$

and a distribution $p_1(x_1, x_2)$ with support $\mathcal{S}_1 \subseteq \mathcal{S}_0$ that satisfies

$$I_1(X_1, X_2; Y) + D(p_1(y)\|p_0(y)) > I_0(X_1, X_2; Y).$$

For each $\lambda \in (0, 1)$, define the distribution $p_\lambda(x_1, x_2)$ as

$$p_\lambda(x_1, x_2) := (1 - \lambda)p_0(x_1)p_0(x_2) + \lambda p_1(x_1, x_2).$$

We have

$$\begin{aligned}
I_\lambda(X_1; X_2) &= \sum_{x_1, x_2} p_\lambda(x_1, x_2) \log \frac{p_\lambda(x_1, x_2)}{p_\lambda(x_1)p_\lambda(x_2)} \\
&= \sum_{x_1, x_2} p_\lambda(x_1, x_2) \log \frac{p_\lambda(x_1, x_2)}{p_0(x_1)p_0(x_2)} + \sum_{x_1, x_2} p_\lambda(x_1, x_2) \log \frac{p_0(x_1)p_0(x_2)}{p_\lambda(x_1)p_\lambda(x_2)} \\
&= \sum_{x_1, x_2} p_\lambda(x_1, x_2) \log \frac{p_\lambda(x_1, x_2)}{p_0(x_1)p_0(x_2)} + \sum_{x_1} p_\lambda(x_1) \log \frac{p_0(x_1)}{p_\lambda(x_1)} + \sum_{x_2} p_\lambda(x_2) \log \frac{p_0(x_2)}{p_\lambda(x_2)} \\
&= D(p_\lambda(x_1, x_2) \| p_0(x_1)p_0(x_2)) - D(p_\lambda(x_1) \| p_0(x_1)) - D(p_\lambda(x_2) \| p_0(x_2))
\end{aligned}$$

Note \mathcal{S}_λ , the support of $p_\lambda(x_1, x_2)$, satisfies

$$\mathcal{S}_\lambda \subseteq \mathcal{S}_0 \cup \mathcal{S}_1 \subseteq \mathcal{S}_0. \quad (17)$$

Therefore, by [22, Theorem 4.1, p. 448] (described in the remark above), we have

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda^2} I_\lambda(X_1; X_2) = K_1, \quad (18)$$

where

$$K_1 := \frac{1}{2 \ln 2} \left(\chi^2(p_1(x_1, x_2), p_0(x_1)p_0(x_2)) - \chi^2(p_1(x_1), p_0(x_1)) - \chi^2(p_1(x_2), p_0(x_2)) \right). \quad (19)$$

Since $\mathcal{S}_1 \subseteq \mathcal{S}_0$, all chi-squared distances in (19) are well-defined. Also, since mutual information is always nonnegative, by (18) we have $K_1 \geq 0$. Fix $\epsilon > 0$, and define the mapping $\delta: [0, 1] \rightarrow \mathbb{R}$ as

$$\delta(\lambda) := I_\lambda(X_1; X_2) + \epsilon \lambda^2.$$

Note that by (18),

$$\lim_{\delta \rightarrow 0^+} \frac{\delta(\lambda)}{\lambda^2} = K_1 + \epsilon > 0.$$

Furthermore, since δ is continuously differentiable in λ , by the inverse function theorem, there exists a function $\lambda^*: [0, \delta_0) \rightarrow [0, 1]$ for some $\delta_0 > 0$ that satisfies

$$\lim_{\delta \rightarrow 0^+} \frac{(\lambda^*(\delta))^2}{\delta} = \frac{1}{K_1 + \epsilon},$$

or equivalently,

$$\lim_{\delta \rightarrow 0^+} \frac{\lambda^*(\delta)}{\sqrt{\delta}} = \frac{1}{\sqrt{K_1 + \epsilon}}.$$

Thus we have

$$\lambda^*(\delta) = \sqrt{\frac{\delta}{K_1 + \epsilon}} + o(\sqrt{\delta}). \quad (20)$$

Now consider $I_\lambda(X_1, X_2; Y)$ for $\lambda \in [0, 1]$. Direct differentiation gives [6, Lemma 15, Part (iii)]

$$I_\lambda(X_1, X_2; Y) = I_0(X_1, X_2; Y) + \left(I_1(X_1, X_2; Y) - I_0(X_1, X_2; Y) + D(p_1(y) \| p_0(y)) \right) \lambda + o(\lambda).$$

If we now set $\lambda = \lambda^*(\delta)$, define constant K as

$$K := \frac{1}{\sqrt{K_1 + \epsilon}} \left(I_1(X_1, X_2; Y) - I_0(X_1, X_2; Y) + D(p_1(y) \| p_0(y)) \right),$$

and apply (20), we get

$$I_{\lambda^*(\delta)}(X_1, X_2; Y) = I_0(X_1, X_2; Y) + K\sqrt{\delta} + o(\sqrt{\delta}).$$

Note that by the definition of σ_1 ,

$$\sigma_1(\delta) \geq I_{\lambda^*(\delta)}(X_1, X_2; Y)$$

$$\sigma_1(0) = I_0(X_1, X_2; Y).$$

This concludes the proof of Lemma IX.1.

B. Proof of Corollary V.2

By Proposition IV.1, we only need to prove one direction. Specifically, here we show that for a fixed MAC and $\mathbf{C}_{\text{in}} \in \mathbb{R}_{>0}^2$, if $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ is not continuous at $\mathbf{C}_{\text{out}} = \mathbf{0}$, then

$$C_{\text{sum}}(\mathbf{0}, \mathbf{0}) > C_{\text{sum,max}}(\mathbf{0}, \mathbf{0}).$$

This is equivalent to showing that if

$$C_{\text{sum}}(\mathbf{0}, \mathbf{0}) = C_{\text{sum,max}}(\mathbf{0}, \mathbf{0}), \tag{21}$$

then $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ is continuous at $\mathbf{C}_{\text{out}} = \mathbf{0}$.

We begin by defining the function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ as

$$f(C_{\text{out}}) := C_{\text{sum,max}}(\mathbf{C}_{\text{in}}^*, (C_{\text{out}}, C_{\text{out}})),$$

where \mathbf{C}_{in}^* is defined in Section V. Then by [5, Theorem 1] for all $C_{\text{out}} > 0$, and by (21) for $C_{\text{out}} = 0$, we have

$$f(C_{\text{out}}) = C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, (C_{\text{out}}, C_{\text{out}})).$$

For each $\mathbf{C}_{\text{out}} \in \mathbb{R}_{\geq 0}^2$, let $C_{\text{out}}^* := \max\{C_{\text{out}}^1, C_{\text{out}}^2\}$. Then for any $\mathbf{C}_{\text{in}} \in \mathbb{R}_{\geq 0}^2$,

$$f(0) \leq C_{\text{sum,max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) \leq f(C_{\text{out}}^*).$$

If we now let $\mathbf{C}_{\text{out}} \rightarrow \mathbf{0}$ and apply Theorem V.1, the continuity of $C_{\text{sum,max}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ at $\mathbf{C}_{\text{out}} = \mathbf{0}$ follows.

C. Proof of Lemma VI.1

Our goal here is to show that for integers $m, n \geq 1$ and $\delta \geq 0$,

$$(m+n)\sigma_{m+n}(\delta) \geq m\sigma_m(\delta) + n\sigma_n(\delta).$$

By the definition of $\sigma_n(\delta)$, for all $\epsilon > 0$, there exist finite alphabets \mathcal{U}_0 and \mathcal{U}_1 and distributions $p_n \in \mathcal{P}_{\mathcal{U}_0}^{(n)}(\delta)$ and $p_m \in \mathcal{P}_{\mathcal{U}_1}^{(m)}(\delta)$ such that

$$I_n(X_1^n, X_2^n; Y^n | U_0) \geq n\sigma_n(\delta) - n\epsilon$$

$$I_m(X_1^m, X_2^m; Y^m | U_1) \geq m\sigma_m(\delta) - m\epsilon.$$

Consider the distribution

$$p_{n+m}(u_0, u_1, x_1^{n+m}, x_2^{n+m}) = p_n(u_0, x_1^n, x_2^n) p_m(u_1, x_1^{n+1:n+m}, x_2^{n+1:n+m}).$$

Let $\mathcal{U} := \mathcal{U}_0 \times \mathcal{U}_1$. Then it is straightforward to show that $p_{n+m} \in \mathcal{P}_{\mathcal{U}}^{(n+m)}(\delta)$, and

$$I_{n+m}(X_1^{n+m}, X_2^{n+m}; Y^{n+m} | U_0, U_1) \geq n\sigma_n(\delta) + m\sigma_m(\delta) - (n+m)\epsilon,$$

which implies the desired result.

D. Proof of Lemma VI.2

Here we prove lower and upper bounds for $C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$ in terms of $\sigma(C_{\text{out}}^1 + C_{\text{out}}^2)$.

We first prove the lower bound. For $i \in \{1, 2\}$, choose C_{dep}^i such that

$$0 \leq C_{\text{dep}}^i \leq C_{\text{out}}^i.$$

The rate C_{dep}^i indicates the amount of information the CF sends to encoder i to enable dependence between the encoders' codewords. (See [6, Section III] for details.) For a finite alphabet \mathcal{U} and positive integer n , let $p(u, x_1^n, x_2^n)$ be a distribution that satisfies

$$I(X_1^n; X_2^n | U) = n(C_{\text{dep}}^1 + C_{\text{dep}}^2).$$

Then applying [6, Corollary 8] to the MAC

$$p(y^n | x_1^n, x_2^n) = \prod_{t \in [n]} p(y_t | x_{1t}, x_{2t}),$$

gives

$$\begin{aligned} nC_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}}) &\geq I(X_1^n, X_2^n; Y^n | U) - n \min\{C_{1d}, C_{2d}\} \\ &\geq I(X_1^n, X_2^n; Y^n | U) - n \min\{C_{\text{out}}^1, C_{\text{out}}^2\}. \end{aligned} \quad (22)$$

This completes the proof of the lower bound.

For the upper bound, consider a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ -codes for the MAC with a $(\mathbf{C}_{\text{in}}^*, \mathbf{C}_{\text{out}})$ -CF. For a fixed positive integer n , let W_1 and W_2 be independent random variables distributed uniformly on $[2^{nR_1}]$ and $[2^{nR_2}]$, respectively. Furthermore, for the blocklength- n code and transmitted message pair (W_1, W_2) , let random variables $Z_1 \in [2^{nC_{\text{out}}^1}]$ and $Z_2 \in [2^{nC_{\text{out}}^2}]$ denote the information the CF sends to encoders 1 and 2 respectively. Note that (Z_1, Z_2) is a deterministic function of (W_1, W_2) . By the data processing inequality,

$$\begin{aligned} I(X_1^n; X_2^n) &\leq I(X_1^n; W_2, Z_2) \\ &\leq I(W_1, Z_1; W_2, Z_2) \\ &= H(W_1, Z_1) + H(W_2, Z_2) - H(W_1, W_2, Z_1, Z_2) \\ &= H(Z_1 | W_1) + H(Z_2 | W_2) \\ &\leq n(C_{\text{out}}^1 + C_{\text{out}}^2). \end{aligned}$$

In addition, from Fano's inequality it follows that there exists a sequence $(\epsilon_n)_{n=1}^{\infty}$ such that

$$H(W_1, W_2|Y^n) \leq n\epsilon_n,$$

and $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. We have

$$\begin{aligned} n(R_1 + R_2) &= H(W_1, W_2) \\ &= I(W_1, W_2; Y^n) + H(W_1, W_2|Y^n) \\ &\leq I(W_1, W_2; Y^n) + n\epsilon_n \\ &= I(X_1^n, X_2^n; Y^n) + n\epsilon_n \\ &\leq n\sigma(C_{\text{out}}^1 + C_{\text{out}}^2) + n\epsilon_n. \end{aligned}$$

Dividing by n and taking the limit as $n \rightarrow \infty$ completes the proof.

E. Proof of Lemma VI.4

For $n \geq 1$, we use the auxiliary random variable U in the definition of σ_n to show that σ_n is concave.

It suffices to prove the result for $n = 1$ since the proof for arbitrary integers follows similarly. We apply the technique from [21, Appendix B]. Recall that

$$\sigma_1(\delta) = \sup_U \max_{p \in \mathcal{P}_{\mathcal{U}}^{(1)}(\delta)} I(X_1, X_2; Y|U).$$

Fix $a, b \geq 0$, $\lambda \in (0, 1)$, and $\epsilon > 0$. By definition, there exist finite sets \mathcal{U}_0 and \mathcal{U}_1 and distributions $p_0 \in \mathcal{P}_{\mathcal{U}_0}^{(1)}(a)$ and $p_1 \in \mathcal{P}_{\mathcal{U}_1}^{(1)}(b)$ satisfying

$$\begin{aligned} I_0(X_1, X_2; Y|U_0) &\geq \sigma_1(a) - \epsilon \\ I_1(X_1, X_2; Y|U_1) &\geq \sigma_1(b) - \epsilon, \end{aligned}$$

respectively. Define the alphabet \mathcal{V} as

$$\mathcal{V} := \{0\} \times \mathcal{U}_0 \cup \{1\} \times \mathcal{U}_1.$$

We denote an element of \mathcal{V} by $v = (v_1, v_2)$. Define the distribution $p_\lambda(v, x_1, x_2)$ as

$$p_\lambda(v, x_1, x_2) = p_\lambda(v_1)p_{v_1}(v_2, x_1, x_2),$$

where

$$p_\lambda(v_1) = \begin{cases} 1 - \lambda & \text{if } v_1 = 0 \\ \lambda & \text{if } v_1 = 1. \end{cases}$$

Then

$$\begin{aligned} I_\lambda(X_1; X_2|V) &= I_\lambda(X_1, X_2|V_1, V_2) \\ &= (1 - \lambda)I(X_1; X_2|V_1 = 0, V_2) + \lambda I(X_1; X_2|V_1 = 1, V_2) \\ &= (1 - \lambda)I_0(X_1; X_2|U_0) + \lambda I_1(X_1; X_2|U_1) \\ &\leq (1 - \lambda)a + \lambda b, \end{aligned}$$

which implies $p_\lambda \in \mathcal{P}_V^{(1)}((1-\lambda)a + \lambda b)$. Similarly,

$$\begin{aligned} I_\lambda(X_1, X_2; Y|V) &= I_\lambda(X_1, X_2; Y|V_1, V_2) \\ &= (1-\lambda)I(X_1, X_2; Y|V_1 = 0, V_2) + \lambda I(X_1, X_2; Y|V_1 = 1, V_2) \\ &= (1-\lambda)I_0(X_1, X_2; Y|U_0) + \lambda I_1(X_1, X_2; Y|U_1) \\ &\geq (1-\lambda)\sigma_1(a) + \lambda\sigma_1(b) - \epsilon. \end{aligned}$$

Therefore,

$$\sigma_1((1-\lambda)a + \lambda b) \geq (1-\lambda)\sigma_1(a) + \lambda\sigma_1(b) - \epsilon.$$

The result now follows from the fact that the above equation holds for all $\epsilon > 0$.

F. Proof of Lemma VI.5 (Dueck's Lemma)

If for all $t \in [n]$, we have

$$I(X_{1t}; X_{2t}|U) \leq \epsilon,$$

then we define $T := \emptyset$. Otherwise, there exists $t_1 \in [n]$ such that

$$I(X_{1t_1}; X_{2t_1}|U) > \epsilon. \quad (23)$$

Let $S_1 := [n] \setminus \{t_1\}$. Then

$$\begin{aligned} I(X_1^n; X_2^n|U) &= I(X_1^n; X_{2t_1}|U) + I(X_1^n; X_2^{S_1}|U, X_{2t_1}) \\ &= I(X_{1t_1}; X_{2t_1}|U) + I(X_1^{S_1}; X_{2t_1}|U, X_{1t_1}) \\ &\quad + I(X_{1t_1}; X_2^{S_1}|U, X_{2t_1}) + I(X_1^{S_1}; X_2^{S_1}|U, X_{1t_1}, X_{2t_1}) \\ &\geq I(X_{1t_1}; X_{2t_1}|U) + I(X_1^{S_1}; X_2^{S_1}|U, X_{1t_1}, X_{2t_1}). \end{aligned}$$

Since $I(X_1^n; X_2^n|U) \leq n\delta$, using (23), we get

$$I(X_1^{S_1}; X_2^{S_1}|U, X_{1t_1}, X_{2t_1}) \leq n\delta - \epsilon.$$

Now if for all $t \in S_1$,

$$I(X_{1t}; X_{2t}|U, X_{1t_1}, X_{2t_1}) \leq \epsilon,$$

then we define $T := \{t_1\}$. Otherwise, there exists $t_2 \in [n]$ such that

$$I(X_{1t_2}; X_{2t_2}|U, X_{1t_1}, X_{2t_1}) > \epsilon.$$

Similar to the above argument, if we define $S_2 := [n] \setminus \{t_1, t_2\}$, then

$$I(X_1^{S_2}; X_2^{S_2}|U, X_{1t_1}, X_{1t_2}, X_{2t_1}, X_{2t_2}) \leq n\delta - 2\epsilon.$$

If we continue this process, we eventually get a set $T := \{t_1, \dots, t_k\}$ such that

$$I(X_1^{T^c}; X_2^{T^c}|U, X_1^T, X_2^T) \leq n\delta - |T|\epsilon, \quad (24)$$

and for all $t \in S_k := T^c$,

$$I(X_{1t}; X_{2t} | U, X_1^T, X_2^T) \leq \epsilon.$$

In addition, from (24) it follows that

$$|T| \leq \frac{n\delta}{\epsilon}. \quad (25)$$

G. Proof of Corollary VI.1

Here we give an upper bound for $\sigma(\delta)$, which is defined as

$$\sigma(\delta) := \lim_{n \rightarrow \infty} \sigma_n(\delta).$$

in terms of $\sigma_1(\delta)$.

Fix a positive integer n . By Lemma VI.6, we can choose $\mathcal{U} = \{a, b\}$ in the definition of $\sigma_n(\delta)$. From Lemma VI.5, it follows that there exists a set $T \subseteq [n]$ such that

$$0 \leq |T| \leq \frac{n\delta}{\epsilon}, \quad (26)$$

and

$$\forall t \notin T: I(X_{1t}; X_{2t} | U, X_1^T, X_2^T) \leq \epsilon.$$

Thus

$$\begin{aligned} I(X_1^n, X_2^n; Y^n | U) &= I(X_1^T, X_2^T; Y^n | U) + I(X_1^{T^c}, X_2^{T^c}; Y^n | U, X_1^T, X_2^T) \\ &\leq |T| \log |\mathcal{X}_1| |\mathcal{X}_2| + I(X_1^{T^c}, X_2^{T^c}; Y^n | U, X_1^T, X_2^T). \end{aligned} \quad (27)$$

We further bound the second term on the right hand side by

$$\begin{aligned} &I(X_1^{T^c}, X_2^{T^c}; Y^n | U, X_1^T, X_2^T) \\ &= I(X_1^{T^c}, X_2^{T^c}; Y^{T^c} | U, X_1^T, X_2^T) + I(X_1^{T^c}, X_2^{T^c}; Y^T | U, X_1^T, X_2^T, Y^{T^c}) \\ &= I(X_1^{T^c}, X_2^{T^c}; Y^{T^c} | U, X_1^T, X_2^T) \\ &\leq \sum_{t \notin T} I(X_{1t}, X_{2t}; Y_t | U, X_1^T, X_2^T) \\ &\leq n \max_{p \in \mathcal{P}_{\mathcal{V}}^{(1)}(\epsilon)} I(X_1, X_2; Y | V) \leq n\sigma_1(\epsilon), \end{aligned} \quad (28)$$

where in (28),

$$\mathcal{V} := \mathcal{U} \times \mathcal{X}_1^{|T|} \times \mathcal{X}_2^{|T|}.$$

Therefore, by (26), (27), and (28),

$$\frac{1}{n} I(X_1^n, X_2^n; Y^n | U) \leq \frac{\delta}{\epsilon} \log |\mathcal{X}_1| |\mathcal{X}_2| + \sigma_1(\epsilon),$$

which completes the proof.

H. Proof of Lemma VI.6

Here we show that in the definition of σ_1 ,

$$\sigma_1(\delta) := \sup_{\mathcal{U}} \max_{p \in \mathcal{P}_{\mathcal{U}}^{(n)}(\delta)} I(X_1, X_2; Y|U)$$

where the supremum is over all finite sets \mathcal{U} , we can instead take the supremum over all sets \mathcal{U} with cardinality at most two.

Let \mathcal{U} be some finite set and let $p^*(u, x_1, x_2) \in \mathcal{P}_{\mathcal{U}}^{(1)}(\delta)$ be a distribution that satisfies

$$I^*(X_1, X_2; Y|U) = \max_{p \in \mathcal{P}_{\mathcal{U}}^{(1)}(\delta)} I(X_1, X_2; Y|U),$$

where the mutual information term $I^*(X_1, X_2; Y|U)$ on the left hand side is calculated with respect to $p^*(u, x_1, x_2)p(y|x_1, x_2)$. Let $\mathcal{Q} \subseteq \mathbb{R}^{|\mathcal{U}|}$ denote the set of all vectors $(q(u))_{u \in \mathcal{U}}$ that satisfy

$$\begin{aligned} q(u) &\geq 0 \text{ for all } u \in \mathcal{U} \\ \sum_{u \in \mathcal{U}} q(u) &= 1 \\ \sum_{u \in \mathcal{U}} q(u) I^*(X_1; X_2|U = u) &= I^*(X_1; X_2|U), \end{aligned} \quad (29)$$

where in (29), $I^*(X_1; X_2|U = u)$ and $I^*(X_1; X_2|U)$ are calculated according to $p^*(x_1, x_2|u)$ and $p^*(u, x_1, x_2)$, respectively. Note that \mathcal{Q} is nonempty since $p^*(u) \in \mathcal{Q}$. Consider the mapping $F: \mathcal{Q} \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$F[q] := \sum_{u \in \mathcal{U}} q(u) I^*(X_1, X_2; Y|U = u), \quad (30)$$

where $I^*(X_1, X_2; Y|U = u)$ is calculated with respect to $p^*(x_1, x_2|u)p(y|x_1, x_2)$. Since $p^*(u) \in \mathcal{Q}$ and for all $q(u) \in \mathcal{Q}$, by (29) we have $q(u)p^*(x_1, x_2|u) \in \mathcal{P}_{\mathcal{U}}^{(1)}(\delta)$, thus

$$\max_{q \in \mathcal{Q}} F[q] = \max_{p \in \mathcal{P}_{\mathcal{U}}^{(1)}(\delta)} I(X_1, X_2; Y|U).$$

Therefore, it suffices to find $q^* \in \mathcal{Q}$ which has at most two non-zero components and at which F obtains its maximal value. To this end, note that since \mathcal{Q} is a nonempty bounded polyhedron, by [24, p. 65, Corollary 2.2] and [24, p. 50, Theorem 2.3], \mathcal{Q} has at least one extreme point. Since F is linear in q and \mathcal{Q} has at least one extreme point, [24, p. 65, Theorem 2.7] shows that there exists an extreme point of \mathcal{Q} , say $q^* \in \mathcal{Q}$, at which F obtains its maximum. Finally, since q^* is an extreme point, applying [24, p. 50, Theorem 2.3] to the definition of \mathcal{Q} implies that we must have $q^*(u) = 0$ for at least $|\mathcal{U}| - 2$ values of u . This completes the proof.

I. Proof of Lemma VI.7

We next show that $\sigma_1(\delta)$ is continuous at $\delta = 0$.

By Lemma VI.6, without loss of generality, we can set $\mathcal{U} := \{a, b\}$. (Our proof below applies for any finite set \mathcal{U} .) For all $\delta \geq 0$, we have

$$\sigma_1(\delta) = \max_{p \in \mathcal{P}_{\mathcal{U}}^{(1)}(\delta)} I(X_1, X_2; Y|U).$$

Fix $\delta > 0$. Let $p^*(u, x_1, x_2)$ be a distribution in $\mathcal{P}_{\mathcal{U}}^{(1)}(\delta)$ achieving the maximum above, and define

$$p_{\text{ind}}^*(x_1, x_2|u) := p^*(x_1|u)p^*(x_2|u).$$

In addition, for each $u \in \mathcal{U}$, let

$$\begin{aligned} D(p^*(x_1, x_2|u) \| p_{\text{ind}}^*(x_1, x_2|u)) &= \sum_{x_1, x_2} p^*(x_1, x_2|u) \log \frac{p^*(x_1, x_2|u)}{p_{\text{ind}}^*(x_1, x_2|u)} \\ \|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1} &= \sum_y |p^*(y|u) - p_{\text{ind}}^*(y|u)| \\ \|p^*(x_1, x_2|u) - p_{\text{ind}}^*(x_1, x_2|u)\|_{L^1} &= \sum_{x_1, x_2} |p^*(x_1, x_2|u) - p_{\text{ind}}^*(x_1, x_2|u)|. \end{aligned}$$

Since

$$\sum_{u \in \mathcal{U}} p^*(u) D(p^*(x_1, x_2|u) \| p_{\text{ind}}^*(x_1, x_2|u)) = I^*(X_1; X_2|U) \leq \delta,$$

by applying [3, Lemma 11.6.1] for every $u \in \mathcal{U}$, we get

$$\sum_{u \in \mathcal{U}} p^*(u) \|p^*(x_1, x_2|u) - p_{\text{ind}}^*(x_1, x_2|u)\|_{L^1}^2 \leq 2\delta \ln 2. \quad (31)$$

In addition,

$$\begin{aligned} &\sum_{u \in \mathcal{U}} p^*(u) \|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1} \\ &\leq \sum_{u \in \mathcal{U}} p^*(u) \sum_{x_1, x_2} p(y|x_1, x_2) |p^*(x_1, x_2|u) - p_{\text{ind}}^*(x_1, x_2|u)| \\ &\leq \sum_{u \in \mathcal{U}} p^*(u) \|p^*(x_1, x_2|u) - p_{\text{ind}}^*(x_1, x_2|u)\|_{L^1} \\ &\leq \sqrt{2\delta \ln 2}, \end{aligned} \quad (32)$$

where (32) follows from (31) and the Cauchy-Schwarz inequality. Define the subset $\mathcal{U}_0 \subseteq \mathcal{U}$ as

$$\mathcal{U}_0 = \left\{ u \in \mathcal{U} : \|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1} \leq 1/2 \right\}.$$

Clearly, by (32),

$$\sum_{u \notin \mathcal{U}_0} p^*(u) \leq 2\sqrt{2\delta \ln 2}. \quad (33)$$

Thus

$$\begin{aligned}
& |H^*(Y|U) - H_{\text{ind}}^*(Y|U)| \\
& \leq \sum_{u \in \mathcal{U}} p^*(u) |H^*(Y|U = u) - H_{\text{ind}}^*(Y|U = u)| \\
& \leq 2\sqrt{2\delta \ln 2} \log |\mathcal{Y}| \tag{34}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{u \in \mathcal{U}_0} p^*(u) \|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1} \log \frac{\|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1}}{|\mathcal{Y}|} \\
& \leq 2\sqrt{2\delta \ln 2} \log |\mathcal{Y}| \tag{35}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{u \in \mathcal{U}} p^*(u) \|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1} \log \frac{\|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1}}{|\mathcal{Y}|} \\
& \leq 2\sqrt{2\delta \ln 2} \log |\mathcal{Y}| - \sqrt{2\delta \ln 2} \log \left(\frac{1}{|\mathcal{Y}|} \sqrt{2\delta \ln 2} \right) \tag{36}
\end{aligned}$$

$$= \sqrt{2\delta \ln 2} \log \frac{|\mathcal{Y}|^3}{\sqrt{2\delta \ln 2}}, \tag{37}$$

where (34) follows from (33) and [3, Theorem 17.3.3], (35) follows from the fact that for all $u \in \mathcal{U}$,

$$- \|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1} \log \frac{\|p^*(y|u) - p_{\text{ind}}^*(y|u)\|_{L^1}}{|\mathcal{Y}|} \geq 0,$$

and for δ satisfying

$$0 < \sqrt{2\delta \ln 2} < \frac{1}{e} |\mathcal{Y}|, \tag{38}$$

(36) follows from (32) and the fact that the mapping $t \mapsto -t \log(t/|\mathcal{Y}|)$ is concave on $(0, \infty)$ and increasing on $(0, |\mathcal{Y}|/e)$. In addition, by (32),

$$\begin{aligned}
& |H^*(Y|U, X_1, X_2) - H_{\text{ind}}^*(Y|U, X_1, X_2)| \\
& \leq \sum_{u, x_1, x_2} |p^*(u, x_1, x_2) - p_{\text{ind}}^*(u, x_1, x_2)| H(Y|X_1 = x_1, X_2 = x_2) \\
& \leq \sum_{u, x_1, x_2} |p^*(u, x_1, x_2) - p_{\text{ind}}^*(u, x_1, x_2)| \log |\mathcal{Y}| \\
& = \left(\log |\mathcal{Y}| \right) \cdot \sum_{u \in \mathcal{U}} p^*(u) \|p^*(x_1, x_2|u) - p_{\text{ind}}^*(x_1, x_2|u)\|_{L^1} \\
& \leq \left(\log |\mathcal{Y}| \right) \sqrt{2\delta \ln 2}. \tag{39}
\end{aligned}$$

Thus by (37) and (39), for all δ satisfying (38),

$$\begin{aligned}
\sigma_1(\delta) &= I^*(X_1, X_2; Y|U) = H^*(Y|U) - H^*(Y|U, X_1, X_2) \\
&\leq |H^*(Y|U) - H_{\text{ind}}^*(Y|U)| + |H^*(Y|U, X_1, X_2) - H_{\text{ind}}^*(Y|U, X_1, X_2)| \\
&\quad + I_{\text{ind}}^*(X_1, X_2; Y|U) \\
&\leq \sqrt{2\delta \ln 2} \log \frac{|\mathcal{Y}|^3}{\sqrt{2\delta \ln 2}} + \left(\log |\mathcal{Y}| \right) \sqrt{2\delta \ln 2} + \sigma_1(0).
\end{aligned}$$

Since $\sigma_1(0) \leq \sigma_1(\delta)$ for all $\delta \geq 0$, the continuity of σ_1 at $\delta = 0^+$ follows.

J. Proof of Lemma VII.1

For a fixed $\mathbf{C}_{\text{out}} \in \mathbb{R}_{\geq 0}^2$, here we show that $C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ is continuous with respect to \mathbf{C}_{in} .

Define the functions $f, g: \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}$ as

$$f(\mathbf{C}_{\text{in}}) := C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) = C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}) - C_{\text{sum}}(\mathbf{0}, \mathbf{0})$$

$$g(\mathbf{C}_{\text{in}}) := C_{\text{in}}^1 + C_{\text{in}}^2 - f(\mathbf{C}_{\text{in}}).$$

Note that since f is concave, g is convex. Thus for all $\lambda \in [0, 1]$ and all $(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}})$,

$$g(\lambda \mathbf{C}_{\text{in}} + (1 - \lambda)\tilde{\mathbf{C}}_{\text{in}}) \leq \lambda g(\mathbf{C}_{\text{in}}) + (1 - \lambda)g(\tilde{\mathbf{C}}_{\text{in}}).$$

Since $g(\mathbf{0}) = 0$, setting $\tilde{\mathbf{C}}_{\text{in}} = \mathbf{0}$ gives

$$g(\lambda \mathbf{C}_{\text{in}}) \leq \lambda g(\mathbf{C}_{\text{in}})$$

Note that by [6, Proposition 6], g is nonnegative. Thus

$$g(\lambda \mathbf{C}_{\text{in}}) \leq g(\mathbf{C}_{\text{in}}),$$

which when written in terms of f , is equivalent to

$$f(\mathbf{C}_{\text{in}}) - f(\lambda \mathbf{C}_{\text{in}}) \leq (1 - \lambda)(C_{\text{in}}^1 + C_{\text{in}}^2). \quad (40)$$

Consider $\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}} \in \mathbb{R}_{\geq 0}^2$. Define the pairs $\underline{\mathbf{C}}_{\text{in}}, \bar{\mathbf{C}}_{\text{in}} \in \mathbb{R}_{\geq 0}^2$ as

$$\forall i \in \{1, 2\}: C_{\text{in}}^i := \min\{C_{\text{in}}^i, \tilde{C}_{\text{in}}^i\}$$

$$\forall i \in \{1, 2\}: \bar{C}_{\text{in}}^i := \max\{C_{\text{in}}^i, \tilde{C}_{\text{in}}^i\}.$$

Next define $\lambda^*(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}}) \in [0, 1]$ as⁶

$$\lambda^* := \min_{i \in \{1, 2\}} C_{\text{in}}^i / \bar{C}_{\text{in}}^i.$$

Then

$$|f(\mathbf{C}_{\text{in}}) - f(\tilde{\mathbf{C}}_{\text{in}})| \leq |f(\mathbf{C}_{\text{in}}) - f(\underline{\mathbf{C}}_{\text{in}})| + |f(\underline{\mathbf{C}}_{\text{in}}) - f(\tilde{\mathbf{C}}_{\text{in}})| \quad (41)$$

$$= f(\mathbf{C}_{\text{in}}) - f(\underline{\mathbf{C}}_{\text{in}}) + f(\tilde{\mathbf{C}}_{\text{in}}) - f(\underline{\mathbf{C}}_{\text{in}}) \quad (42)$$

$$\leq f(\mathbf{C}_{\text{in}}) - f(\lambda^* \mathbf{C}_{\text{in}}) + f(\tilde{\mathbf{C}}_{\text{in}}) - f(\lambda^* \tilde{\mathbf{C}}_{\text{in}}) \quad (43)$$

$$\leq (1 - \lambda^*)(C_{\text{in}}^1 + C_{\text{in}}^2) + (1 - \lambda^*)(\tilde{C}_{\text{in}}^1 + \tilde{C}_{\text{in}}^2), \quad (44)$$

where (41) follows from the triangle inequality, (42) follows from the definition of $\underline{\mathbf{C}}_{\text{in}}$, (43) follows from the definition of λ^* , and (44) follows from (40). Finally, if we let $\tilde{\mathbf{C}}_{\text{in}} \rightarrow \mathbf{C}_{\text{in}}$ in (44), we see that $f(\tilde{\mathbf{C}}_{\text{in}}) \rightarrow f(\mathbf{C}_{\text{in}})$, since

$$\lim_{\tilde{\mathbf{C}}_{\text{in}} \rightarrow \mathbf{C}_{\text{in}}} \lambda^*(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}}) = 1.$$

⁶If for some $i \in \{1, 2\}$, say $i = 1$, $\bar{C}_{\text{in}}^i = 0$, set $\lambda^* := \min\{1, C_{\text{in}}^2 / \bar{C}_{\text{in}}^2\}$. If $\bar{C}_{\text{in}}^1 = \bar{C}_{\text{in}}^2 = 0$, set $\lambda^* = 1$. These definitions ensure the continuity of λ^* in $(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{in}})$.

K. Proof of Lemma VII.2

Here we derive a necessary and sufficient condition for the continuity of $C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ with respect to \mathbf{C}_{out} for a fixed \mathbf{C}_{in} .

Recall that we only need to verify continuity on the boundary of $\mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0}^2$; namely, the set of all points $(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}})$ where at least one of C_{in}^1 , C_{in}^2 , C_{out}^1 , or C_{out}^2 is zero.

We first show that

$$\lim_{\tilde{\mathbf{C}}_{\text{out}} \rightarrow \mathbf{C}_{\text{out}}} C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) = C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{C}_{\text{out}}).$$

holds if $\mathbf{C}_{\text{out}} = \mathbf{0}$, or if either $C_{\text{in}}^1 = 0$ or $C_{\text{in}}^2 = 0$. For the case $\mathbf{C}_{\text{out}} = \mathbf{0}$, note that

$$\begin{aligned} C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, \mathbf{0}) &= C_{\text{sum}}(\mathbf{C}_{\text{in}}, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{0}, \mathbf{0}) \\ &\leq C_{\text{sum}}(\mathbf{C}_{\text{in}}^*, \tilde{\mathbf{C}}_{\text{out}}) - C_{\text{sum}}(\mathbf{0}, \mathbf{0}), \end{aligned}$$

which goes to zero as $\tilde{\mathbf{C}}_{\text{out}} \rightarrow \mathbf{0}$ by Theorem V.1.

Next suppose $C_{\text{in}}^2 = 0$. In this case, we have

$$C_{\text{sum}}((C_{\text{in}}^1, 0), \tilde{\mathbf{C}}_{\text{out}}) = C_{\text{sum}}((C_{\text{in}}^1, 0), (0, \tilde{C}_{\text{out}}^2)).$$

Let $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ denote the function

$$f(C_{\text{out}}^2) := C_{\text{sum}}((C_{\text{in}}^1, 0), (0, C_{\text{out}}^2)).$$

Note that f is continuous on $\mathbb{R}_{> 0}$ since it is concave. To prove the continuity of f at $C_{\text{out}}^2 = 0$, observe that $f(C_{\text{out}}^2)$ equals the sum-capacity of a MAC with a $(C_{12}, 0)$ -conference [4], where

$$C_{12} := \min\{C_{\text{in}}^1, C_{\text{out}}^2\}.$$

From the capacity region given in [4], we have

$$f(C_{\text{out}}^2) \leq f(0) + \min\{C_{\text{in}}^1, C_{\text{out}}^2\},$$

which implies that f is continuous at $C_{\text{out}}^2 = 0$. The case where $C_{\text{in}}^1 = 0$ follows similarly.

Finally, consider the case where $C_{\text{out}}^2 = 0$, but $C_{\text{out}}^1 > 0$. In this case, we apply the next lemma for concave functions that are nondecreasing as well.

Lemma IX.2. *Let $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be concave and nondecreasing. Then if $|x - y| \leq \min\{x, y\}$,*

$$|f(x) - f(y)| \leq f(|x - y|) - f(0).$$

We have

$$\begin{aligned} &|C_{\text{sum}}(\mathbf{C}_{\text{in}}, (\tilde{C}_{\text{out}}^1, \tilde{C}_{\text{out}}^2)) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, 0))| \\ &\stackrel{(a)}{\leq} |C_{\text{sum}}(\mathbf{C}_{\text{in}}, (\tilde{C}_{\text{out}}^1, \tilde{C}_{\text{out}}^2)) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, \tilde{C}_{\text{out}}^2))| + |C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, \tilde{C}_{\text{out}}^2)) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, 0))| \\ &\stackrel{(b)}{\leq} |C_{\text{sum}}(\mathbf{C}_{\text{in}}, (|\tilde{C}_{\text{out}}^1 - C_{\text{out}}^1|, \tilde{C}_{\text{out}}^2)) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, (0, \tilde{C}_{\text{out}}^2))| \\ &\quad + |C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, \tilde{C}_{\text{out}}^2)) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, (C_{\text{out}}^1, 0))|, \end{aligned}$$

where (a) follows from the triangle inequality, and (b) follows from Lemma IX.2. If we now let $\tilde{\mathbf{C}}_{\text{out}} \rightarrow (C_{\text{out}}^1, 0)$, Corollary V.1 implies

$$\lim_{\tilde{\mathbf{C}}_{\text{out}} \rightarrow (C_{\text{out}}^1, 0)} \left| C_{\text{sum}}(\mathbf{C}_{\text{in}}, (|\tilde{\mathbf{C}}_{\text{out}}^1 - C_{\text{out}}^1|, \tilde{\mathbf{C}}_{\text{out}}^2)) - C_{\text{sum}}(\mathbf{C}_{\text{in}}, (0, \tilde{\mathbf{C}}_{\text{out}}^2)) \right| = 0,$$

from which our result follows. An analogous proof applies in the case where $C_{\text{out}}^1 = 0$ and $C_{\text{out}}^2 > 0$.

L. Proof of Claim VIII.1

The result follows from the definition of W^{n_1} . Consider entry i of $w_2 \in \{0, 1\}^{n_1}$, $w_{2;i} \in \{0, 1\}$, and its corresponding entry $\bar{w}_{2;i} \in \{0, 1\}$ in $\bar{w}_2 \in \{0, 1\}^{n_1}$. Let $x_1^{n_1} = f_{1;1}(w_1)$, and let $x_{1;i}$ be the i 'th entry of $x_1^{n_1}$. It holds (by a simple exhaustive case analysis) that exactly one of the values $W_1(x_{1;i}, w_{2;i})$ and $W_1(x_{1;i}, \bar{w}_{2;i})$ is in the set $\{c, C\}$. Thus, the number of entries in $W_1^{n_1}(f_{1;1}(w_1), w_2)$ that are in the set $\{c, C\}$ plus the number of entries in $W_1^{n_1}(f_{1;1}(w_1), \bar{w}_2)$ that are in the set $\{c, C\}$ equals n_1 . If the former exceeds $n_1/2$ (i.e., $W^{n_1}(f_{1;1}(w_1), w_2)$ is bad) then the latter is less than $n_1/2$ (i.e., $W^{n_1}(f_{1;1}(w_1), \bar{w}_2)$ is good).

M. Proof of Claim VIII.2

Consider a good pair $(y_1^{n_1}, y_2^{n_1})$. As the pair $(y_1^{n_1}, y_2^{n_1})$ is good, we start by noting that the set $(W^{-1})^{n_1}(y_1^{n_1}, y_2^{n_1})$ of preimages $(x_1^{n_1}, x_2^{n_1}) \in \mathcal{X}_1^n \times \mathcal{X}_2^n$ for which $W^{n_1}(x_1^{n_1}, x_2^{n_1}) = (y_1^{n_1}, y_2^{n_1})$ satisfies that $x_1^{n_1}$ must be one of at most $2^{n_1/2}$ vectors in $\mathcal{X}_1^{n_1}$. Denote this latter subset of $\mathcal{X}_1^{n_1}$ by $\text{Pre}_1(y_1^{n_1})$. In addition, for $W^{n_1}(f_{1;1}(w_1), f_{2;1}(w_2, \psi_{2;1}(w_1, w_2)))$ to be equal to $(y_1^{n_1}, y_2^{n_1})$ it must be the case that w_2 equals $y_2^{n_1}$ or its complement. Thus, the number of message pairs (w_1, w_2) such that $W^{n_1}(f_{1;1}(w_1), f_{2;1}(w_2, \psi_{2;1})) = (y_1^{n_1}, y_2^{n_1})$ equals 2 times the number of messages w_1 for which $f_{1;1}(w_1) \in \text{Pre}_1(y_1^{n_1})$. We analyze this latter quantity.

Recall that the codebook $\{x_1^{n_1}(w_1)\}_{w_1 \in [2^{nR_1}]}$ is chosen uniformly at random from the collection of all subsets of size 2^{nR_1} of $\mathcal{X}_1^{n_1}$. Namely, one can choose the codebook $\{x_1^{n_1}(w_1)\}_{w_1 \in [2^{nR_1}]}$ in an iterative manner, where in iteration w_1 , $x_1^{n_1}(w_1)$ is chosen uniformly from $\mathcal{X}_1^{n_1} \setminus \{x_1^{n_1}(w'_1)\}_{w'_1 < w_1}$. Here, we use the standard order on $[2^{nR_1}]$. For any w_1 , conditioning on the value of $f_{1;1}(w'_1)$ for all $w'_1 < w_1$, the probability (over the choice of $f_{1;1}(w_1) = x_1^{n_1}(w_1)$) that $f_{1;1}(w_1) \in \text{Pre}_1(y_1^{n_1})$ is at most

$$\frac{2^{n_1/2}}{4^{n_1} - 2^{3n_1/2 - \delta n_1}} \leq 2 \cdot 2^{-3n_1/2}$$

for sufficiently large n . We use the fact that $n_1 = (1 - \epsilon)n$, that in each iteration at most $2^{nR_1} = 2^{n_1(3/2 - \delta)}$ codewords have been chosen so far, and that our choices are without repetition. Thus, the probability that there exist ℓ messages $\{w_{1,1}, \dots, w_{1,\ell}\}$ in $[2^{nR_1}]$ such that for all $i = 1, \dots, \ell$ it holds that $f_{1;1}(w_{1,i}) \in \text{Pre}_1(y_1^{n_1})$ is at most

$$\begin{aligned} \binom{2^{3n_1/2 - \delta n_1}}{\ell} 2^\ell 2^{-3\ell n_1/2} &\leq 2^{3\ell n_1/2 - \delta n_1 \ell} 2^\ell 2^{-3\ell n_1/2} \\ &= 2^{\ell 2 - \delta n_1 \ell} \end{aligned}$$

Setting $\ell = \lceil 3/\delta \rceil + 1$, we have, for sufficiently large n (and thus n_1), that the above probability is at most 2^{-3n_1} . Finally, taking the union bound over possible $y_1^{n_1}$, we conclude that with probability at least $1 - |\mathcal{Y}_1|^{n_1} \cdot 2^{-3n_1} =$

$1 - 2^{-n_1(3-\log 6)} \geq 1 - 2^{-0.4n_1}$ (over $f_{1:1}$) for any good pair $(y_1^{n_1}, y_2^{n_1})$, the number of messages w_1 for which $f_{1:1}(w_1) \in \text{Pre}_1(y_1^{n_1})$ is bounded by $\lceil 3/\delta \rceil$. This, in turn, implies that for any good pair $(y_1^{n_1}, y_2^{n_1})$ the list size obtained by the decoder is bounded by $2\lceil 3/\delta \rceil$.

X. SUMMARY

Consider a network consisting of a discrete MAC and a CF that has full knowledge of the messages. In this work, we show that the average-error sum-capacity of such a network is always a continuous function of the CF output link capacities; this is in contrast to our previous results on maximal-error sum-capacity [5] and our current result using only a constant number of cooperation bits. For the average case analysis, our proof method relies on finding lower and upper bounds on the sum-capacity and then using a modified version of a technique developed by Dueck [7] to demonstrate continuity. Our result on maximal-error considers first the maximal-error list-decoding capacity, and then reduces list-decoding to unique-decoding. Our result strongly relies on the precise functionality of Dueck's MAC [19] and on the fact that it has a maximal-error capacity region that differs from its average-error capacity region. A deeper understanding of the family of MACs for which a constant number of bits (or even a single bit) of cooperation can affect the maximal-error capacity region is left for future research.

The edge removal problem opens the door to a host of related questions on general multi-terminal networks. Each question seeks to determine whether adding a δ capacity noiseless channel e to a memoryless network \mathcal{N} results in an average-error capacity region that is not continuous at $\delta = 0$. These questions help us pinpoint whether the addition of asymptotically negligible cooperation can ever have a non-negligible impact on average-error capacity as it can on maximal-error capacity.

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